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A global higher regularity result for the static relaxed micromorphic model on smooth domains

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We derive a global higher regularity result for weak solutions of the linear relaxed micromorphic model on smooth domains. The governing equations consist of a linear elliptic system of partial differential equations that is coupled with a system of Maxwell-type. The result is obtained by combining a Helmholtz decomposition argument with regularity results for linear elliptic systems and the classical embedding of $H(\mathrm{div};\Omega)\cap H_0(\mathrm{curl};\Omega)$ into $H^1(\Omega)$.

Keywords: global regularity; smooth domain; relaxed micromorphic model; elasticity coupled with Maxwell system; Helmholtz decomposition; generalized continua; dislocation model

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1. Introduction

The relaxed micromorphic model is a novel generalized continuum model allowing to describe size effects and band-gap behaviour of microstructured solids with effective equations ignoring the detailed microstructure [**2**, **6**, **8**, **21**, **25**, **28**]. As a micromorphic model it couples the classical displacement $u:\Omega\subset\mathbb{R}^3\to\mathbb{R}^3$ with a non-symmetric tensor field $P:\Omega\subset\mathbb{R}^3\to\mathbb{R}^{3\times 3}$ called the microdistortion through the variational problem

$$\begin{aligned}\int_\Omega \Big(&\big\langle \mathbb{C}_{\mathrm{e}}\, \mathrm{sym}(\mathrm{D}u - P), \mathrm{sym}(\mathrm{D}u - P)\big\rangle \\ &+ \big\langle \mathbb{C}_{\mathrm{micro}}\, \mathrm{sym}\, P, \mathrm{sym}\, P\big\rangle + \big\langle \mathbb{L}_{\mathrm{c}}\, \mathrm{Curl}\, P, \mathrm{Curl}\, P\big\rangle \\ &- \langle f, u\rangle - \langle M, P\rangle \Big)\, \mathrm{d}x \quad \longrightarrow \min \text{ w.r.t. } (u, P),\end{aligned} \tag{1.1}$$

subject to suitable boundary conditions. The tensor $\mathbb{L}_{\mathrm{c}}$ introduces a size-dependence into the model in the sense that smaller samples respond relatively stiffer. The existence and uniqueness in the static case follows from the incompatible Korn's inequality [**13**, **18**–**20**, **27**]. The constitutive tensors $\mathbb{C}_{\mathrm{e}}$, $\mathbb{C}_{\mathrm{micro}}$ and $\mathbb{L}_{\mathrm{c}}$ are to be found by novel homogenization strategies [**23**, **35**–**37**]. Letting $\mathbb{C}_{\mathrm{micro}} \to +\infty$ the models response tends to the linear Cosserat model [**9**]. A range of engineering relevant analytical solutions are already available for the relaxed micromorphic model [**34**]. The solution is naturally found as $u \in H^1(\Omega)$ and $P \in H(\mathrm{Curl}; \Omega)$, thus the microdistortion P may have jumps in normal direction. The implementation in the finite element context needs standard element formulations for the displacement u, but e.g. Nédélec - spaces for P in order to achieve optimal convergence rates [**33**, **37**–**40**].

However, it is sometimes preferred to circumvent the Nédélec - framework and to work with $H^1(\Omega)$ for the microdistortion tensor P. For these cases it is mandatory to clarify in advance whether the regularity of P allows for a faithful result. In this spirit, we continue here the investigation of regularity in the static case and we will be able to derive a global higher regularity result for weak solutions of the relaxed micromorphic model (global as opposed to only interior regularity). It extends the local result from [**15**] to smooth domains. The latter is formulated on a bounded domain $\Omega \subset \mathbb{R}^3$ and the Euler-Lagrange equations to (1.1) read as follows ([**2**, **8**, **25**, **28**, **29**]): given positive definite and symmetric material dependent coefficient tensors $\mathbb{C}_e : \Omega \to \mathrm{Lin}(\mathrm{Sym}(3), \mathrm{Sym}(3))$, $\mathbb{C}_{\mathrm{micro}} : \Omega \to \mathrm{Lin}(\mathrm{Sym}(3), \mathrm{Sym}(3))$ and $\mathbb{L}_c : \Omega \to \mathrm{Lin}(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$ determine a displacement field $u : \Omega \to \mathbb{R}^3$ and a non-symmetric microdistortion tensor $P : \Omega \to \mathbb{R}^{3\times 3}$ satisfying

$$\begin{aligned} 0 &= \mathrm{Div}\left(\mathbb{C}_e\, \mathrm{sym}(\mathrm{D}u - P)\right) + f \quad \text{in } \Omega, \\ 0 &= -\,\mathrm{Curl}\left(\mathbb{L}_c\, \mathrm{Curl}\, P\right) + \mathbb{C}_e\, \mathrm{sym}(\mathrm{D}u - P) - \mathbb{C}_{\mathrm{micro}}\, \mathrm{sym}\, P + M \quad \text{in } \Omega \end{aligned} \tag{1.2}$$

together with suitable boundary conditions. Here, $f : \Omega \to \mathbb{R}^3$ is a given volume force density, $M : \Omega \to \mathbb{R}^{3\times 3}$ a body moment tensor and $\sigma = \mathbb{C}_e\, \mathrm{sym}(\mathrm{D}u - P)$ is the symmetric force stress tensor while $m = \mathbb{L}_{\mathrm{c}}\, \mathrm{Curl}\, P$ is the non-symmetric moment tensor.

The main result of the present contribution is theorem 4.1 which states that on smooth domains and with smooth coefficients weak solutions of (1.2) are more regular and satisfy

$$u \in H^2(\Omega), \quad P \in H^1(\Omega)\,, \quad \sigma \in H^1(\Omega)\,, \quad m \in H(\mathrm{Curl}; \Omega)\,, \quad \mathrm{Curl}\, m \in H^1(\Omega), \tag{1.3}$$

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where the last regularity in (1.3) follows from equation (1.2). Moreover, if $\mathbb{L}_c$ has a special block diagonal structure (see corollary 4.2), then we in addition have

$$m \in H^1(\Omega), \qquad \operatorname{Curl} P \in H^1(\Omega). \tag{1.4}$$

The results (1.3) are obtained by a combination of the Helmholtz decomposition for the matrix field P, regularity results for linear elliptic systems of elasticity-type and the classical Maxwell embedding recalled in theorem 2.4. The additional regularity formulated in (1.4) relies on a weighted version of the Maxwell embedding theorem, [**42**].

