AN EXISTENCE THEOREM FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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It may sometimes be desirable to start a lecture course on the second order linear differential equation with an existence theorem without having recourse to the complete set of existence theorems for systems of differential equations or for non-linear equations of the nth order.

We give below a short and simple treatment leading to the following existence theorem for the second order linear differential equation:

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0. \qquad \text{.........................(1)}$$

Theorem. *A solution of the differential equation* (1) *exists for all values of x for which both $P(x)$ and $Q(x)$ are regular.*

The use of the integrating factor $w(x)=e^{\int P(x)dx}$ enables us to write (1) in the form

$$w\frac{d}{dx}\left(w\frac{dy}{dx}\right)+w^2Qy=0. \qquad \text{.............................(2)}$$

If $P(x)$, $Q(x)$ and therefore $w(x)$ are regular at $x=a$ the change of variable

$$t=\int_a^x \frac{dx}{w(x)}$$

makes $t=0$ correspond to $x=a$. Since $P(x)=\dfrac{w'(x)}{w(x)}$ it is clear that $w(a)\neq 0$. Equation (2) now becomes

$$\frac{d^2y}{dt^2}-q(t)y=0, \qquad \text{..................................(3)}$$

where $q(t)$ is regular at $t=0$.

Thus $q(t)=\sum_0^\infty q_nt^n$ has a positive radius of convergence k' and $|q_n|<1/k^n$ for any $k<k'$ and some $n>n_0$. We can, however, find a constant K such that

$$\frac{d^2u}{dt^2}-\frac{Ku}{1-\dfrac{t}{k}}=0 \qquad \text{..................................(4)}$$

is a dominant for (3) in the sense that $|q_n|<K/k^n$ for every n.

If $y(t)=\sum_0^\infty b_nt^n$ is a solution of (3) then it is sufficient for its validity that it has a positive radius of convergence. We show that the corresponding solution of (4), $u(t)=\sum_0^\infty c_nt^n$ has a positive radius of convergence and that

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$c_n \geqq | b_n |$ for every n. From this it will follow that the series for $y(t)$ has a positive radius of convergence.

Substituting the series for $y(t)$ in (3) and for $u(t)$ in (4), we get on identifying coefficients,

$$2b_2 = q_0 b_0, \quad 6b_3 = q_0 b_1 + q_1 b_0, \quad 12b_4 = q_0 b_2 + q_1 b_1 + q_2 b_0,$$
$$20b_5 = q_0 b_3 + q_1 b_2 + q_2 b_1 + q_3 b_0, \; \ldots \quad \ldots\ldots\ldots(5)$$

and

$$2c_2 = Kc_0, \quad 6c_3 = Kc_1 + Kc_0/k, \quad 12c_4 = Kc_2 + Kc_1/k + Kc_0/k^2,$$
$$20c_5 = Kc_3 + Kc_2/k + Kc_1/k^2 + Kc_0/k^3, \; \ldots \quad \ldots\ldots\ldots(6)$$

It follows that b_0, b_1, c_0, c_1 are arbitrary. We therefore choose $b_0 = c_0$ and $b_1 = c_1$ to be both positive. Now, while the quantities q_n in (5) may be complex, or if real, either positive or negative, the corresponding quantities in (6) are all positive. Hence, every term in (6) is positive and so is every c_n. Moreover, every $| q_n | < K/k^n$. From these it follows at once that $c_n \geqq | b_n |$ for every n.

Substituting $u(t) = \sum_0^\infty c_n t^n$ in (4), we get

$$\left(1 - \frac{t}{k}\right) \sum_0^\infty (n+2)(n+1) c_{n+2} t^n - K \sum_0^\infty c_n t^n = 0. \quad \ldots\ldots\ldots(7)$$

Equating coefficients in (7) gives

$$(n+2)(n+1)c_{n+2} - \frac{1}{k}(n+1)nc_{n+1} - Kc_n = 0$$

or

$$c_{n+2} - \frac{1}{k}\frac{n}{n+2}c_{n+1} - \frac{K}{(n+2)(n+1)}c_n = 0. \quad \ldots\ldots\ldots(8)$$

The limiting form of the coefficients in (8) is given by the coefficients of the equation $\lambda^2 - \dfrac{\lambda}{k} = 0$, and, by a well-known theorem of Poincaré,†

$$\lim_{n\to\infty} c_{n+1}/c_n = \frac{1}{k},$$

which proves that the radius of convergence of the series for $u(t)$ is $k > 0$.

Hence, since the series $\sum_0^\infty c_n t^n$ is a dominant for $\sum_0^\infty b_n t^n$, it follows that the radius of convergence of the series for $y(t)$ is at least k.

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† N. E. Nörlund, *Vorlesungen über Differenzenrechnung* (Berlin, 1924), p. 305. Guldberg und Wallenberg, *Theorie der linearen Differenzengleichungen* (Leipzig, 1911), p. 203.

Poincaré's result can be obtained quite simply in this elementary case. For, put $v_n = c_{n+1}/c_n$ in (8) to get $\left(v_{n+1} - \dfrac{1}{k}\dfrac{n}{n+2}\right)v_n = \dfrac{K}{(n+2)(n+1)}$. Then only three cases are possible as $n \to \infty$. Either $\lim\limits_{n\to\infty} v_n = 0$ or $\lim\limits_{n\to\infty} v_n = \dfrac{1}{k}$, or $\overline{\lim\limits_{n\to\infty}}\, v_n = \dfrac{1}{k}$ and $\underline{\lim\limits_{n\to\infty}}\, v_n = 0$.

In all these cases the radius of convergence of $\sum_0^\infty c_n t^n$ is at least k.