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ON WAVE MOTION IN A TWO-LAYERED LIQUID OF INFINITE DEPTH IN THE PRESENCE OF SURFACE AND INTERFACIAL TENSION

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Abstract

In this paper various two-dimensional motions are determined for waves in a stratified region of infinite total depth with a free surface containing two superposed liquids, allowing for the effects of surface and interfacial tension. The fundamental set of wave-source potentials for the two layers is used to construct the set of slope potentials that produce discontinuous free-surface and interface slopes. The latter potentials are then utilized to obtain the potentials for waves due to both heaving vertical plates and incident progressive waves against a vertical wall. The underlying assumption of small time-harmonic motion pertains, described by a pair of velocity potentials for the two layers satisfying coupled linearized boundary-value problems, and all solutions are obtained in terms of their matching basic solutions. The technique for applying Green's theorem in the two layers is developed for use with the wave-source potentials, which themselves are found to obey a generalised reciprocity principle. Familiar results for a single liquid of infinite depth are hereby extended, but the new feature emerges of there being two types of progressive waves in all solutions. For ease of presentation the solutions are obtained for a particular relationship between surface and interfacial tension.

1. Introduction

Following the extensive study of small wave motion in a single liquid, considerable interest has now developed in extending the theory to multi-layered or stratified liquids; see for example the papers of Gorgui [3], Gorgui and Kassem [4], Rhodes-Robinson [11], Mandal [8], Kassem [6], Chakrabarti and

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Mandal [2], Chakrabarti [1], and Mandal and Chakrabarti [9, 10] in their order of appearance. Most of these investigations involve two superposed liquids with an interface and possibly a free surface undergoing time-harmonic motion that is described by a pair of velocity potentials satisfying coupled linearized boundary-value problems in the two layers. Interest has centred on the determination of the potentials for two-dimensional wave sources or multipoles and axisymmetric multipoles, which are all fairly tedious to obtain (particularly if the effects of surface and interfacial tension are included). Their forms have been determined from trial solutions, although in one special case for two symmetric liquids another technique relates the potentials to others for a single liquid that are known; see Rhodes-Robinson [11]. These fundamental singularities have the same importance as those for a single liquid, so any that can be found are candidates for use in practical problems.

In this paper we consider various time-harmonic problems for two dimensional motion in a two-layered liquid of infinite total depth with a free surface, allowing for the effects of surface and interfacial tension. First we determined the two pairs of potentials for wave sources at internal points of each layer in terms of basic solutions for progressive and non-propagating waves; these potentials have already been determined in [9] but only in complicated forms involving contour integrals that are unsuitable for most practical applications and must be transformed to alternative useful forms, just as for a single liquid. The now more complicated calculation requiring deformations of contour and integral representation of the singular term is made to obtain the potential in the upper layer for a source in the same layer; this is sufficient as the other three potentials may then be deduced easily from this by matching the basic solutions at the interface and using a new generalized reciprocity principle that is established in and between the layers by a double application of Green's theorem. Next we determine the two pairs of slope potentials at the free surface and interface, which produce a symmetric slope discontinuity at the relevant boundary. These potentials may be constructed in terms of basic solutions using formal relationships involving the previous wave-source potentials, also established by Green's theorem; again by matching at the interface the calculation can be minimized, however.

Two applications are then made to problems involving the radiation and reflexion of waves in the presence of immersed boundaries, which have known solutions for a single liquid. The problems both have solutions expressible in terms of the pairs of slope potentials. The first involves the symmetric capillarity-dependent motion produced by various heaving vertical plates that intersect one or both of the free surface and interface at pairs of opposite edges, for which there are three possible configurations with plates partially immersed in the upper or both liquids and a plate completely submerged in both liquids; this problem was solved in Rhodes-Robinson [13] for a single liquid in similar form and involves just a partially immersed plate, after being considered originally by Hocking [5] to illustrate the use of a dynamical edge condition that pertains in the presence of surface tension. The other application involves reflexion of incident waves by a vertical wall, and is treated similarly; this problem was solved in Rhodes-Robinson [12] for single liquid in similar form. The solutions obtained in both these problems involve edge-slope constants, which are evaluated using the same dynamical edge condition; this procedure was used in [12, 13] in order to quickly identify the form of the solutions in terms of the known slope potential for a single liquid.

For practical reasons of presentation all calculations herein are done only for the case when the ratios of surface and interfacial tension to the density change across the free surface and interface respectively are equal; in principle the more complicated results are obtainable in any situation, however. In this case the basic solutions involve progressive waves of two types, which appear in all solutions and must be taken into account in the use of Green's theorem.

2. Basic formulation and solutions

Consider a region of infinite total depth containing two superposed immiscible liquids in horizontal layers with equilibrium free surface $y = 0$ and interface $y = h$. The upper liquid ('layer 1') has finite depth h and the lower liquid ('layer 2') has infinite depth; the liquids have densities ρ_1, ρ_2 respectively, where $0 < \rho_1 < \rho_2$. The stratified liquid is undergoing small time-harmonic wave motion of angular frequency σ under the action of gravity with acceleration g and surface and interfacial tensions T_1, T_2 respectively. The motion is described by a pair of velocity potentials of the form $\mathrm{R}[\phi_m e^{-i\sigma t}]$ in layer m, where ϕ_m $(m = 1, 2)$ are complex-valued potential functions of position and t is time. The motion is two-dimensional and Cartesian coordinates x, y are used in directions horizontally along and vertically down from the free surface. The basic requirements giving the differential equations and boundary conditions for $\phi_1, \phi_2(x, y)$ may then be taken as

$$\nabla^2\phi_1 = 0 \quad \text{in} \quad 0 < y < h, \qquad \nabla^2\phi_2 = 0 \quad \text{in} \quad y > h, \tag{2.1}$$

$$K\phi_1 + \phi_{1y} + M_1\phi_{1yyy} = 0 \quad \text{on} \quad y = 0, \tag{2.2}$$

$$\left.\begin{aligned} \phi_{1y} &= \phi_{2y}, \\ \rho_1(K\phi_1 + \phi_{1y} + M_2\phi_{1yyy}) &= \rho_2(K\phi_2 + \phi_{2y} + M_2\phi_{2yyy}) \end{aligned}\right\} \text{ on } \quad y = h, \tag{2.3}$$

$$\phi_2 \to 0 \quad \text{as} \quad y \to \infty, \tag{2.4}$$

where $K = \sigma^2/g$ and $M_1 = T_1/\rho_1 g$, $M_2 = T_2/(\rho_2 - \rho_1)g$. In specific problems other boundary conditions are needed to complete the specification of a pair of coupled boundary-value problems, from which solutions for ϕ_1, ϕ_2 are to be determined and known results for a single liquid of infinite depth extended. In the problems considered herein it is found to be a lot simpler to now suppose that $M_1 = M_2 = M$, say; it will be noted later how the results may vary for $M_1 \neq M_2$ but due to their extremely complicated forms no details will be given. Simpler results in the absence of surface and interfacial tension correspond to $M = 0$, provided they remain valid.