2. Background from function space theory

2.1. Notation, assumptions

For vectors $a, b \in \mathbb{R}^3$, we define the scalar product $\langle a, b\rangle := \sum_{i=1}^3 a_i b_i$, the Euclidean norm $|a|^2 := \langle a, a\rangle$ and the dyadic product $a \otimes b = (a_i b_j)_{i,j=1}^3 \in \mathbb{R}^{3\times 3}$, where $\mathbb{R}^{3\times 3}$ will denote the set of real 3×3 matrices. For matrices $P, Q \in \mathbb{R}^{3\times 3}$, we define the standard Euclidean scalar product $\langle P, Q\rangle := \sum_{i=1}^3 \sum_{j=1}^3 P_{ij} Q_{ij}$ and the Frobenius-norm $\|P\|^2 := \langle P, P\rangle$. $P^T \in \mathbb{R}^{3\times 3}$ denotes the transposition of the matrix $P \in \mathbb{R}^{3\times 3}$ and for $P \in \mathbb{R}^{3\times 3}$, the symmetric part of P will be denoted by $\operatorname{sym} P = 1/2(P + P^T) \in \operatorname{Sym}(3)$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. As a minimal requirement, we assume in this paper that the boundary $\partial\Omega$ is Lipschitz continuous, meaning that it can locally be described as the graph of a Lipschitz continuous function, see [**14**] for a precise definition. In a similar spirit we speak of C^1 or $C^{1,1}$-regular boundaries. For a function $u = (u^1, u^2, u^3)^T : \Omega \to \mathbb{R}^3$, the differential $\mathrm{D}u$ is given by

$$\mathrm{D}u = \begin{pmatrix} \mathrm{D}u^1 \\ \mathrm{D}u^2 \\ \mathrm{D}u^3 \end{pmatrix} \in \mathbb{R}^{3\times 3},$$

with $(\mathrm{D}u^k)_l = \partial_{x_l} u^k$ for $1 \leqslant k \leqslant 3$ and $1 \leqslant \ell \leqslant 3$ and with $\mathrm{D}u^k \in \mathbb{R}^{1\times 3}$. For a vector field $w : \Omega \to \mathbb{R}^3$, the divergence and the curl are given as

$$\operatorname{div} w = \sum_{i=1}^3 w^i_{,x_i}, \quad \operatorname{curl} w = \left(w^3_{,x_2} - w^2_{,x_3}, w^1_{,x_3} - w^3_{,x_1}, w^1_{,x_2} - w^2_{,x_1}\right).$$

For tensor fields $Q : \Omega \to \mathbb{R}^{3\times 3}$, $\operatorname{Curl} Q$ and $\operatorname{Div} Q$ are defined row-wise:

$$\operatorname{Curl} Q = \begin{pmatrix} \operatorname{curl} Q^1 \\ \operatorname{curl} Q^2 \\ \operatorname{curl} Q^3 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \quad \text{and } \operatorname{Div} Q = \begin{pmatrix} \operatorname{div} Q^1 \\ \operatorname{div} Q^2 \\ \operatorname{div} Q^3 \end{pmatrix} \in \mathbb{R}^3,$$

where Q^i denotes the i-th row of Q. With these definitions, for $u : \Omega \to \mathbb{R}^3$ we have consistently $\operatorname{Curl} \mathrm{D}u = 0 \in \mathbb{R}^{3\times 3}$.

The Sobolev spaces [1, 11] used in this paper are

$$H^1(\Omega) = \{u \in L^2(\Omega)\,|\,\mathrm{D}u \in L^2(\Omega)\}\,, \quad \|u\|^2_{H^1(\Omega)} := \|u\|^2_{L^2(\Omega)} + \|\mathrm{D}u\|^2_{L^2(\Omega)},$$
$$H(\mathrm{curl};\Omega) = \{v \in L^2(\Omega;\mathbb{R}^d)\,|\; \mathrm{curl}\, v \in L^2(\Omega)\}, \quad \|v\|^2_{H(\mathrm{curl};\Omega)} := \|v\|^2_{L^2(\Omega)} + \|\mathrm{curl}\, v\|^2_{L^2(\Omega)},$$
$$H(\mathrm{div};\Omega) = \{v \in L^2(\Omega;\mathbb{R}^d)\,|\; \mathrm{div}\, v \in L^2(\Omega)\}, \quad \|v\|^2_{H(\mathrm{div};\Omega)} := \|v\|^2_{L^2(\Omega)} + \|\,\mathrm{div}\, v\|^2_{L^2(\Omega)},$$

spaces for tensor valued functions are denoted by $H(\mathrm{Curl};\Omega)$ and $H(\mathrm{Div};\Omega)$. Moreover, $H^1_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ with respect to the H^1-norm and $H_0(\mathrm{curl};\Omega)$ and $H_0(\mathrm{div};\Omega)$ are the completions of $C^\infty_0(\Omega)$ with respect to the $H(\mathrm{curl})$-norm and the $H(\mathrm{div})$-norm, respectively. By $H^{-1}(\Omega)$ we denote the dual of $H^1_0(\Omega)$. Finally we define

$$\begin{aligned} H(\mathrm{div}, 0;\Omega) &= \{u \in H(\mathrm{div};\Omega)\,|\; \mathrm{div}\, u = 0\},\\ H(\mathrm{curl}, 0;\Omega) &= \{u \in H(\mathrm{curl};\Omega)\,|\; \mathrm{curl}\, u = 0\} \end{aligned} \tag{2.1}$$

and set

$$\begin{aligned} H_0(\mathrm{div}, 0;\Omega) &= H_0(\mathrm{div};\Omega) \cap H(\mathrm{div}, 0;\Omega),\\ H_0(\mathrm{curl}, 0;\Omega) &= H_0(\mathrm{curl};\Omega) \cap H(\mathrm{curl}, 0;\Omega). \end{aligned} \tag{2.2}$$

Assumption A: We assume that the coefficient functions $\mathbb{C}_{\mathrm{e}}$, $\mathbb{C}_{\mathrm{micro}}$ and $\mathbb{L}_{\mathrm{c}}$ in (1.2) are fourth order elasticity tensors from $C^{0,1}(\overline{\Omega};\mathrm{Lin}(\mathbb{R}^{3\times3};\mathbb{R}^{3\times3}))$ and are symmetric and positive definite in the following sense

(i) For every σ, $\tau \in \mathrm{Sym}(3)$, η_1, $\eta_2 \in \mathbb{R}^{3\times3}$ and all $x \in \overline{\Omega}$:

$$\langle \mathbb{C}_e(x)\sigma, \tau\rangle = \langle \sigma, \mathbb{C}_e(x)\tau\rangle, \quad \langle \mathbb{C}_{\mathrm{micro}}(x)\sigma, \tau\rangle = \langle \sigma, \mathbb{C}_{\mathrm{micro}}(x)\tau\rangle,$$
$$\langle \mathbb{L}_c(x)\eta_1, \eta_2\rangle = \langle \eta_1, \mathbb{L}_c(x)\eta_2\rangle. \tag{2.3}$$

