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A NOTE ON DERIVATIONS

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In this note we prove two results about derivations. The first result holds in any ring, whereas the second one is valid in prime rings. From the nature of the results one would expect them to be known, but we have not been able to locate them in the literature.

We begin with

THEOREM 1. *Let R be any ring, d a derivation of R such that $d^3 \neq 0$. Then A, the subring generated by all $d(r)$, $r \in R$, contains a non-zero ideal of R.*

Proof. Since $d^3 \neq 0$ and $d(R) \subset A$, $d^2(A) \neq 0$. Pick $y \in A$ such that $d^2(y) \neq 0$. If $x \in R$ then $A \ni d(xy) = d(x)y + xd(y)$, and since both y and $d(x)$ are in A we end up with $xd(y) \in A$, which is to say, $Rd(y) \subset A$. Similarly, $d(y)R \subset A$.

If $r, s \in R$ then $A \ni d\ (rd(y)s) = d(r)d(y)s + rd^2(y)s + rd(y)d(s)$. But by the above $d(y)s \in A$ and $rd(y) \in A$; this all boils down to $rd^2(y)s \in A$ for all r, $s \in R$. Since, by the above, $Rd^2(y) \subset A$, $d^2(y)R \subset A$ and $Rd^2(y)R \subset A$ we have that the ideal of R generated by $d^2(y) \neq 0$ must be in A. This proves the theorem.

If $d^3 = 0$ the result need not be true. Let R be any prime ring having nilpotent elements, and let $a \neq 0 \in R$ be such that $a^2 = 0$. Let d: $R \to R$ be defined by $d(x) = ax - xa$. Then $B = aR + RA$ is a subring of R (since $a^2 = 0$) and contains $d(R)$. Also, $d^3 = 0$, $d^2 \neq 0$ (if char $R \neq 2$). Yet B contains no non-zero ideal of R, for $aBa = 0$.

The next theorem, although somewhat special, is of some independent interest. It does not depend on Theorem 1.

THEOREM 2. *Let R be a prime ring, $d \neq 0$ a derivation of R such that $d(x)d(y) = d(y)d(x)$ for all x, $y \in R$. Then, if char $R \neq 2$, R is a commutative integral domain, and if char $R = 2$, R is commutative or is an order in a simple algebra which is 4-dimensional over its center.*

Proof. Let A be the subring of R generated by all $d(x)$, for $x \in R$. By our hypothesis, A is a commutative subring of R.

If $a \in A$ and $x \in R$ then $d(a)x + ad(x) = d(ax) \in A$, hence centralizes A. So, if $b \in A$, $0 = bd(ax) - d(ax)b = d(a)(bx - xb)$. If $A \not\subset Z$, the center of R, we

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must have $d(a)=0$, for the annihilator of all $bx=xb$, $x \in R$, is an ideal of R by Lemma 1.1.7 of [1].

Suppose, for the moment, that $A \not\subset Z$. By the above, $d(A)=0$, hence $d^2(R) \subset d(A)=0$, that is, $d^2(x)=0$ for all $x \in R$. Using Leibniz' rule we obtain $2d(x)d(y)=0$, so, if char $R \neq 2$, $d(x)d(y)=0$. Using $y=zx$ leads us to $d(x)Rd(x)=0$ and so $d(x)=0$ for all $x \in R$. This contradicts $d \neq 0$. Hence, if $A \not\subset Z$ we must have that char $R=2$.

Let $T=\{x \in R \mid d(x)=0\}$; by the above, $T \supset A$. If $t \in T$, $x \in R$ then $A \ni d(tx)=d(t)x+td(x)=td(x)$, hence $td(x)$ centralizes A. If $a \in A$ then a commutes with $d(x) \in A$ and $td(x)$, therefore $(at-ta)d(x)=0$ for all $x \in R$. Since R is prime this gives us that $at-ta=0$, hence A centralizes T.

T is clearly a subring of R. Moreover, it is a Lie ideal of R. For, if $t \in T$, $x \in R$ then $d(tx-xt)=td(x)-d(x)t$ (since $d(t)=0$) $=0$ since $d(x) \in A$ centralizes T. Therefore $tx-xt \in T$.

Since R is prime and T is a subring of R, and a Lie ideal of R, by Theorem 2.1.2 of [1], either T contains a non-zero ideal of R or T is commutative and $t^2 \in Z$ for all $t \in T$. If T contains a non-zero ideal of R, A must centralize this ideal since A centralizes all of T; in a prime ring this forces $A \subset Z$, contrary to supposition. Hence we have that T is commutative and $t^2 \in Z$ for all $t \in T$.

Let $a \in A$, $a \notin Z$ then $a \in T$ and $ax-xa \in T$; thus $a^2 \in Z$ and $(ax-xq)^2 \in Z$ for all $x \in R$. It follows easily from Theorem 1.4 of [2] that R must be an order in a simple algebra which is 4-dimensional over its center. Hence, if $A \not\subset Z$, the theorem is proved.

Suppose now that $A \subset Z$. Thus $d(x) \in Z$ for all $x \in R$. Hence, if $x, y \in R$, $Z \ni d(xy)=xd(y)+d(x)y$. Commuting this with x, and using $d(x)$, $d(y)$ in Z, we obtain $d(x)(xy-yx)=0$. However, if $d(x) \neq 0$ since it is in Z it is not a zero divisor. Thus we have, in that case, that $xy=yx$ for all $y \in R$. In short, if $d(x) \neq 0$ then $x \in Z$. If $d(x)=0$, since $d(R) \neq 0$, pick x_0 such that $d(x_0) \neq 0$; then $x_0 \in Z$ by the above. Also, $d(x+x_0)=d(x_0) \neq 0$, hence $x+x_0 \in Z$. This leaves us with $x \in Z$. In other words, if $A \subset Z$ then R is commutative; since R is prime it must be an integral domain. The theorem is now completely proved.

One might wonder if one could generalize the above result to more general polynomial identities. In particular, if $s_k(x_1, \ldots, x_k)$ is the standard identity of degree k, and if $d \neq 0$ is a derivation of a prime ring R such that $s_k(d(x_1), \ldots, d(x_k))=0$ for all $x_1, \ldots, x_k \in R$ can we conclude that R must be rather special or must satisfy s_k?

References

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