Note that the depression of the free surface and interface at $(x, 0)$ and (x, h) are given by $\mathrm{R}[\eta_r(x)e^{-i\sigma t}]$ $(r = 0, 1)$ respectively for the two-dimensional motion, where

$$\eta_0 = (-i\sigma)^{-1}\phi_{1y}(x, 0)$$

and

$$\eta_1 = (-i\sigma)^{-1}\phi_{1y}(x, h) = (-i\sigma)^{-1}\phi_{2y}(x, h);$$

these relations allow the shapes of these nearly horizontal boundaries to be determined after ϕ_1, ϕ_2 are obtained. Note also that their slopes depend on η_r' $(r = 0, 1)$, which therefore involve $\phi_{1xy}(x, 0)$ and $\phi_{1xy}(x, h) = \phi_{2xy}(x, h)$; these relations are needed to formulate other boundary conditions used below.

Relevant basic solutions of (2.1)–(2.4) for $M_1 = M_2 = M$ only are now noted, on which other solutions depend. For convenience let $s = \rho_1/\rho_2$ $(0 < s < 1)$ and put $t = 1 - s$, $\mu = (1 - s)/(1 + s)$.

There are two types of solution for *progressive waves* in the positive or negative x directions with wave numbers κ, κ^*. The faster solutions satisfy $K\phi_m + \phi_{my} + M\phi_{myyy} = 0$ $(m = 1, 2)$ identically in respect of (2.2), (2.3) for $M_1 = M_2 = M$ and are continuous across the interface. These are given by identical constant multiples of

$$\phi_m = e^{-\kappa y \pm i\kappa x} \qquad (m = 1,\ 2) \tag{2.5}$$

respectively in both layers, where κ satisfies the equation $\kappa(1 + M\kappa^2) = K$. Likewise the slower solutions are given by

$$\phi_m = F_m(y)e^{\pm i\kappa^* x} \qquad (m = 1,\ 2), \tag{2.6}$$

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where

$$F_1(y) = t\left[\kappa^*(1+M\kappa^{*2})\cosh\kappa^* y - K\sinh\kappa^* y\right],$$
$$F_2(y) = -sKe^{\kappa^*(2h-y)}$$

in layers 1, 2 and κ^* satisfies $[t\kappa^*(1+M\kappa^{*2}) - sK]\tanh\kappa^* h = K$. (Note that $\kappa^* > K/\mu > K > \kappa$.) There is an interesting discussion of these waves in Lamb [7, Section 231 (note also Section 233)] for $M = 0$.

Other *non-propagating* waves are given by identical linear integral superpositions for $k : 0 \to \infty$ of the solutions

$$\phi_m = e^{\pm kx} f_m(y; k) \qquad (m = 1,\ 2) \tag{2.7}$$

respectively, where

$$f_1(y; k) = K\left[k(1-Mk^2)\cos ky - K\sin ky\right],$$
$$f_2(y; k) = f_1(y; k) + t\left[k^2(1-Mk^2)^2 + K^2\right]\sin kh\cos k(y-h)$$

$(k > 0)$ in layers 1, 2.

In general for pairs of solutions satisfying (2.1)–(2.4) for $M_1 = M_2 = M$ expressed in terms of (2.5)–(2.7) there will be matching at the interface; and if the solution for one layer is known, the solution for the other layer is easily deduced. This *matching principle* is used to advantage herein.

3. Green's theorem

Double applications are to be made for the superposed liquid of Green's theorem for harmonic functions, used often for a single liquid. The modified procedure for this is now determined, and a special result established for progressive waves. Apply Green's theorem to the potential pairs $\phi_m = \Phi_m, \Psi_m$ satisfying (2.1)–(2.4) for $M_1 = M_2 = M$ around matching rectangular contours C_m determined by the horizontal boundaries of each of the layers m $(m = 1, 2)$ and any two fixed vertical lines $x = a, b$ $(a < b)$ in the region, indented if necessary about any singularity in the usual way; then

$$I_m^* = \oint_{C_m} \mathbf{n}\cdot(\Phi_m\nabla\Psi_m - \Psi_m\nabla\Phi_m)ds = 0 \qquad (m = 1,\ 2), \tag{3.1}$$

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where the unit normal $\mathbf{n}$ is inward for each layer (say). To formally evaluate contributions to I_m^* in (3.1) define the Green's integrals

$$I_m(y) = \int_a^b (\Phi_m \Psi_{my} - \Psi_m \Phi_{my})dx \qquad (m = 1,\ 2) \tag{3.2}$$

and

$$\begin{aligned} J_1(x) &= \int_0^h (\Phi_1 \Psi_{1x} - \Psi_1 \Phi_{1x})dy, \\ J_2(x) &= \int_h^\infty (\Phi_2 \Psi_{2x} - \Psi_2 \Phi_{2x})dy, \end{aligned} \tag{3.3}$$

where x or y as appropriate are fixed during integration. The contributions R_m to (3.1) from the *rectangular boundaries* (only) of C_m $(m = 1,\ 2)$ are then

$$\begin{aligned} R_1 &= I_1(0) - I_1(h) - J_1(x)|_a^b, \\ R_2 &= I_2(h) - I_2(\infty) - J_2(x)|_a^b, \end{aligned} \tag{3.4}$$

in layers 1, 2 respectively.

