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ON LOCALIZING ORDERABLE MODULES

COLIN D. FOX

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Abstract

If A is a T-orderable R-module and S is a multiplicative subsemigroup of R, each $s \in S$ acting as a monomorphism of A, then it is possible sometimes to embed A in a T-orderable R-module on which each $s \in S$ acts as an automorphism. We show that such an embedding does not always exist and, by generalizing a theorem of Kokorin and Kopytov, provide a partial answer to the question "when is such an embedding possible?"

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1. Introduction

Kokorin and Kopytov (1972) amply demonstrate the usefulness of ordered modules in dealing with certain ordered group problems. In this paper we correct a proposition, generalize a theorem and discuss a question of Kokorin and Kopytov (op. cit.).

Let A be an R-module (throughout our modules will be left unitary) and let X be a subset of R. We call A an *X-orderable R-module* if A can be made a fully ordered group in such a way that for all $a \in A$ and $r \in X$, $a \geqslant 0$ implies $ra \geqslant 0$. Such an order will be called an *X-order*. A is an *X-complete R-module* if for all $a \in A$ and $r \in X$ there is $a' \in A$ satisfying $ra' = a$.

The problem we are considering is the following:

Given a T-orderable R-module, A, and a multiplicative subsemigroup, S, of regular elements of R, each $s \in S$ acting as a monomorphism of A, is it always possible to embed A in a T-orderable R-module, B, which is S-complete? (See Theorem 1, §3, Chapter V, and Problem 18 of Kokorin and Kopytov, op. cit.)

Using localization techniques we generalize the above-mentioned theorem and, with examples, provide negative answers to two variants of this question.

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2. Preliminaries

For a subset, T, of the ring, R, denote by $\mathscr{S}(T)$ the subsemiring of R generated by T, 0 and 1.

LEMMA 1. *A is a T-orderable R-module if, and only if, A is an $\mathscr{S}(T)$-orderable R-module.*

PROOF. Suppose A is T-orderable and take any T-order, $\leqslant$, of A. For $a \geqslant 0$ in A, $xa \geqslant 0$ where x is a finite product of elements of $T \cup \{0, 1\}$. So, for the products $x_1, \ldots, x_k$, $(x_1 + \ldots + x_k)a = x_1 a + \ldots + x_k a \geqslant 0$. Hence, A is $\mathscr{S}(T)$-orderable. The converse is obvious.

In the sequel (except Section 4), whenever we have a T-orderable R-module, we shall assume that $T = \mathscr{S}(T)$. For elements $a_1, \ldots, a_n$ of an R-module, A, and for a subset, T, of R, we denote by $T\langle a_1, \ldots, a_n \rangle$ the closure in A of $\{a_1, \ldots, a_n\}$ with respect to addition and the action of elements of T. We omit the straightforward proof of the next lemma.

LEMMA 2. *When $T = \mathscr{S}(T)$, $T\langle a_1, \ldots, a_n \rangle = \{t_1 a_1 + \ldots + t_n a_n : t_1, \ldots, t_n \in T\}$.*

The following theorem, incorrectly stated by both Kokorin and Kopytov (op. cit., Proposition 1, §3, Chapter V) and Botto Mura and Rhemtulla (1975, Lemma A1), is analogous to the local theorem for orderable rings (see Fuchs, 1963). Call a subset, X, of an abelian group *conic* if $0 \in X$ and $X \cap -X = \{0\}$.

THEOREM. *The R-module, A, is T-orderable if, and only if, for all $a_1, \ldots, a_n \in A$ there exist $\varepsilon_1, \ldots, \varepsilon_n$, each equalling ± 1, such that $T\langle \varepsilon_1 a_1, \ldots, \varepsilon_n a_n \rangle$ is conic.*

(The incorrect version of this theorem asserts that, given $a_1, \ldots, a_n \in A$ there are $\varepsilon_1, \ldots, \varepsilon_n$ each ± 1 such that $0 \notin T\langle \varepsilon_1 a_1, \ldots, \varepsilon_n a_n \rangle$. This is incorrect as T may contain an element which does not act as a monomorphism on A.)

The final result of this section is a technical lemma which will enable us to extract Kokorin and Kopytov's theorem from our main theorem.

LEMMA 3. *Let $\varphi: R \to R_1$ be a ring homomorphism and let B be an $R\varphi$-module. Making B an R-module in the usual way we have for any subset, X, of R:*

(i) *if B is $X\varphi$-complete, then B is X-complete, and*

(ii) *if B is $X\varphi$-orderable, then B is X-orderable.*

PROOF. The action of R on B is given by $rb = (r\varphi)b$ $(r \in R, b \in B)$. So if B is $X\varphi$-complete, then for all $b \in B$ and $r \in X$ there is $b' \in B$ such that $rb' = (r\varphi)b' = b$. If B is $X\varphi$-orderable, then, taking any $X\varphi$-order, $\leqslant$, of B, we have $rb = (r\varphi)b \geqslant 0$ for all $r \in X$ and $b \geqslant 0$ in B.