(ii) There exists positive constants C_{e}, C_{micro} and L_{c} such that for all $x \in \overline{\Omega}$, $\sigma \in \mathrm{Sym}(3)$ and $\eta \in \mathbb{R}^{3\times3}$:

$$\langle \mathbb{C}_{\mathrm{e}}(x)\sigma, \sigma\rangle \geqslant C_{\mathrm{e}}|\sigma|^2, \quad \langle \mathbb{C}_{\mathrm{micro}}(x)\sigma, \sigma\rangle \geqslant C_{\mathrm{micro}}|\sigma|^2\,, \quad \langle \mathbb{L}_{\mathrm{c}}(x)\eta, \eta\rangle \geqslant L_{\mathrm{c}}|\eta|^2. \tag{2.4}$$

2.2. Helmholtz decomposition, embeddings, elliptic regularity

Based on the results from Section 3.3 of the book [11] (Corollary 3.4), see also [4, Theorem 5.3], the following version of the Helmholtz decomposition will be used:

THEOREM 2.1 Helmholtz decomposition. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary. Then*

$$L^2(\Omega;\mathbb{R}^3) = \mathrm{D}H^1_0(\Omega) \oplus H(\mathrm{div}, 0;\Omega)$$

and hence, for every $p \in L^2(\Omega;\mathbb{R}^3)$ there exist unique $v \in H^1_0(\Omega)$ and $q \in H$ $(\mathrm{div},\, 0;\Omega)$ such that $p = \mathrm{D}v + q$.

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It is worth mentioning that in the case of the L^2-theory, theorem 2.1 holds for any bounded domain $\Omega \subset \mathbb{R}^3$ without having a Lipschitz boundary, see e.g. [**7**]. An immediate consequence of the Helmholtz decomposition theorem is

PROPOSITION 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let $p \in L^2(\Omega)$ with $p = \mathrm{D}v + q$, where $v \in H_0^1(\Omega)$ and $q \in H(\mathrm{div},\, 0;\Omega)$ are given according to the Helmholtz decomposition. Then $\mathrm{D}v \in H_0(\mathrm{curl},\, 0;\Omega)$.*

Proof. Notice that it is sufficient to prove that

$$\mathrm{D}H_0^1(\Omega) \subset H_0(\mathrm{curl}, 0;\Omega)\,.$$

For this, let us assume that $v \in H_0^1(\Omega)$ and let $\{v_n\} \subset C_0^\infty(\Omega)$ such that $v_n \to v$ in $H^1(\Omega)$. Then, for all $\phi \in C_0^\infty(\Omega)$ we obtain

$$\int_\Omega \langle \mathrm{D}v, \mathrm{curl}\,\phi\rangle \,\mathrm{d}x \leftarrow \int_\Omega \langle \mathrm{D}v_n, \mathrm{curl}\,\phi\rangle \,\mathrm{d}x = -\int_\Omega \langle v_n, \mathrm{div}\,\mathrm{curl}\,\phi\rangle \,\mathrm{d}x = 0$$

and $\mathrm{D}v \in H(\mathrm{curl},\, 0;\Omega)$. Moreover, $\{\mathrm{D}v_n\} \subset C_0^\infty(\Omega) \cap H_0(\mathrm{curl},\, 0;\Omega)$ and $\mathrm{D}v_n \to \mathrm{D}v$ in $H(\mathrm{curl};\Omega)$. As $H_0(\mathrm{curl},\, 0;\Omega)$ is a closed subspace of $H(\mathrm{curl};\Omega)$ we conclude that $\mathrm{D}v \in H_0(\mathrm{curl},\, 0;\Omega)$. □

In an analogous way to proposition 2.2, we can prove the following lemma, from which additional regularities will be derived for the solution of system (1.2) (see corollary 4.2).

LEMMA 2.3. *For any bounded domain $\Omega \subset \mathbb{R}^3$ it holds*

$$\overline{\mathrm{curl}\, H_0(\mathrm{curl};\Omega)} \subset H_0(\mathrm{div}, 0;\Omega).$$

Proof. Let us assume that $\{E_n\} \subset C_0^\infty(\Omega)$ such that $E_n \to E$ in $H(\mathrm{curl};\Omega)$. Then, for all $\phi \in C_0^\infty(\Omega)$ we obtain

$$\int_\Omega \langle \mathrm{curl}\, E, \mathrm{D}\phi\rangle \,\mathrm{d}x \leftarrow \int_\Omega \langle \mathrm{curl}\, E_n, \mathrm{D}\phi\rangle \,\mathrm{d}x = -\int_\Omega \langle \mathrm{div}\,\mathrm{curl}\, E_n, \phi\rangle \,\mathrm{d}x = 0$$

and $\mathrm{curl}\, E \in H(\mathrm{div},\, 0;\Omega)$. Moreover, $\{\mathrm{curl}\, E_n\} \subset C_0^\infty(\Omega) \cap H_0(\mathrm{div},\, 0;\Omega)$ and $\mathrm{curl}\, E_n \to \mathrm{curl}\, E$ in $H(\mathrm{div};\Omega)$. As $H_0(\mathrm{div},\, 0;\Omega)$ is a closed subspace of $H(\mathrm{div};\Omega)$ we conclude that $\mathrm{curl}\, E \in H_0(\mathrm{div},\, 0;\Omega)$. □

The next embedding theorem is for instance proved in [**11**, Sections 3.4, 3.5] and [**16**] (see [**14**] for properties of convex sets):

THEOREM 2.4 Embedding theorem. *Let $\Omega \subset \mathbb{R}^3$ be a convex domain or a domain with a $C^{1,1}$-smooth boundary $\partial\Omega$. Then*

$$H(\mathrm{curl};\Omega) \cap H_0(\mathrm{div};\Omega) \subset H^1(\Omega), \qquad H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega) \subset H^1(\Omega)$$

and there exists a constant $C > 0$ such that for every $p \in H(\mathrm{curl};\Omega) \cap H_0(\mathrm{div};\Omega)$ or $p \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ we have

$$\|p\|_{H^1(\Omega)} \leqslant C(\|p\|_{H(\mathrm{curl};\Omega)} + \|p\|_{H(\mathrm{div};\Omega)})\,.$$

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A version of this result for Lipschitz domains showing $H^{1/2}(\Omega)$- regularity is given in [5] (similar results for bounded strong Lipschitz domains with strong Lipschitz boundary interface parts may be found in [31, 32]). In addition, it can be noted that convex domains are strongly Lipschitz, cf. [14]. A proof for convex domains including a family of relevant boundary conditions can be found in [30]. For the previous embedding theorem there are also some versions with weights, and we cite here Theorem 2.2 from [42] with $k = 1$ and $\ell = 0$. We assume that the weight function $\varepsilon : \overline{\Omega} \to \mathbb{R}^{3\times3}$ for every $x \in \overline{\Omega}$ is symmetric and positive definite, uniformly in x.

