Canad. Math. Bull. Vol. 17 (4), 1974

VECTOR BUNDLES OVER SUSPENSIONS

BY
W. M. CHAN* AND P. HOFFMAN

We consider finite dimensional complex vector bundles over a compact connected Hausdorff space X, as defined, for example, in [1]. It is well known that if ξ is such a bundle, then there is a bundle η such that $\xi\oplus\eta$ is trivial.

LEMMA 1. *If $X=\mathbf{CP}^k$, complex projective k-space, and ξ is the canonical line bundle, then $\xi\oplus\eta$ is non-trivial for any bundle η of fibre dimension less than k.*

Proof. Suppose η has dimension $r-1$, and $\xi\oplus\eta$ is trivial. Then we have

$$\binom{r}{i} = \lambda^i(\xi\oplus\eta) = \lambda^i(\eta)+\xi\lambda^{i-1}(\eta).$$

Thus

$$\lambda^i(\eta) = \sum_{j=0}^{i}(-1)^i\binom{r}{i-j}\xi^i \quad \text{in } \mathbf{K}(X).$$

The set $\{1, \xi, \xi^2, \ldots, \xi^k\}$ is a basis for $\mathbf{K}(X)$, so $\lambda^i(\eta)\neq 0$ for $i\leqq k$. Thus dim $\eta\geqq k$, as required.

LEMMA 2. *If $X=S^{2n}$, there is a bundle ξ of fibre dimension n such that $\xi\oplus\eta$ is non-trivial for any bundle η of fibre dimension less than n.*

Proof. There is an element $x\in\tilde{\mathbf{K}}(S^{2n})$ with $ch^n(x)\neq 0$, where ch^n is the nth Chern character [2]. Since the inclusion

$$GL(r;\mathbf{C})\to GL(r+1,\mathbf{C})$$

induces an isomorphism

$$\pi_{2n-1}[GL(r,\mathbf{C})]\to\pi_{2n-1}[GL(r+1;\mathbf{C})]$$

for all $r\geqq n$, there is a bundle ξ of dimension n whose image in $\mathbf{K}(S^{2n})$ is $x+n$. Therefore $ch^n(\xi)\neq 0$, and hence $C_n(\xi)\neq 0$, where C_n is the nth Chern class. Now suppose $\xi\oplus\eta$ is trivial. Then we have

$$0 = C_n(\xi\oplus\eta) = \sum_{i+j=n} C_i(\xi)C_j(\eta) = C_n(\xi)+C_n(\eta).$$

Thus $C_n(\eta)\neq 0$, so *dim* $\eta\geqq n$, as required.

THEOREM. *If $X=\Sigma Y$, where Y is compact Hausdorff, and ξ is a bundle over X, then there is a bundle η, whose fibre dimension equals that of ξ, such that $\xi\oplus\eta$ is trivial.*

Received by the editors May 9, 1973 and, in revised form, May 30, 1973.
* The first author was partially supported by NRC grant A4840.

https://doi.org/10.4153/CMB-1974-085-0 Published online by Cambridge University Press

Note. Adding trivial line bundles to ξ in Lemma 1 shows the hypothesis that X is a suspension to be necessary. η cannot have fibre dimension less than that of ξ by Lemma 2. The proof below is a combination of the isomorphism of [1, 1.4.9] with the homotopy in [1, 2.4.6].

Proof. Regard ΣY as $C^+Y \cup C^-Y$, where $C^+Y=[0,1]\times Y/\{1\}\times Y$, $C^-Y=[-1,0]\times Y/\{-1\}\times Y$, and Y is identified with $\{0\}\times Y$. Since C^+Y and C^-Y are contractible, we can find isomorphisms

$$\alpha^+:\xi \mid C^+Y \to C^+Y\times \mathbf{C}^n = \gamma^+$$

$$\alpha^-:\xi \mid C^-Y \to C^-Y\times \mathbf{C}^n = \gamma^-$$

Let $\beta=[\alpha^- \mid (\xi \mid Y)]\circ[(\alpha^+)^{-1} \mid Y\times\mathbf{C}^n]: Y\times\mathbf{C}^n\to Y\times\mathbf{C}^n$. α^+ and α^- induce an isomorphism

$$\alpha:\xi \to \gamma^+ \bigcup_{\beta} \gamma^-.$$

Let $\eta=\gamma^+ \bigcup_{\beta^{-1}} \gamma^-$. If β corresponds to a map

$$\Phi: Y \to GL(n,\mathbf{C})$$

then β^{-1} corresponds to $\bar{\Phi}$, where $\bar{\Phi}(y)=[\Phi(y)]^{-1}$. Now

$$\xi\oplus\eta \cong (\gamma^+\oplus\gamma^+)\bigcup_{\beta\oplus\beta^{-1}}(\gamma^-\oplus\gamma^-),$$

and $\beta\oplus\beta^{-1}$ corresponds to the map

$$\Psi: Y \to GL(2n,\mathbf{C})$$

given by

$$\Psi(y) = \begin{pmatrix} \Phi(y) & 0 \\ 0 & \bar{\Phi}(y) \end{pmatrix}$$

The technique of [1, Lemma 2.4.6] then shows that Ψ is homotopic to Γ, where $\Gamma(x)$ is the identity matrix for all x. Thus

$$\xi\oplus\eta \cong (\gamma^+\oplus\gamma^+) \bigcup_{1_{Y\times\mathbf{C}^{2n}}} (\gamma^-\oplus\gamma^-) \cong \Sigma Y\times\mathbf{C}^{2n},$$

as required.

References

1. M. F. Atiyah, *K-theory*, Benjamin, 1967.

2. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. **3**, 7–38, Amer. Math. Soc., Providence, R.I., 1961.

Department of Pure Mathematics,
University of Waterloo,
Waterloo, Ontario