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3. Ordering a well-known localization

To provide a partial answer to the question under consideration, Kokorin and Kopytov assume S acts centrally on A. We shall assume that R satisfies a left Ore condition to obtain a suitable (and well-known) R-module and make a further assumption (see (iv) in the theorem below) to prove this R-module orderable. Whether or not (iv) is necessary remains undecided.

Recall that a subset, S, of R is called a *multiplicative set* if $1 \in S$ and S is a multiplicative subsemigroup of R.

MAIN THEOREM. *Let A be a T-orderable R-module and let S be a multiplicative set of regular elements of R. If S satisfies*

(i) *for all $a \in A$, $s \in S$ with $a \neq 0$ we have $sa \neq 0$;*

(ii) *for all $r \in R$, $s \in S$ there exist $r' \in R$, $s' \in S$ such that $r's = s'r$;*

(iii) *for all $s_1, s_2 \in S$ there exist $s_1', s_2' \in S$ such that $s_1' s_1 = s_2' s_2$; and*

(iv) *for all $s \in S$ there exist $s', s'' \in S$ such that $s'T \subseteq Ts$ and $sT \subseteq Ts''$,*

then there is an S-complete T-orderable R-module, B, containing A and having the property:

(*) *for all $b \in B$ there exists $s \in S$ such that $sb \in A$.*

Moreover, any T-order of A which is also an S-order extends to a T-order of B.

PROOF. Using (i), (ii) and (iii) we construct the R-module described on p. 262 of Jacobson (1964). (Note that (iii) is superfluous if S is the set of all regular elements of R.) So B is all the "fractions" $a/s\,(a \in A, s \in S)$ with $a_1/s_1 = a_2/s_2$ if $s_1' a_1 = s_2' a_2$ whenever $s_1' s_1 = s_2' s_2$ $(s_1', s_2' \in S)$. Addition in B is given by

$$a_1/s_1 + a_2/s_2 = (s_1' a_1 + s_2' a_2)/s_1' s_1$$

$(s_1', s_2'$ as above) and the action of R on B is given by $r(a/s) = (r'a)/s'$, where $r's = s'r\,(r' \in R, s' \in S)$. The mapping $a \to (sa)/s\,(s \in S)$ embeds A in B and we identify A with its image in B. B is S-complete since for all $a/s \in B$ and $s_0 \in S$, $s_0(a/ss_0) = a/s$ and B satisfies (*) since, for all $a/s \in B$, $s(a/s) = (sa)/s \in A$.

Turning to the T-orderability of B, consider first the easy case (for which (iv) is not necessary). That is, let $\leqslant$ be a T-order of A which is also an S-order. It is not difficult to check that the order

$$a/s \geqslant' 0 \quad \text{if, and only if,} \quad a \geqslant 0$$

is a T-order of B and that $\leqslant'$ is an extension of $\leqslant$. In general, however, we cannot use this method. (For instance, if there are $s \in S$ and $a \in A$ such that $sa = -a \neq 0$, then no order of A is an S-order.)

We shall need the following lemma whose proof is very easy and omitted.

LEMMA 4. *For all* $s_1, \dots, s_n \in S$ *there are* $s, s'_1, \dots, s'_n \in S$ *such that for all* $a_1, \dots, a_n \in A$

$$a_1/s_1 + \dots + a_n/s_n = (s'_1 a_1 + \dots + s'_n a_n)/s.$$

Take any $a_1/s_1, \dots, a_n/s_n \in B$ and write $X = T\langle a_1/s_1, \dots, a_n/s_n\rangle$. Then

$$\begin{aligned} X &= \{t_1(a_1/s_1) + \dots + t_n(a_n/s_n) \colon t_i \in T\} \quad \text{(Lemma 2)} \\ &\subseteq \{(t'_1 a_1)/s'_1 + \dots + (t'_n a_n)/s'_n \colon t'_i \in T\} \\ &\qquad \text{where } s'_i \in S \text{ satisfies } s'_i T \subseteq T s_i \\ &= \{(s''_1 t'_1 a_1 + \dots + s''_n t'_n a_n)/s \colon t'_i \in T\} \\ &\qquad \text{where } s, s''_i \text{ are given by Lemma 4} \\ &\subseteq \{(t''_1 s'''_1 a_1 + \dots + t''_n s'''_n a_n)/s \colon t''_i \in T\} \\ &\qquad \text{where } s'''_i \in S \text{ satisfies } s''_i T \subseteq T s'''_i \\ &= \{a/s \colon a \in T\langle s'''_1 a_1, \dots, s'''_n a_n\rangle\}. \end{aligned}$$

Similarly, for $\varepsilon_1, \dots, \varepsilon_n = \pm 1$,

$$T\langle \varepsilon_1(a_1/s_1), \dots, \varepsilon_n(a_n/s_n)\rangle \subseteq \{a/s \colon a \in T\langle \varepsilon_1 s'''_1 a_1, \dots, \varepsilon_n s'''_n a_n\rangle\}.$$

Since $\varepsilon_1, \dots, \varepsilon_n = \pm 1$ can be chosen to make $T\langle \varepsilon_1 s'''_1 a_1, \dots, \varepsilon_n s'''_n a_n\rangle$ conic and since $a/s = 0$ if, and only if, $a = 0$, it follows that, with the same choice of $\varepsilon_1, \dots, \varepsilon_n = \pm 1$, $T\langle \varepsilon_1(a_1/s_1), \dots, \varepsilon_n(a_n/s_n)\rangle$ is conic. So, by the theorem quoted in Section 2, B is T-orderable.