Now if we let

$$f(x, y) = \Psi_{1y}\Phi_{1xy} - \Phi_{1y}\Psi_{1xy} \tag{3.5}$$

in layer 1, then it is found readily by integration from (2.2)–(2.4) for $M_1 = M_2 = M$ (or else is obvious) that

$$\begin{aligned} I_1(0) &= (M/K)f(x,0)|_a^b, \\ \rho_1 I_1(h) - \rho_2 I_2(h) &= -[M(\rho_2 - \rho_1)/K]f(x,h)|_a^b, \\ I_2(\infty) &= 0. \end{aligned} \tag{3.6}$$

Hence on forming a suitable linear combination we have

$$\begin{aligned} \rho_1 R_1 + \rho_2 R_2 &= \rho_1 I_1(0) - [\rho_1 I_1(h) - \rho_2 I_2(h)] - \rho_2 I_2(\infty) \\ &\qquad - [\rho_1 J_1(x) + \rho_2 J_2(x)]|_a^b, \\ &= F(x)|_a^b \end{aligned} \tag{3.7a}$$

from (3.4), (3.6), where

$$F(x) = (M/K)[\rho_1 f(x,0) + (\rho_2 - \rho_1)f(x,h)] - \rho_1 J_1(x) - \rho_2 J_2(x). \tag{3.8}$$

Note that in obtaining (3.6), (3.7a) it is assumed that Φ_{1xy}, Ψ_{1xy} and therefore $f(x, y)$ in (3.5) are continuous on $y = 0, h$ for $a < x < b$; however, if amongst these there is a jump discontinuity at $x = c$ (say) representing a discontinuous

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free-surface or interface slope then the corresponding evaluation in (3.6) needs to be split so that (3.7a) is replaced by

$$\rho_1 R_1 + \rho_2 R_2 = F(x)\left[|_a^b - |_{c-}^{c+}\right]. \tag{3.7b}$$

The results (3.7) for the superposed liquids give the contribution from the rectangular boundaries to the now significant linear combination

$$\rho_1 I_1^* + \rho_2 I_2^* = 0 \tag{3.9}$$

from (3.1), although there may be others from indentations in either layer to evaluate; they represent the extension of a similar result for a single liquid.

To obtain a special result involving the two types of progressive waves in (2.5), (2.6) now take

$$\Phi_m = Ae^{-\kappa y + i\kappa x} + BF_m(y)e^{i\kappa^* x}, \qquad \Psi_m = Ce^{-\kappa y + i\kappa x} + DF_m(y)e^{i\kappa^* x}$$

$(m = 1, 2)$ in the above, where A, B, C, D are constants. Then

$$J_m(x) = \beta_m G(x) \quad (m = 1,\ 2) \qquad \text{and} \qquad f(x, y) = \kappa e^{-\kappa y} F_1'(y) G(x)$$

from (3.3), (3.5), where

$$G(x) = i(\kappa^* - \kappa)(AD - BC)e^{i(\kappa + \kappa^*)x}$$

and

$$\beta_1 = \int_0^h e^{-\kappa y} F_1(y)dy, \qquad \beta_2 = \int_h^\infty e^{-\kappa y} F_2(y)dy;$$

thus $F(x) = \alpha G(x)$ from (3.8) so that

$$\rho_1 R_1 + \rho_2 R_2 = \alpha G(x)|_a^b \tag{3.10}$$

from (3.7a), where

$$\alpha = M\kappa\left[\rho_1 F_1'(0) + (\rho_2 - \rho_1)e^{-\kappa h} F_1'(h)\right]/K - \rho_1\beta_1 - \rho_2\beta_2 \tag{3.11}$$

is a constant. There are no further contributions to (3.9) so that by inserting (3.10) we obtain the result $\alpha G(x)|_a^b = 0$ which holds for *any* a, b; thus

$$\alpha = 0 \tag{3.12}$$

in (3.11), since $G(a) \neq G(b)$ in general. [Then $F(x) = 0$.]

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It now follows for potential pairs Φ_m, Ψ_m $(m = 1, 2)$ that represent outgoing waves as $x \to \infty$ (described as above) that $F(b) \sim \alpha G(b)$ and therefore by (3.12)

$$F(b) \to 0 \tag{3.13a}$$

in the contribution to (3.9) in (3.7) on letting $b \to \infty$; similarly if there are outgoing waves as $x \to -\infty$ then

$$F(a) \to 0 \tag{3.13b}$$

on letting $a \to -\infty$. The results (3.13) represent the extension of a similar but immediately obvious result for a single liquid, which possesses progressive waves of only one type.

To conclude, note that the result (3.12) may be verified directly using the calculated values $F_1'(0)/K = -t\kappa^*$, $F_1'(h)/K = s\kappa^* e^{\kappa^* h}$ and

$$\beta_1 = t\left[(1 + M\kappa^2 + M\kappa^{*2})\sinh \kappa^* h + M\kappa\kappa^* \cosh \kappa^* h\right] e^{-\kappa h} - tM\kappa\kappa^*,$$
$$\beta_2 = -st(1 + M\kappa^2 - M\kappa\kappa^* + M\kappa^{*2})e^{-\kappa h} \sinh \kappa^* h$$

in (3.11), taken in the form $\alpha/\rho_2 = M\kappa\left[sF_1'(0) + te^{-\kappa h}F_1'(h)\right]/K - s\beta_1 - \beta_2$.

4. Slope and wave-source potentials

Two symmetric singular potentials familiar in the theory of a single liquid have analogues ϕ_m $(m = 1,\ 2)$ for the stratified liquid, satisfying certain additional boundary conditions in unbounded horizontal layers that include outgoing waves at infinity as in (2.5), (2.6).

Slope potentials exist only for $M > 0$ and produce a symmetric slope discontinuity at a point of either the free surface or interface: for fixed $r = 0, 1$ the two pairs $\phi_m = G_{r;m}(x, y)$ at $x = 0$ on $y = 0$ $(r = 0)$ and $y = h$ $(r = 1)$ satisfy the conditions

$$\phi_{1xy}(0\pm, 0) = \pm\pi/M \qquad (r = 0), \tag{4.1}$$
$$\phi_{1xy}(0\pm, h) = \phi_{2xy}(0\pm, h) = \pm\pi/M \qquad (r = 1) \tag{4.2}$$

—whereas

$$\phi_{1xy}(0, h) = \phi_{2xy}(0, h) = 0 \qquad (r = 0), \tag{4.3}$$
$$\phi_{1xy}(0, 0) = 0 \qquad (r = 1); \tag{4.4}$$

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also, for both pairs,

$$\phi_m \to \text{linear combination of } e^{-\kappa y+i\kappa|x|},\ F_m(y)e^{i\kappa^*|x|} \quad \text{as} \quad |x| \to \infty \tag{4.5}$$

and ϕ_m is symmetrical in x ($m = 1, 2$).