THEOREM 2.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a C^2-smooth boundary and let $\varepsilon \in C^1(\overline{\Omega}, \mathbb{R}^{3\times3})$ be a symmetric and positive definite weight function. Assume that $p : \Omega \to \mathbb{R}^3$ belongs to one of the following spaces:*

$$p \in H_0(\mathrm{curl}; \Omega) \quad \text{and} \quad \varepsilon\, p \in H(\mathrm{div}; \Omega) \tag{2.5}$$

or

$$p \in H(\mathrm{curl}; \Omega) \quad \text{and} \quad \varepsilon\, p \in H_0(\mathrm{div}; \Omega)\,. \tag{2.6}$$

Then $p \in H^1(\Omega)$ and there exists a constant $C > 0$ (independent of p) such that

$$\|p\|_{H^1(\Omega)} \leqslant C\Big(\|p\|_{H(\mathrm{curl};(\Omega)} + \|\,\mathrm{div}(\varepsilon p)\|_{L^2(\Omega)}\Big)\,.$$

REMARK 2.6. Assuming higher regularity on the weight function ε and the smoothness of $\partial\Omega$ (i.e. $\varepsilon \in C^k(\overline{\Omega}, \mathbb{R}^{3\times3})$ and $\partial\Omega \in C^{k+1}$), Theorem 2.2 from [42] guarantees a corresponding higher regularity of p.

In the proof of theorem 4.1 we will decompose the microdistortion tensor P as $P = \mathrm{D}q + Q$ and apply theorem 2.4 to Q. The regularity for the displacement field u and the vector q then is a consequence of an elliptic regularity result that we discuss next.

Let us consider the following auxiliary bilinear form: for $(u,\, q) \in H^1_0(\Omega; \mathbb{R}^{3+3})$ and $(u,\, v) \in H^1_0(\Omega; \mathbb{R}^{3+3})$ we define

$$\tilde{a}\left(\begin{pmatrix} u \\ q \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right) = \int_\Omega \langle \mathbb{A}(x) \begin{pmatrix} \mathrm{sym}\, \mathrm{D}u \\ \mathrm{sym}\, \mathrm{D}q \end{pmatrix}, \quad \begin{pmatrix} \mathrm{sym}\, \mathrm{D}v \\ \mathrm{sym}\, \mathrm{D}w \end{pmatrix} \rangle \, \mathrm{d}x, \tag{2.7}$$

where $\mathbb{A} : \Omega \to (\mathrm{Lin}(\mathrm{Sym}(3), \mathrm{Sym}(3)))^4$ is defined by the following formula

$$\mathbb{A}(x) = \begin{pmatrix} \mathbb{C}_{\mathrm{e}}(x) & -\mathbb{C}_{\mathrm{e}}(x) & -\mathbb{C}_{\mathrm{e}}(x) & \mathbb{C}_{\mathrm{e}}(x) + \mathbb{C}_{\mathrm{micro}}(x) \end{pmatrix}. \tag{2.8}$$

COROLLARY 2.7. *Let Assumption $\mathbf{A}_{\mathbf{(ii)}}$ be satisfied. Then, there exists a positive constant $C_{\mathbb{A}}$ such that for all $x \in \overline{\Omega}$ and $\sigma = (\sigma_1,\, \sigma_2) \in \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ we have:*

$$\langle \mathbb{A}(x)\sigma, \sigma \rangle \geqslant C_{\mathbb{A}} |\sigma|^2\,. \tag{2.9}$$

Proof. Fix $x \in \overline{\Omega}$ and $\sigma = (\sigma_1, \sigma_2) \in \mathrm{Sym}(3) \times \mathrm{Sym}(3)$, then

$$\langle \mathbb{A}(x)\sigma, \sigma\rangle = \langle \mathbb{C}_{\mathrm{e}}(\sigma_1 - \sigma_2), \sigma_1 - \sigma_2\rangle + \langle \mathbb{C}_{\mathrm{micro}}(\sigma_2), \sigma_2\rangle .$$

Assumption $\mathbf{A}_{\mathbf{(ii)}}$ implies

$$\begin{aligned}\langle \mathbb{A}(x)\sigma, \sigma\rangle &\geqslant C_{\mathrm{e}}|\sigma_1 - \sigma_2|^2 + C_{\mathrm{micro}}|\sigma_2|^2 \geqslant \min\{C_{\mathrm{e}}, C_{\mathrm{micro}}\}\big(|\sigma_1 - \sigma_2|^2 + |\sigma_2|^2\big)\\ &\geqslant \frac{2}{9}\min\{C_{\mathrm{e}}, C_{\mathrm{micro}}\}\big(|\sigma_1|^2 + |\sigma_2|^2\big) = \frac{2}{9}\min\{C_{\mathrm{e}}, C_{\mathrm{micro}}\}|\sigma|^2\end{aligned}$$

and the proof is completed. □

Now, for all $(u, q) \in H^1_0(\Omega; \mathbb{R}^{3+3})$ we have

$$\begin{aligned}\tilde{a}\left(\begin{pmatrix} u \\ q\end{pmatrix}, \begin{pmatrix} u \\ q\end{pmatrix}\right) &\geqslant C_{\mathbb{A}}\big(\|\operatorname{sym} \mathrm{D}u\|^2_{L^2(\Omega)} + \|\operatorname{sym} \mathrm{D}q\|^2_{L^2(\Omega)}\big)\\ &\geqslant C_{\mathbb{A}}\, C_K\big(\|u\|^2_{H^1_0(\Omega)} + \|q\|^2_{H^1_0(\Omega)}\big), \end{aligned} \tag{2.10}$$

where the constant C_K is a constant resulting from the standard Korn's inequality [**22**]. This shows that the bilinear form (2.7) is coercive on the space $H^1_0(\Omega; \mathbb{R}^{3+3})$. The form (2.7) defines the following auxiliary problem: for $(F_1, F_2) \in L^2(\Omega; \mathbb{R}^{3+3})$ find $(u, q) \in H^1_0(\Omega; \mathbb{R}^{3+3})$ with

$$\tilde{a}\big(\big(\begin{smallmatrix} u \\ q\end{smallmatrix}\big), \big(\begin{smallmatrix} v \\ w\end{smallmatrix}\big)\big) = \int_\Omega \big\langle \big(\begin{smallmatrix} F_1 \\ F_2\end{smallmatrix}\big), \big(\begin{smallmatrix} v \\ w\end{smallmatrix}\big)\big\rangle\, \mathrm{d}x \tag{2.11}$$

for all $(v, w) \in H^1_0(\Omega; \mathbb{R}^{3+3})$. Modifying the results concerning the regularity of elliptic partial differential equations, it would be possible to obtain the existence of a solution for system (2.11) with the regularity $(u, q) \in H^1_0(\Omega; \mathbb{R}^{3+3}) \cap H^2(\Omega; \mathbb{R}^{3+3})$ (see for example [**10**, Theorem 9.15, Section 9.6]). However, system (2.11) fits perfectly into the class considered in [**26**]. There, the global regularity of weak solutions to a quasilinear elliptic system with a rank-one-monotone nonlinearity was investigated. As an application of the result from [**26**] we obtain