(Note that by virtue of Lemma 3 and using the homomorphism $\varphi \colon R \to \operatorname{end}(A)$ given by $(r\varphi)a = ra$ $(r \in R, a \in A)$, we can generalize our main theorem as follows:

We assume that R acts on A in such a way that $R\varphi$ satisfies (i)–(iv). That is, (i) remains unchanged, (ii) is replaced by

(ii)′ *for all* $r \in R$, $s \in S$ *there exist* $r' \in R$, $s' \in S$ *such that for all* $a \in A$ $r'sa = s'ra$,

and (iii) and (iv) are replaced similarly.

Bearing this in mind, Kokorin and Kopytov's theorem follows easily.)

COROLLARY (Kokorin and Kopytov, 1972). *Let A be a T-orderable R-module and let S be a multiplicative subsemigroup of R satisfying* (i) *of the Main Theorem and also*

$$\text{for all } s \in S,\ r \in R,\ a \in A \ \ sra = rsa.$$

Then A can be embedded in an S-complete T-orderable R-module, B, satisfying (*).

4. Two examples

If, in our question, we do not insist that S be a set of regular elements, then the following simple example answers the question negatively. (Note that in posing the problem, Kokorin and Kopytov (1972) do not explicitly require the elements of S to be regular.)

Let A be the free abelian group (written additively) on generators $a_1, a_2, \ldots$. Let $s, t \in \operatorname{end}(A)$ be given by

$$sa_i = a_{i+1} \quad \text{for all } i = 1, 2, \ldots$$

and

$$ta_i = \begin{cases} 2a_i & \text{if } i = 1, \\ a_i & \text{if } i = 2, 3, \ldots. \end{cases}$$

Let R be the subring of $\operatorname{end}(A)$ generated by 1, s and t, let $T = \{t\}$ and let $S = \{s, s^2, \ldots\}$. Then $ts = s$ in R (hence s is not regular) and t preserves (amongst others) the lexicographic order of A. Now A cannot be embedded in any R-module, B, which is S-complete (let alone a T-orderable R-module, B). For if so, take $b \in B$ such that $sb = a_1$. So $tsb = 2a_1 \neq sb$, a contradiction.

However, in Kokorin and Kopytov's theorem, the elements of S are regular. Furthermore, they act as monomorphisms of B, a fact used in all applications of the theorem. As can be easily checked, the construction used in our Main Theorem also yields this latter property.

The example below demonstrates that the answer to our question is "No" if we require of B not only that it be S-complete but also that each $s \in S$ acts as a monomorphism of B.

Let S be the cancellative semigroup of Malcev (1937) which cannot be embedded in a group, let S_1 be the semigroup obtained by adjoining an identity, 1, to S and let R be the semigroup ring $Z(S_1)$ (where Z is the ring of integers). Regarding R as a left R-module over itself and setting $T = S$, we make R a T-orderable R-module as follows:

Give S the full order described by Chehata (1953) and extend this order to S_1 by defining for $s, s' \in S_1$

$$s \leqslant s' \quad \text{if, and only if,} \quad \text{either} \quad s, s' \in S \quad \text{and} \quad s \leqslant s'$$
$$\text{or} \quad s = 1 \quad \text{and} \quad s' \in S_1.$$

That S_1 with this full order is a fully ordered semigroup can be checked using the fact that for all $s, s' \in S$, $s < ss'$ and $s < s's$.

Now the additive group of R, being the free abelian group with S_1 as a set of free generators, can be ordered lexicographically using the above order of S_1. That is, for $r = m_1 s_1 + \ldots + m_k s_k \in R$ ($m_i \in Z$, $s_i \in S_1$ with $s_1 < \ldots < s_k$) we define

$$r \geqslant 0 \quad \text{if, and only if,} \quad \text{either} \quad r = 0$$
$$\text{or} \quad r \neq 0 \quad \text{and} \quad m_1 > 0.$$

It is straightforward to check that, ordered in this way, R is a T-orderable R-module and that the remaining hypotheses of the question hold.

However, if it were possible to perform the required embedding (even an embedding where B is not T-orderable), then S would be embedded in a group, namely the automorphism group of B.

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Department of Mathematics
La Trobe University
Bundoora, Victoria 3083
Australia