Wave-source potentials have the usual logarithmic singularity at an internal point of one layer: for fixed $n = 1, 2$ the two pairs $\phi_m = G_{m,n}(x, y; S)$ for a source at $S(X, Y)$ in layer n satisfy the conditions

$$\phi_n \sim \ln \rho \quad \text{as} \quad \rho = \left[(x-X)^2 + (y-Y)^2\right]^{1/2} \to 0; \tag{4.6}$$

also for both pairs

$$\phi_m \to \text{linear combination of } e^{-\kappa y+i\kappa|x-X|},\ F_m(y)e^{i\kappa^*|x-X|} \quad \text{as} \quad |x-X| \to \infty \tag{4.7}$$

and ϕ_m depends symmetrically on the single horizontal coordinate $x - X$; ($m = 1, 2$). Note that $0 < Y < h$ ($n = 1$) or $Y > h$ ($n = 2$).

5. Reciprocity principle

There is a generalized reciprocity principle for wave-source potentials, extending that for a single liquid. To determine this now apply Green's theorem as in Section 3 to the potential pairs

$$\Phi_m = G_{m,r}(x, y; S'), \qquad \Psi_m = G_{m,n}(x, y; S)$$

for fixed $r, n = 1, 2$ (S, S' distinct) around the infinite rectangular boundary of each of the layers m ($m = 1, 2$); then $a \to -\infty$, $b \to \infty$ and $F(a)$, $F(b) \to 0$ by (3.13) because of (4.7) so that

$$\rho_1 R_1 + \rho_2 R_2 = 0 \tag{5.1}$$

from (3.7a). Here there are contributions to (3.9) from indentations about both S, S' given by

$$2\pi\left[\rho_n G_{n,r}(S; S') - \rho_r G_{r,n}(S'; S)\right] \tag{5.2}$$

from (4.6) in the usual way, so that by inserting (5.1), (5.2) in (3.9) we obtain the result

$$\rho_n G_{n,r}(S; S') - \rho_r G_{r,n}(S'; S) \tag{5.3}$$

($r, n = 1, 2$), which is the reciprocity principle either in or between the layers for wave-source potentials in the stratified liquid.

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6. Construction of slope potentials

There is also a formal construction for the slope potentials from the wave-source potentials, analogous to that for a single liquid. To determine this now apply Green's theorem as in Section 3 to the potential pairs

$$\Phi_m = G_{r;m}(x, y), \qquad \Psi_m = G_{m,n}(x, y; S)$$

for fixed $r = 0, 1$ and $n = 1, 2$ around the same infinite rectangular boundary of each of the layers m $(m = 1, 2)$; then again $a \to -\infty$, $b \to \infty$ and $F(a), F(b) \to 0$ by (3.13) because of (4.5) so that

$$\rho_1 R_1 + \rho_2 R_2 = -F(x)|_{0-}^{0+} = -(2M/K)[\rho_1 f(0+, 0) + (\rho_2 - \rho_1) f(0+, h)] \tag{6.1}$$

from (3.7b) with $c = 0$ and (3.8), allowing for a symmetric discontinuity in $f(x, 0)$ or $f(x, h)$ at $x = 0$ and noting that J_1, $J_2(x)$ are continuous. Now from (3.5)

$$\begin{aligned} f(0+, 0) &= (\pi/M) G_{1,ny}(0, 0; S), \\ f(0+, h) &= f(0, h) = 0 \end{aligned} \tag{6.2}$$

$(r = 0)$ by (4.1), (4.3) and

$$\begin{aligned} f(0+, 0) &= f(0, 0) = 0, \\ f(0+, h) &= (\pi/M) G_{1,ny}(0, h; S) \end{aligned} \tag{6.3}$$

$(r = 1)$ by (4.2), (4.4); thus (6.1) becomes

$$\rho_1 R_1 + \rho_2 R_2 = -2\pi(\rho_1/K) G_{1,ny}(0, 0; S) \tag{6.4}$$

$$= -2\pi[(\rho_2 - \rho_1)/K] G_{1,ny}(0, h; S) \tag{6.5}$$

for $r = 0, 1$ respectively. Here again there is a contribution to (3.9) from the indentation about S given by

$$2\pi \rho_n G_{r;n}(S) \tag{6.6}$$

from (4.6) so that by inserting (6.4)–(6.6) in (3.9) we obtain the results

$$G_{0;n}(S) = (\rho_1/K\rho_n) G_{1,ny}(0, 0; S) \tag{6.7}$$

and

$$G_{1;n}(S) = [(\rho_2 - \rho_1)/K\rho_n] G_{1,ny}(0, h; S) \tag{6.8}$$

$(n = 1, 2)$, which are the relationships for determining the slope potentials from the wave-source potentials for the stratified liquid.

7. Determination of $G_{1,1}$

Specific forms in terms of the basic solutions (2.5)–(2.7) are now obtained for the sets of wave-source potentials $G_{m,n}$ $(m, n = 1, 2)$ and slope potentials $G_{r;m}$ $(r = 0, 1$ and $m = 1, 2)$. For the wave-source potentials it is sufficient to calculate $G_{1,1}$ only, since $G_{2,1}$ and $G_{m,2}$ $(m = 1, 2)$ can then be deduced from it by using the matching and reciprocity principles. The slope potentials that depend on these are then easily calculated using (6.7), (6.8); but note that use of $G_{1,1}$ alone is again sufficient, since $G_{r;1}$ can be calculated from it and then $G_{r;2}$ deduced using the matching principle $(r = 0, 1)$. The form for the wave-source potential $G_{1,1}$ is now determined by our only long calculation.