LEMMA 2.8. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a $C^{1,1}$-smooth boundary $\partial\Omega$. Let furthermore $(F_1, F_2) \in L^2(\Omega; \mathbb{R}^{3+3})$ and $\mathbb{C}_{\mathrm{e}}, \mathbb{C}_{\mathrm{micro}} \in C^{0,1}(\overline{\Omega}; \mathrm{Lin}(\mathbb{R}^{3\times3}; \mathbb{R}^{3\times3}))$. Then the problem (2.11) has a unique solution $(u, q) \in H^1_0(\Omega; \mathbb{R}^{3+3}) \cap H^2(\Omega; \mathbb{R}^{3+3})$.*

Proof. Coercivity (2.10) of the bilinear form (3.1) and the Lax-Milgram theorem imply the existence of exactly one weak solution $(u, q) \in H^1_0(\Omega; \mathbb{R}^{3+3})$. In order to prove higher regularity of this solution we will use Theorem 5.2 of [**26**].

Let us introduce $\mathbb{B} : \overline{\Omega} \times \mathbb{R}^{(3+3)\times 3} \to \mathbb{R}^{(3+3)\times 3}$ as the unique x-dependent linear mapping satisfying

$$\Big\langle \mathbb{B}\Big(x, \big(\begin{smallmatrix} A_1 \\ A_2\end{smallmatrix}\big)\Big), \big(\begin{smallmatrix} B_1 \\ B_2\end{smallmatrix}\big)\Big\rangle = \Big\langle \mathbb{A}(x)\begin{pmatrix} \operatorname{sym} A_1 \\ \operatorname{sym} A_2\end{pmatrix}, \quad \begin{pmatrix} \operatorname{sym} B_1 \\ \operatorname{sym} B_2\end{pmatrix}\Big\rangle \tag{2.12}$$

for all $A_1, A_2, B_1, B_2 \in \mathbb{R}^{3\times3}$ and $x \in \overline{\Omega}$. Then for every $x \in \overline{\Omega}$, $A = (A_1, A_2) \in \mathbb{R}^{(3+3)\times 3}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{3+3}$ and $\eta \in \mathbb{R}^3$ we have thanks to the positive

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definiteness of $\mathbb{A}$

$$\begin{aligned}&\Big\langle\, \mathbb{B}\Big(x, \big(\begin{smallmatrix}A_1\\A_2\end{smallmatrix}\big) + \xi\otimes\eta\Big)\\ &\qquad - \mathbb{B}\Big(x, \big(\begin{smallmatrix}A_1\\A_2\end{smallmatrix}\big)\Big), \xi\otimes\eta\,\Big\rangle \\ &\quad \geqslant C_{\mathbb{A}}\big(\|\operatorname{sym}(\xi_1\otimes\eta)\|^2 + \|\operatorname{sym}(\xi_2\otimes\eta)\|^2\big)\,.\end{aligned} \tag{2.13}$$

Since $\|\operatorname{sym}(\xi_i\otimes\eta)\|^2 = 1/2(\|\xi_i\|^2\|\eta\|^2 + \|\langle\xi_i,\,\eta\rangle\|^2)$, this ultimately implies that $\mathbb{B}$ is strongly rank-one monotone/satisfies the Legendre-Hadamard condition. Due to the Lipschitz continuity of $\mathbb{C}_{\mathrm{e}}$ and $\mathbb{C}_{\mathrm{micro}}$ the remaining assumptions of Theorem 5.2 of [**26**] can easily be verified. Hence, [**26**, Theorem 5.2] implies $(u,\,q) \in H^2(\Omega;\mathbb{R}^{3+3})$. □

3. Weak formulation of the relaxed micromorphic model

For $u,\,v \in H_0^1(\Omega,\,\mathbb{R}^3)$ and $P,\,W \in H_0(\operatorname{Curl};\Omega)$ the following bilinear form is associated with the system (1.2)

$$\begin{aligned} a\big((u,P),(v,W)\big) &= \int_\Omega \Big(\big\langle \mathbb{C}_{\mathrm{e}}\operatorname{sym}(\mathrm{D}u - P), \operatorname{sym}(\mathrm{D}v - W)\big\rangle \\ &\qquad + \big\langle \mathbb{C}_{\mathrm{micro}}\operatorname{sym}P, \operatorname{sym}W\big\rangle + \big\langle \mathbb{L}_{\mathrm{c}}\operatorname{Curl}P, \operatorname{Curl}W\big\rangle\Big)\,\mathrm{d}x \\ &\equiv \int_\Omega \langle \mathbb{A}\begin{pmatrix}\operatorname{sym}\mathrm{D}\\ \operatorname{sym}P\end{pmatrix}, \quad \begin{pmatrix}\operatorname{sym}\mathrm{D}v\\ \operatorname{sym}W\end{pmatrix}\rangle + \big\langle \mathbb{L}_{\mathrm{c}}\operatorname{Curl}P, \operatorname{Curl}W\big\rangle\Big)\,\mathrm{d}x \end{aligned} \tag{3.1}$$

where the tensor $\mathbb{A}$ is defined in (2.8). Here, homogeneous boundary conditions $u\big|_{\partial\Omega} = 0$ and $(P\times n)\big|_{\partial\Omega} = 0$ are considered.

THEOREM 3.1 Existence of weak solutions. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with a Lipschitz boundary and assume that $\mathbb{C}_{\mathrm{e}}$, $\mathbb{C}_{\mathrm{micro}}$, $\mathbb{L}_{\mathrm{c}} \in L^\infty(\Omega;\operatorname{Lin}(\mathbb{R}^{3\times3};\mathbb{R}^{3\times3}))$ comply with the symmetry and positivity properties formulated in (2.3)–(2.4). Then for every $f\in H^{-1}(\Omega)$ and $M \in (H_0(\operatorname{Curl};\Omega))^*$ there exists a unique pair $(u,\,P) \in H_0^1(\Omega)\times H_0(\operatorname{Curl};\Omega)$ such that*

$$\forall (v,W) \in H_0^1(\Omega)\times H_0(\operatorname{Curl};\Omega): \qquad a\big((u,P),(v,W)\big) = \int_\Omega \langle f, v\rangle + \langle M, W\rangle\,\mathrm{d}x. \tag{3.2}$$

Proof. For a bounded domain $\Omega\subset\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ the incompatible Korn's inequality [**12**, **19**, **20**, **27**] implies that there is a constant $\tilde{c} > 0$ such that

$$\|P\|^2_{L^2(\Omega)} \leqslant \tilde{c}\left(\|\operatorname{sym}P\|^2_{L^2(\Omega)} + \|\operatorname{Curl}P\|^2_{L^2(\Omega)}\right) \tag{3.3}$$

for all $P\in \mathrm{H}_0(\operatorname{Curl};\Omega)$. Positive definiteness of the tensors $\mathbb{A}$ and $\mathbb{L}_{\mathrm{c}}$ entail

$$a\big((u,P),(u,P)\big) \geqslant C_{\mathbb{A}}(\|\operatorname{sym}\mathrm{D}u\|^2_{L^2(\Omega)} + \|\operatorname{sym}P\|^2_{L^2(\Omega)}) + L_{\mathrm{c}}\|\operatorname{Curl}P\|^2_{L^2(\Omega)}. \tag{3.4}$$

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Hence, by (3.3) and Korn's inequality the bilinear form (3.1) is coercive on $H_0^1(\Omega) \times H_0(\mathrm{Curl};\Omega)$ and the Lax-Milgram theorem finishes the proof. □

Thanks to the Helmholtz decomposition, weak solutions can equivalently be characterized as follows:

LEMMA 3.2. *Let the assumptions of theorem 3.1 be satisfied, $f \in H^{-1}(\Omega)$ and $M \in (H_0(\mathrm{Curl};\Omega))^*$. Let furthermore $(u,\, P) \in H_0^1(\Omega) \times H_0(\mathrm{Curl};\Omega)$ and let $(q,\, Q) \in H_0^1(\Omega;\mathbb{R}^3) \times H(\mathrm{Div},\, 0;\Omega)$ such that $P = \mathrm{D}q + Q$. Then the following (a) and (b) are equivalent:*

(a) *$(u,\, P)$ is a weak solution of (1.2) in the sense of (3.2).*

(b) *For all $(v,\, W) \in H_0^1(\Omega) \times H_0(\mathrm{Curl};\Omega)$ the triple $(u,\, q,\, Q)$ satisfies*

$$\int_\Omega \langle \mathbb{A} \begin{pmatrix} \mathrm{sym}\,\mathrm{D}u \\ \mathrm{sym}\,\mathrm{D}q + \mathrm{sym}\, Q \end{pmatrix}, \quad \begin{pmatrix} \mathrm{sym}\,\mathrm{D}v \\ \mathrm{sym}\, W \end{pmatrix} \rangle \,\mathrm{d}x \\ + \int_\Omega \langle \mathbb{L}_c\, \mathrm{Curl}\, Q, \mathrm{Curl}\, W\rangle \,\mathrm{d}x = \int_\Omega \langle f, v\rangle + \langle M, W\rangle \,\mathrm{d}x. \tag{3.5}$$

Proof. $(a) \implies (b)$: Assume that $(u,\, P) \in H_0^1(\Omega) \times H_0(\mathrm{Curl};\Omega)$ is a weak solution of (1.2) in the sense of (3.2). Then theorem 2.1 implies that for $i = 1,\, 2,\, 3$ there exists unique $q_i \in H_0^1(\Omega)$ and $Q_i \in H_0(\mathrm{Div},\, 0;\Omega)$ such that $P_i = \mathrm{D}q_i + Q_i$, where P_i denotes the rows of the matrix P. Inserting $P = (\nabla q_1 + Q_1,\, \nabla q_2 + Q_2,\, \nabla q_3 + Q_3)^T$ into (3.2) we obtain (3.5), where $Q = (Q_1,\, Q_2,\, Q_3)^T$.

$(b) \implies (a)$: Let for all $(v,\, W) \in H_0^1(\Omega) \times H_0(\mathrm{Curl};\Omega)$ the triple $(u,\, q,\, Q) \in H_0^1(\Omega;\mathbb{R}^3) \times H_0^1(\Omega;\mathbb{R}^3) \times H(\mathrm{Div},\, 0;\Omega)$ satisfy (3.5). Then (3.5) can be written in the form

$$\int_\Omega \langle \mathbb{A} \begin{pmatrix} \mathrm{sym}\,\mathrm{D}u \\ \mathrm{sym}\,\mathrm{D}q + \mathrm{sym}\, Q \end{pmatrix}, \begin{pmatrix} \mathrm{sym}\,\mathrm{D}v \\ \mathrm{sym}\, W \end{pmatrix} \rangle \,\mathrm{d}x \\ + \int_\Omega \langle \mathbb{L}_c\, \mathrm{Curl}(\mathrm{D}q + Q), \mathrm{Curl}\, W\rangle \,\mathrm{d}x = \int_\Omega \langle f, v\rangle + \langle M, W\rangle \,\mathrm{d}x \tag{3.6}$$

and the function $(u,\, \mathrm{D}q + Q)$ satisfies (3.2) for all $(v,\, W) \in H_0^1(\Omega) \times H_0(\mathrm{Curl};\Omega)$. Uniqueness of a weak solution of the problem (1.2) implies that $P = \mathrm{D}q + Q$. □

4. Global regularity on smooth domains

The aim of this section is to prove the following regularity theorem

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a $C^{1,1}$-smooth boundary. Moreover, in addition to the assumptions of theorem 3.1 let $\mathbb{C}_e,\, \mathbb{C}_{\mathrm{micro}},\, \mathbb{L}_c \in C^{0,1}(\overline{\Omega};\mathrm{Lin}(\mathbb{R}^{3\times3};\mathbb{R}^{3\times3}))$. Finally, we assume that $f \in L^2(\Omega)$ and $M \in H(\mathrm{Div};\Omega)$.*

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Then for every weak solution $(u,\, P) \in H_0^1(\Omega) \times H_0(\mathrm{Curl};\Omega)$ *we have*

$$u \in H^2(\Omega), \quad P \in H^1(\Omega), \quad \mathbb{L}_c \operatorname{Curl} P \in H(\mathrm{Curl};\Omega) \tag{4.1}$$

and there exists a constant $C > 0$ *(independent of* f *and* M*) such that*

$$\|u\|_{H^2(\Omega)} + \|P\|_{H^1(\Omega)} + \|\mathbb{L}_c \operatorname{Curl} P\|_{H(\mathrm{Curl};\Omega)} \leqslant C(\|f\|_{L^2(\Omega)} + \|M\|_{H(\mathrm{Div};\Omega)}). \tag{4.2}$$

The proof relies on the Helmholtz decomposition of P, the embedding theorem 2.4 and theorem 2.8 about the global regularity for the auxiliary problem (2.11).

Proof. Let $(u,\, P) \in H_0^1(\Omega) \times H_0(\mathrm{Curl};\Omega)$ satisfy (3.2) with $f \in L^2(\Omega)$ and $M \in H(\mathrm{Div};\Omega)$.