Detailed results for the wave-source potentials have been derived for general values of M_1, M_2 in largely unsimplified forms involving contour integrals by Mandal and Chakrabarti [9], similar to the results for $M_1 = M_2 = 0$ of Chakrabarti and Mandal [2]. These forms are not of practical usefulness but do provide a basis for obtaining useful forms in terms of basic solutions by transformation, as for a single liquid. Taking $X = 0$, $0 < Y < h$ their first result for the symmetric $G_{1,1}$ in terms of s, t, μ (see Section 2) can be simplified and for $M_1 = M_2 = M$ is

$$\begin{aligned} G_{1,1} = {} & \tfrac{1}{2}\sum_{j=0}^{\infty}(-\mu)^j \ln\frac{x^2 + (y - Y - 2jh)^2}{x^2 + (y + Y + 2jh)^2} \\ & + \tfrac{1}{2}\sum_{j=1}^{\infty}(-\mu)^j \ln\frac{x^2 + (y - Y + 2jh)^2}{x^2 + (y + Y - 2jh)^2} \\ & - 2t \oint_0^{\infty} \frac{(1 + Mk^2)q(y, Y; k)}{(\cosh kh + s\sinh kh)\Delta(k)}\cos kx\, dk \\ & + 2sK \oint_0^{\infty} \frac{(1 + Mk^2)e^{-k(y+Y-h)}}{[k(1 + Mk^2) - K]\Delta(k)}\cos kx\, dk, \end{aligned} \tag{7.1}$$

where $\Delta(k) = [tk(1 + Mk^2) - sK]\sinh kh - K\cosh kh$ and as indicated the contours are indented below the poles $k = \kappa, \kappa^*$ (see Section 2); also

$$q(y, Y; k) = \cosh k(h - y)\cosh k(h - Y) + s\sinh kh\cosh k(y + Y - h).$$

Some further manipulation then allows the integral terms in (7.1) to be taken as

$$2t \oint_0^{\infty} \frac{(1 + Mk^2)\sinh ky \sinh kY}{(\cosh kh + s\sinh kh)\Delta(k)}\cos kx\, dk$$

$$-2 ⨍_0^\infty \frac{p(y, Y; k)}{[k(1+Mk^2) - K]\Delta(k)} \cos kx \, dk \tag{7.2}$$

$(0 < y < h)$, where

$$p(y, Y; k) = [tk(1+Mk^2) - K] \cosh k(y+Y-h) + sK \sinh k(y+Y-h),$$

and these are now ready for transformation.

The alternative form for $x > 0$, say, is now found by first putting $2\cos kx = e^{ikx} + e^{-ikx}$ in (7.2) and rotating the contours in the integrals so formed to contours along the positive and negative imaginary axes respectively so that we must include residue terms at $k = \kappa, \kappa^*$ for the first (all other poles have negative real part so are not crossed). The *integral terms* obtained are

$$2st \int_0^\infty \frac{(1-Mk^2)e^{-kx} \sin ky \sin kY}{(\cos^2 kh + s^2 \sin^2 kh)D(k)} \sin kh \Delta_2(k) dk$$
$$+ \mathrm{R}\left[2i \int_0^\infty \frac{(1-Mk^2)e^{-kx}}{[k(1-Mk^2) + iK]\Delta_1(k)} p_1(y, Y; k) dk\right], \tag{7.3}$$

where

$$\Delta_1(k) = -\Delta(ik) = tk(1-Mk^2) \sin kh + K \cos kh + isK \sin kh$$

and

$$D(k) = |\Delta_1(k)|^2 = [tk(1-Mk^2) \sin kh + K \cos kh]^2 + s^2 K^2 \sin^2 kh;$$

also $\Delta_2(k) = tk(1-Mk^2) \sin kh + 2K \cos kh$ and

$$p_1(y, Y; k) = -ip(y, Y; ik)$$
$$= [tk(1-Mk^2) + iK] \cos k(y+Y-h) + sK \sin k(y+Y-h)$$

$(k > 0)$.

An integral representation of similar form for the series of logarithmic terms in (7.1) containing the singularity is determined as

$$-2s \int_0^\infty \frac{e^{-kx} \sin ky \sin kY}{k(\cos^2 kh + s^2 \sin^2 kh)} dk \qquad (x > 0) \tag{7.4}$$

in the Appendix; thus combining (7.3), (7.4) the non-propagating part of $G_{1,1}$ is found to be

$$-2sK \int_0^\infty \frac{e^{-kx}}{D(k)} \left[\frac{K \sin ky \sin kY}{k} + \frac{(1-Mk^2) f_1(y+Y; k)}{k^2(1-Mk^2)^2 + K^2}\right] dk$$
$$= -2s \int_0^\infty \frac{e^{-kx} f_1(y; k) f_1(Y; k)}{k[k^2(1-Mk^2)^2 + K^2] D(k)} dk, \tag{7.5}$$

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in terms of the basic solutions (2.7) for $m = 1$.

The remaining *residue terms* referred to above give the outgoing waves of $G_{1,1}$ as $x \to \infty$: those with wave number κ are given by

$$-2\pi i s(1 + M\kappa^2)A_0 e^{-\kappa(y+Y-h)+i\kappa x} \tag{7.6}$$

in terms of the basic solutions (2.5) for m = 1, where

$$A_0 = 1/(1 + 3M\kappa^2)[(s - t)\sinh\kappa h + \cosh\kappa h];$$

and those with wave number κ^* by

$$\begin{aligned} &2\pi i \frac{1 + M^{*2}}{D_1}\left[\frac{t\sinh\kappa^* y\sinh\kappa^* Y}{\cosh\kappa^* h + s\sinh\kappa^* h} - \frac{p(y, Y;^*)}{\kappa^*(1 + M^{*2}) - K}\right]e^{i\kappa^* x} \\ &\quad = 2\pi i \frac{1 + M^{*2}}{D_1}\left[\frac{t\sinh\kappa^* y\sinh\kappa^* Y}{\cosh\kappa^* h + s\sinh\kappa^* h} - \frac{e^{-\kappa^* h}F_1(y + Y)}{\kappa^*(1 + M^{*2}) - K}\right]e^{i\kappa^* x} \\ &\quad = -2\pi i A_1 F_1(y)F_1(Y)e^{-\kappa^* h + i\kappa^* x} \end{aligned} \tag{7.7}$$

in terms of the basic solutions (2.6) for m = 1, where

$$A_1 = 1/t\kappa^*[\kappa^*(1 + M\kappa^{*2}) - K]D_1$$

and

$$D_1 = [t(1 + 3M\kappa^{*2}) - Kh]\sinh\kappa^* h + [t\kappa^*(1 + M\kappa^{*2}) - sK]h\cosh\kappa^* h.$$

This completes the calculation for the first wave-source potential. Hence on adding the contributions (7.5)–(7.7) the required form for $0 < Y < h$ is

$$\begin{aligned} G_{1,1}(x, y; 0, Y) = -2s\int_0^\infty \frac{e^{-kx}f_1(y;k)f_1(Y;k)}{k[k^2(1 - Mk^2)^2 + K^2]D(k)}dk \\ -2\pi i\big[s(1 + M\kappa^2)A_0 e^{-\kappa(y+Y-h)+i\kappa x} \\ + A_1 F_1(y)F_1(Y)e^{-\kappa^* h + i\kappa^* x}\big] \end{aligned} \tag{7.8}$$

$(0 < y < h)$ in terms of the basic solutions (2.5)–(2.7) for $m = 1$.