We first show that $(u,\, P) \in H^2(\Omega) \times H^1(\Omega)$. Let $P = \mathrm{D}q + Q$, where $q \in H_0^1(\Omega;\mathbb{R}^3)$ and $Q \in H(\mathrm{Div},\, 0;\Omega)$ are given according to the Helmholtz decomposition. By proposition 2.2 it follows that $Q \in H(\mathrm{Div},\, 0;\Omega) \cap H_0(\mathrm{Curl};\Omega)$ and thanks to the assumed regularity of $\partial\Omega$, theorem 2.4 implies that $Q \in H^1(\Omega)$. Next, choosing $W = \mathrm{D}w$ for $w \in C_0^\infty(\Omega;\mathbb{R}^d)$, the weak form (3.5) in combination with a density argument implies that for all $v,\, w \in H_0^1(\Omega)$ we have

$$\begin{aligned} &\int_\Omega \left\langle \mathbb{A} \begin{pmatrix} \operatorname{sym} \mathrm{D}u \\ \operatorname{sym} \mathrm{D}q \end{pmatrix}, \quad \begin{pmatrix} \operatorname{sym} \mathrm{D}v \\ \operatorname{sym} \mathrm{D}w \end{pmatrix} \right\rangle \mathrm{d}x \\ &\quad = \int_\Omega \langle \mathbb{C}_\mathrm{e} \operatorname{sym} Q, \operatorname{sym} \mathrm{D}v \rangle\, \mathrm{d}x - \int_\Omega \langle (\mathbb{C}_\mathrm{e} + \mathbb{C}_\mathrm{micro}) \operatorname{sym} Q, \operatorname{sym} \mathrm{D}w \rangle\, \mathrm{d}x \\ &\qquad + \int_\Omega \langle f, v \rangle + \langle M, \mathrm{D}w \rangle\, \mathrm{d}x. \end{aligned} \tag{4.3}$$

Since $Q \in H^1(\Omega)$ and $M \in H(\mathrm{Div};\Omega)$, by partial integration the right hand side of (4.3) can be rewritten as $\int_\Omega \langle F_1,\, v\rangle + \langle F_2,\, w\rangle \mathrm{d}x$ with functions $F_1,\, F_2 \in L^2(\Omega)$. theorem 2.8 implies that $u,\, q \in H^2(\Omega)$ and hence $P = \mathrm{D}q + Q \in H^1(\Omega)$.

Let us next choose $v = 0$ and $W \in C_0^\infty(\Omega)$ in (3.1). Rearranging the terms we find that

$$\begin{aligned} \int_\Omega \langle \mathbb{L}_\mathrm{c} \operatorname{Curl} P, \operatorname{Curl} W \rangle\, \mathrm{d}x &= \int_\Omega \langle M, W \rangle\, \mathrm{d}x \\ &\quad + \int_\Omega \langle \mathbb{C}_\mathrm{e} \operatorname{sym} \mathrm{D}u - (\mathbb{C}_\mathrm{e} + \mathbb{C}_\mathrm{micro}) \operatorname{sym} P\,,\, \operatorname{sym} W \rangle\, \mathrm{d}x \end{aligned}$$

which implies that $\operatorname{Curl}(\mathbb{L}_\mathrm{c} \operatorname{Curl} P) \in L^2(\Omega)$ and $\mathbb{L}_\mathrm{c} \operatorname{Curl} P \in H(\mathrm{Curl};\Omega)$. □

If we additionally assume that $\mathbb{L}_c$ has a block-diagonal structure, we may also achieve $\operatorname{Curl} P \in H^1(\Omega)$ by applying the weighted embedding theorem 2.5.

Corollary 4.2. *In addition to the assumptions of theorem 4.1 let* $\mathbb{L}_c \in C^1(\overline{\Omega}; \mathrm{Lin}(\mathbb{R}^{3\times3},\, \mathbb{R}^{3\times3}))$ *be of block diagonal structure, meaning that there exist* $\mathbb{L}_i \in C^1(\overline{\Omega};\mathbb{R}^{3\times3})$, $1 \leqslant i \leqslant 3$, *such that for every* $W \in \mathbb{R}^{3\times3}$ *we have* $(\mathbb{L}_c W)_{i\text{-}th\ row} = \mathbb{L}_i(W_{i\text{-}th\ row})$. *Then* $\mathbb{L}_c \operatorname{Curl} P \in H^1(\Omega)$ *and* $\operatorname{Curl} P \in H^1(\Omega)$.

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Proof. We focus on the i-th row P^i of P. Let $\varepsilon = (\mathbb{L}_i)^{-1}$. Then $\varepsilon \in C^1(\overline{\Omega}; \mathbb{R}^{3\times 3})$ and $\varepsilon(x)$ is symmetric and uniformly positive definite with respect to $x \in \overline{\Omega}$.

We know that $P^i \in H_0(\mathrm{curl}; \Omega)$, hence lemma 2.3 implies immediately that

$$\varepsilon\, \mathbb{L}_i \,\mathrm{curl}\, P^i = \mathrm{curl}\, P^i \in H_0(\mathrm{div}, 0; \Omega)\,.$$

The weighted embedding theorem 2.5 implies $\mathbb{L}_i \,\mathrm{curl}\, P^i \in H^1(\Omega)$, and since $\varepsilon = \mathbb{L}_i^{-1}$ is a multiplyer on $H^1(\Omega)$, we finally obtain $\mathrm{curl}\, P^i \in H^1(\Omega)$. □

Remark 4.3. The previous result may be applied to the simple uni-constant isotropic curvature case $L_c^2 \,\|\,\mathrm{Curl}\, P\|^2$.

Remark 4.4. It is clear that the same higher regularity result can be established for the linear Cosserat model [**9**].

5. Global regularity for a gauge-invariant incompatible elasticity model

The method presented above for obtaining regularity of solution for the static relaxed micromorphic model can be directly applied to the following gauge-invariant incompatible elasticity model [**17**, **24**]

$$0 = -\,\mathrm{Curl}[\mathbb{L}_c \,\mathrm{Curl}\, e] - \mathbb{C}_e \,\mathrm{sym}\, e - \mathbb{C}_c \,\mathrm{skew}\, e + M, \tag{5.1}$$

where the unknown function is the non-symmetric incompatible elastic distortion $e : \Omega \to \mathbb{R}^{3\times 3}$ while $M : \Omega \to \mathbb{R}^{3\times 3}$ is a given body moment tensor. The constitutive tensors $\mathbb{C}_e$, $\mathbb{L}_c$ are positive definite fourth order tensors (fulfilling the Assumption A) while $\mathbb{C}_c : \mathfrak{so}(3) \to \mathfrak{so}(3)$ is positive semi-definite. The system (5.1) is considered with homogeneous tangential boundary conditions, i.e.

$$e_i(x) \times\, n(x) = 0 \text{ for } x \in \partial\Omega, \tag{5.2}$$

where $\times$ denotes the vector product, n is the unit outward normal vector at the surface $\partial\Omega$, e_i $(i = 1,\, 2,\, 3)$ are the rows of the tensor e. Problem (5.1) generalizes the time-harmonic Maxwell-type eigenvalue problem [**3**, **43**] from the vectorial to the tensorial setting. On the other hand, equation (5.1) corresponds to the second equation of (1.2) upon setting $\mathbb{C}_{\mathrm{micro}} \equiv 0$, assuming $\mathbb{C}_c \equiv 0$, identifying the elastic distortion e with $e = \mathrm{D}u - P$ and observing that

$$-\,\mathrm{Curl}\, \mathbb{L}_c \,\mathrm{Curl}\, P = \mathrm{Curl}\, \mathbb{L}_c \,\mathrm{Curl}(-P) = \mathrm{Curl}\, \mathbb{L}_c \,\mathrm{Curl}(\mathrm{D}u - P) = \mathrm{Curl}\, \mathbb{L}_c \,\mathrm{Curl}\, e\,. \tag{5.3}$$

Smooth solutions of (5.1) satisfy the balance of linear momentum equation

$$\mathrm{Div}(\mathbb{C}_e \,\mathrm{sym}\, e + \mathbb{C}_c \,\mathrm{skew}\, e) = \mathrm{Div}\, M. \tag{5.4}$$

Gauge-invariance means here that the solution e is invariant under

$$\mathrm{D}u \to \mathrm{D}u + \mathrm{D}\tau\,, \qquad\qquad P \to P + \mathrm{D}\tau \tag{5.5}$$

which invariance is only possible since $\mathbb{C}_{\mathrm{micro}} \equiv 0$ ($\mathbb{C}_{\mathrm{micro}} > 0$ breaks the gauge-invariance). Here τ is a space-dependent (or local) translation vector. The elastic

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energy can then be expressed as

$$\int_{\Omega} \big\langle \mathbb{C}_{\mathrm{e}} \operatorname{sym} e, \operatorname{sym} e \big\rangle + \big\langle \mathbb{L}_{\mathrm{c}} \operatorname{Curl} e, \operatorname{Curl} e \big\rangle - \langle M, e\rangle \,\mathrm{d}x \tag{5.6}$$

in which the first term accounts for the energy due to elastic distortion, the second term takes into account the energy due to incompatibility in the presence of dislocations and the last term is representing the forcing.

For $e,\, v \in H_0(\operatorname{Curl};\Omega)$ the following bilinear form is associated with the system (5.1)

$$b(e,v) = \int_{\Omega} \big\langle \mathbb{L}_{\mathrm{c}} \operatorname{Curl} e, \operatorname{Curl} v \big\rangle + \big\langle \mathbb{C}_{\mathrm{e}} \operatorname{sym} e, \operatorname{sym} v \big\rangle + \big\langle \mathbb{C}_{\mathrm{c}} \operatorname{skew} e, \operatorname{skew} v \big\rangle \,\mathrm{d}x\,. \tag{5.7}$$

Let us assume that $M \in H(\operatorname{Div};\Omega)$. Coercivity of the bilinear form (5.7) (the generalized incompatible Korn's inequality (3.3)) and the Lax-Milgram theorem imply the existence of exactly one weak solution $e \in H_0(\operatorname{Curl};\Omega)$ of the system (5.1). The Helmholtz decomposition yields that $e = \mathrm{D}q + Q$, where $q \in H^1_0(\Omega;\mathbb{R}^3)$ and $Q \in H(\operatorname{Div}, 0;\Omega)$. By proposition 2.2 it follows that $Q \in H(\operatorname{Div}, 0;\Omega) \cap H_0(\operatorname{Curl};\Omega)$ and theorem 2.4 implies that $Q \in H^1(\Omega)$. Inserting the decomposed form of the tensor e into the weak form of system (5.1), we obtain

$$\begin{aligned} &\int_{\Omega} \langle \mathbb{C}_{\mathrm{e}} \operatorname{sym}(\mathrm{D}q + Q), \operatorname{sym} W\rangle \mathrm{d}x + \int_{\Omega} \langle \mathbb{C}_{\mathrm{c}} \operatorname{skew}(\mathrm{D}q + Q), \operatorname{skew} W\rangle \mathrm{d}x \\ &\quad + \int_{\Omega} \langle \mathbb{L}_{\mathrm{c}} \operatorname{Curl}(\mathrm{D}q + Q), \operatorname{Curl} W\rangle \,\mathrm{d}x = \int_{\Omega} \langle M, W\rangle \,\mathrm{d}x \end{aligned} \tag{5.8}$$

for all $W \in H_0(\operatorname{Curl};\Omega)$. Again, choosing $W = \mathrm{D}w$ for $w \in C_0^\infty(\Omega;\mathbb{R}^3)$, the weak form (5.8) in combination with a density argument implies that for all $w \in H^1_0(\Omega)$ we have

$$\begin{aligned} &\int_{\Omega} \langle \mathbb{C}_{\mathrm{e}} \operatorname{sym} \mathrm{D}q, \operatorname{sym} \mathrm{D}w\rangle \mathrm{d}x + \int_{\Omega} \langle \mathbb{C}_{\mathrm{c}} \operatorname{skew} \mathrm{D}q, \operatorname{skew} \mathrm{D}w\rangle \mathrm{d}x \\ &\quad = \int_{\Omega} \langle M, \mathrm{D}w\rangle \,\mathrm{d}x - \int_{\Omega} \langle \mathbb{C}_{\mathrm{e}} \operatorname{sym} Q, \operatorname{sym} \mathrm{D}w\rangle \mathrm{d}x - \int_{\Omega} \langle \mathbb{C}_{\mathrm{c}} \operatorname{skew} Q, \operatorname{skew} \mathrm{D}w\rangle \,\mathrm{d}x \end{aligned} \tag{5.9}$$

Since $Q \in H^1(\Omega)$ and $M \in H(\operatorname{Div};\Omega)$, by partial integration the right hand side of (4.3) can be rewritten as $\int_{\Omega} \langle F_2,\, w\rangle \mathrm{d}x$ with functions $F_2 \in L^2(\Omega)$. Note that the auxiliary problem (??) obtained this time is a problem from standard linear elasticity. Thus, in this case we do not need to go through lemma 2.8: just from the standard theory of regularity in the linear elasticity we obtain that $q \in H^2(\Omega)$ ([**41**]). The result is that on smooth domains and with smooth coefficients the weak

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solution of (5.1) is more regular and satisfies

$$e \in H^1(\Omega), \quad \mathbb{L}_c \operatorname{Curl} e \in H(\operatorname{Curl};\Omega), \quad \operatorname{Curl}(\mathbb{L}_c \operatorname{Curl} e) \in H^1(\Omega), \tag{5.10}$$

where the last regularity in (5.10) follows from equation (5.1). Moreover, if $\mathbb{L}_c$ has a special block diagonal structure, then we have

$$\mathbb{L}_c \operatorname{Curl} e \in H^1(\Omega), \quad \operatorname{Curl} e \in H^1(\Omega). \tag{5.11}$$

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