8. Determination of $G_{2,1}$ and $G_{m,2}$ $(m = 1, 2)$

The corresponding forms for the three remaining wave-source potentials are now easily deduced in succession; thus by matching at the interface using (2.5)–

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(2.7) we find for $0 < Y < h$ that

$$G_{2,1}(x, y; 0, Y) = -2s \int_0^\infty \frac{e^{-kx} f_2(y; k) f_1(Y; k)}{k[k^2(1 - Mk^2)^2 + K^2]D(k)} dk \\ -2\pi i \big[s(1 + M\kappa^2) A_0 e^{-\kappa(y+Y-h)+i\kappa x} \\ + A_1 F_2(y) F_1(Y) e^{-\kappa^* h + i\kappa^* x} \big] \quad (8.1)$$

$(y > h)$ from (7.8) in terms of the basic solutions (2.5)–(2.7) for $m = 2$.

Further, by reciprocity between the layers and the simpler properties we have

$$sG_{1,2}(x, y; 0, Y) = G_{2,1}(0, Y; x, y) = G_{2,1}(-x, Y; 0, y) = G_{2,1}(x, Y; 0, y)$$

from (5.3) so that for $Y > h$

$$G_{1,2}(x, y; 0, Y) = -2 \int_0^\infty \frac{e^{-kx} f_1(y; k) f_2(Y; k)}{k[k^2(1 - Mk^2)^2 + K^2]D(k)} dk \\ -2\pi i \big[(1 + M\kappa^2) A_0 e^{-\kappa(y+Y-h)+i\kappa x} \\ + A_1 F_1(y) F_2(Y) e^{-\kappa^* h + i\kappa^* x} \big] \quad (8.2)$$

$(0 < y < h)$ from (8.1) in terms of the basic solutions (2.5)–(2.7) for $m = 1$; thus by matching again

$$G_{2,2}(x, y; 0, Y) = -2 \int_0^\infty \frac{e^{-kx} f_2(y; k) f_2(Y; k)}{k[k^2(1 - Mk^2)^2 + K^2]D(k)} dk \\ -2\pi i \big[(1 + M\kappa^2) A_0 e^{-\kappa(y+Y-h)+i\kappa x} \\ + A_1 F_2(y) F_2(Y) e^{-\kappa^* h + i\kappa^* x} \big] \quad (8.3)$$

$(y > h)$ from (8.2) in terms of the basic solutions (2.5)–(2.7) for $m = 2$.

This completes the determination of the required forms for the wave-source potentials $G_{m,n}$ $(m, n = 1, 2)$ in $x > 0$ when $X = 0$, from which those for any source position are easily deduced. Note that $G_{1,1}$, $G_{2,2}$ then exhibit the required reciprocity in their layers. The results for $M = 0$, also new, are obtainable as a special case.

9. Determination of $G_{r;m}$ $(r = 0,\ 1$ and $m = 1,\ 2)$

The corresponding forms for the slope potentials can now be calculated. Hence

$$G_{0;1}(x, y) = 2sK \int_0^\infty \frac{e^{-kx} f_1(y; k)}{[k^2(1 - Mk^2)^2 + K^2]D(k)} dk \\ +2\pi i \big[sA_0 e^{\kappa(h-y)+i\kappa x} + t\kappa^* A_1 F_1(y) e^{-\kappa^* h + i\kappa^* x} \big] \quad (9.1)$$

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$(0 < y < h)$ from (6.7), (7.8); thus by matching at the interface

$$G_{0;2}(x, y) = 2sK \int_0^\infty \frac{e^{-kx} f_2(y; k)}{[k^2(1 - Mk^2)^2 + K^2]D(k)} dk + 2\pi i \left[sA_0 e^{\kappa(h-y)+i\kappa x} + t\kappa^* A_1 F_2(y) e^{-\kappa^* h + i\kappa^* x} \right] \quad (9.2)$$

$(y > h)$ from (9.1).

Likewise

$$G_{1;1}(x, y) = 2t \int_0^\infty \frac{e^{-kx} f_1(y; k)}{[k^2(1 - Mk^2)^2 + K^2]D(k)} \Delta^*(k) dk + 2\pi i t \left[A_0 e^{-\kappa y + i\kappa x} - \kappa^* A_1 F_1(y) e^{i\kappa^* x} \right] \quad (9.3)$$

$(0 < y < h)$ from (6.8), (7.8), where $\Delta^*(k) = k(1 - Mk^2) \sin kh + K \cos kh$ $(k > 0)$; thus by matching again

$$G_{1;2}(x, y) = 2t \int_0^\infty \frac{e^{-kx} f_2(y; k)}{[k^2(1 - Mk^2)^2 + K^2]D(k)} \Delta^*(k) dk + 2\pi i t \left[A_0 e^{-\kappa y + i\kappa x} - \kappa^* A_1 F_2(y) e^{i\kappa^* x} \right] \quad (9.4)$$

$(y > h)$ from (9.3).

This completes the determination of the required forms for the slope potentials $G_{r;m}$ $(r = 0, 1$ and $m = 1, 2)$ in $x > 0$ in terms of the basic solutions (2.5)–(2.7) for $m = 1, 2$; as noted earlier, these only exist for $M > 0$.

10. Heaving-plate problems

An application of the preceding results is now made in the problem to determine the various motions in unbounded horizontal layers of a stratified liquid with outgoing waves at infinity due to the vertical oscillations (heaving) of a vertical plate along part of $x = 0$. To produce this motion for $M > 0$ (none eventuates if $M = 0$) the plate must have at least one point in the free surface or interface, and we suppose that it actually *intersects* that boundary (so has a pair of opposite 'edges'); otherwise, the plate may terminate at any internal point of a layer. The three possible configurations shown in Figure 1 that produce different motions therefore involve plates partially immersed in the upper or both liquids (plates A, C) and a plate completely submerged in both liquids (plate B). The plates are assumed to be uniform and the motion symmetrical about any gap on $x = 0$ not

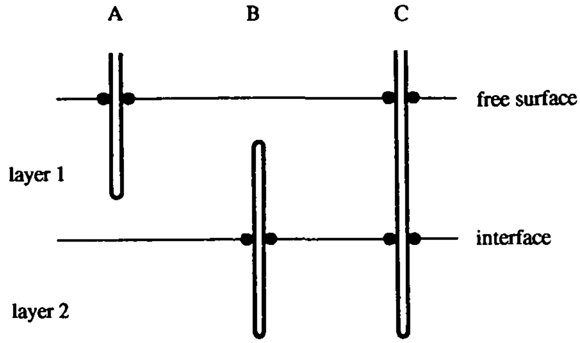

FIGURE 1. The heaving plates A, B, C with edges, as marked. The zero plate thicknesses are exaggerated for clarity.

occupied by the plate. Then the potentials ϕ_m $(m = 1, 2)$ respectively satisfy the additional boundary conditions

$$\phi_{1xy}(0\pm, 0) = \pm\pi\nu_0, \tag{10.1}$$

$$\phi_{1xy}(0\pm, h) = \phi_{2xy}(0\pm, h) = \pm\pi\nu_1 \tag{10.2}$$

formally specifying the free-surface and interface slopes at the plate if there is a relevant edge, where ν_0, ν_1 are appropriate edge-slope constants, whereas

$$\phi_{1xy}(0, 0) = 0, \tag{10.3}$$

$$\phi_{1xy}(0, h) = \phi_{2xy}(0, h) = 0 \tag{10.4}$$

if there is no free-surface or interface edge; also for both pairs

$$\phi_m \to \text{linear combination of } e^{-\kappa y + i\kappa|x|},\ F_m(y)e^{i\kappa^*|x|} \quad \text{as} \quad |x| \to \infty \tag{10.5}$$

and ϕ_m is symmetrical in x $(m = 1, 2)$.

The constants ν_0, ν_1 in (10.1), (10.2) are evaluated after obtaining formal solutions by applying the dynamical edge conditions

$$\phi_{1y}(0, 0) - V = iC_0\nu_0/\sigma, \tag{10.6}$$

$$\phi_{1y}(0, h) - V = \phi_{2y}(0, h) - V = iC_1\nu_1/\sigma \tag{10.7}$$

for the relative motion (slipping) at free-surface or interface edges if the plate has downward heave velocity $\mathrm{R}[Ve^{-i\sigma t}]$, where V is a complex-valued constant and C_0, C_1 are material constants at pairs of opposite edges that are regarded

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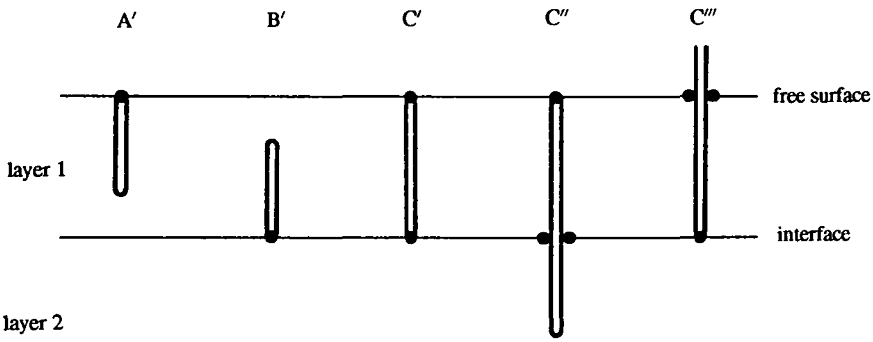

FIGURE 2. The heaving plates A′, B′, C′ with tips and C″, C‴ with tips and edges, as marked. The zero plate thicknesses are exaggerated for clarity.

as known. This procedure to render a formal solution involving edge-slope constants realistic has been used for a single liquid of infinite depth, in particular for a partially immersed heaving vertical plate as described in Rhodes-Robinson [13, Section 4]; the problem is generalized here and was first considered by Hocking [5], who gave a comprehensive discussion of dynamical edge conditions including the type used here that expresses the specific linear relationship between relative velocity of the contact line and free-surface or interface slope at an edge (suitable for time-harmonic motion).

Note that if a plate terminates at the free surface or interface (so has a 'tip') we must instead apply the respective simpler dynamical tip conditions

$$\phi_{1y}(0,0) = V, \tag{10.8}$$

$$\phi_{1y}(0,h) = \phi_{2y}(0,h) = V \tag{10.9}$$

($|V|/\sigma$ small) for no relative motion (sticking); it is then obvious from (10.6), (10.7) that solutions for the five possible limiting configurations shown in Figure 2 (plates A′, B′, C′, C″, C‴) can be deduced from the previous solutions by formally putting $C_0 = 0$ or $C_1 = 0$ as appropriate for any pair of opposite edges that become a tip.

The main heaving-plate problems involving (10.1)–(10.5) above all have obvious formal solutions in terms of the slope potentials (9.1)–(9.4), which satisfy (4.1)–(4.5); these are now listed.

For plate A partially immersed in the upper liquid the solutions are

$$\phi_m = M\nu_0 G_{0;m} \qquad (m = 1,\ 2) \tag{10.10a}$$

in terms of ν_0; this is evaluated from a linear equation obtained by applying the condition (10.6) for $m = 1$ as

$$\nu_0 = V/(MA_{0,0} - iC_0/\sigma), \tag{10.10b}$$

where $A_{0,0} = G_{0;1y}(0, 0)$.

For plate B completely submerged in both liquids the solutions are

$$\phi_m = M\nu_1 G_{1;m} \qquad (m = 1,\ 2) \tag{10.11a}$$

in terms of ν_1; this is evaluated from a linear equation obtained by applying the condition (10.7) for $m = 1$ or $m = 2$ as

$$\nu_1 = V/(MA_{1,1} - iC_1/\sigma), \tag{10.11b}$$

where $A_{1,1} = G_{1;1y}(0, h) = G_{1;2y}(0, h)$.

For plate C partially immersed in both liquids the solutions are

$$\phi_m = M(\nu_0 G_{0;m} + \nu_1 G_{1;m}) \qquad (m = 1,\ 2) \tag{10.12a}$$

in terms of ν_0, ν_1; these are evaluated from the linear equations

$$\begin{aligned} (MA_{0,0} - iC_0/\sigma)\nu_0 + MA_{1,0}\nu_1 &= V, \\ MA_{0,1}\nu_0 + (MA_{1,1} - iC_1/\sigma)\nu_1 &= V \end{aligned} \tag{10.12b}$$

obtained by applying both the conditions (10.6), (10.7) for $m = 1,2$, where $A_{1,0} = G_{1;1y}(0, 0)$ and $A_{0,1} = G_{0;1y}(0, h) = G_{0;2y}(0, h)$. The results (10.10)–(10.12) extend those for the partially immersed plate in a single liquid of infinite depth as obtained in [13]. Integral expressions for the set of constants $A_{r,s}$ ($r, s = 0, 1$) are obtained from (9.1)–(9.4) but are difficult to evaluate analytically. The limiting heaving-plate solutions may be deduced as noted; thus for plate A′ in the upper liquid terminating at the free surface the solutions are $\phi_m = (V/A_{0,0})G_{0;m}$ ($m = 1, 2$) from (10.10) on putting $C_0 = 0$, etc.

11. Reflexion at a wall

Another application is made in the problem to determine the motions in semi-infinite horizontal layers of a stratified liquid occupying $x > 0$ due to progressive waves of one or both types normally incident upon a fixed vertical wall $x = 0$ with two edges, at which reflexion is incomplete in the presence of surface and interfacial tension. Then for incident waves with potentials $e^{-\kappa y - i\kappa x}$ (say) as $x \to \infty$ the potentials ϕ_m ($m = 1, 2$) satisfy the additional boundary conditions

$$\phi_{1xy}(0\pm, 0) = \pi\lambda_0,$$
$$\phi_{1xy}(0\pm, h) = \phi_{2xy}(0\pm, h) = \pi\lambda_1,$$

where λ_0, λ_1 are edge-slope constants; also for both potentials

$$\phi_{mx} = 0 \quad \text{on} \quad x = 0,$$
$$\phi_m - e^{-\kappa y - i\kappa x} \to \text{linear combination of } e^{-\kappa y + i\kappa x},\ F_m(y)e^{i\kappa^* x} \quad \text{as} \quad x \to \infty$$

($m = 1, 2$). The constants λ_0, λ_1 are evaluated using the dynamical edge conditions

$$\phi_{1y}(0, 0) = iC_0\lambda_0/\sigma,$$
$$\phi_{1y}(0, h) = \phi_{2y}(0, h) = iC_1\lambda_1/\sigma,$$

where C_0, C_1 are as before.

The solutions are

$$\phi_m = 2e^{-\kappa y}\cos\kappa x + M(\lambda_0 G_{0;m} + \lambda_1 G_{1;m}) \qquad (m = 1, 2)$$

and λ_0, λ_1 are evaluated from the equations

$$(MA_{0,0} - iC_0/\sigma)\lambda_0 + MA_{1,0}\lambda_1 = 2\kappa,$$
$$MA_{0,1}\lambda_0 + (MA_{1,1} - iC_1/\sigma)\lambda_1 = 2\kappa e^{-\kappa h}.$$

These results (and others like them) extend those for a wall in a single liquid of infinite depth obtained in Rhodes-Robinson [12]. Note that both types of outgoing waves are usually reflected even if only one type is incident. For $M = 0$ there is complete reflexion to give simple standing waves (no edge conditions are now needed).

A familiar problem in the same region of the stratified liquid that involves both slope and wave-source potentials in its solutions is that for the classical *vertical wave-maker* along $x = 0$. These have the form of a distribution throughout the two layers of wave-source potentials on the wave-maker boundary and discrete slope potentials at the two edges. The edge-slope constants are now a little harder to evaluate in the easily obtained formal solutions. For $M = 0$ there are only wave-source potentials in the solutions.

12. Conclusion

In principle the results obtained herein for $M_1 = M_2$ can be extended to the general situation $M_1 \neq M_2$ but the forms are extremely complicated in the specific calculations. The wave numbers for the two types of progressive waves no longer satisfy explicit separate equations and their nature is not readily apparent. The procedure for Green's theorem still applies. The slope potentials in the free surface and interface are constructed from the wave-source potentials using the same relationships if the slope terms in the boundary conditions are taken as $\pm\pi/M_1$, $\pm\pi/M_2$ respectively. The reciprocity principle for wave-source potentials continues to hold.

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Appendix
Integral representation

For $x > 0$ the series of logarithmic terms in $G_{1,1}$ are

$$\frac{1}{2}\sum_{j=0}^{\infty}(-\mu)^j \ln\frac{x^2+(y-Y-2jh)^2}{x^2+(y+Y+2jh)^2}+\frac{1}{2}\sum_{j=1}^{\infty}(-\mu)^j \ln\frac{x^2+(y-Y+2jh)^2}{x^2+(y+Y-2jh)^2}$$
$$=-2\sum_{j=0}^{\infty}(-\mu)^j\int_0^{\infty}\frac{e^{-kx}\sin ky\sin k(Y+2jh)}{k}dk$$
$$-2\sum_{j=1}^{\infty}(-\mu)^j\int_0^{\infty}\frac{e^{-kx}\sin ky\sin k(Y-2jh)}{k}dk$$

on using a familiar integral representation in each term

$$=-2\int_0^{\infty}\frac{e^{-kx}\sin ky\sin kY}{k}\left[1+2\sum_{j=1}^{\infty}(-\mu)^j\cos 2jkh\right]dk$$
$$=-2\int_0^{\infty}\frac{e^{-kx}\sin ky\sin kY}{k}\left[1-\frac{2\mu(\cos 2kh+\mu)}{1+2\mu\cos 2kh+\mu^2}\right]dk$$
$$=-2(1-\mu^2)\int_0^{\infty}\frac{e^{-kx}\sin ky\sin kY}{k(1+2\mu\cos 2kh+\mu^2)}dk$$
$$=-2s\int_0^{\infty}\frac{e^{-kx}\sin ky\sin kY}{k(\cos^2 kh+s^2\sin^2 kh)}dk \tag{A.1}$$

on simplification, recalling that $\mu=(1-s)/(1+s)$.

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