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A Combinatorial Reciprocity Theorem for Hyperplane Arrangements

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Abstract. Given a nonnegative integer m and a finite collection $\mathcal{A}$ of linear forms on $\mathbb{Q}^d$, the arrangement of affine hyperplanes in $\mathbb{Q}^d$ defined by the equations $\alpha(x) = k$ for $\alpha \in \mathcal{A}$ and integers $k \in [-m, m]$ is denoted by $\mathcal{A}^m$. It is proved that the coefficients of the characteristic polynomial of $\mathcal{A}^m$ are quasi-polynomials in m and that they satisfy a simple combinatorial reciprocity law.

1 Introduction

Let V be a d-dimensional vector space over the field $\mathbb{Q}$ of rational numbers and $\mathcal{A}$ be a finite collection of linear forms on V which spans the dual vector space V^*. We denote by $\mathcal{A}^m$ the essential arrangement of affine hyperplanes in V defined by the equations $\alpha(x) = k$ for $\alpha \in \mathcal{A}$ and integers $k \in [-m, m]$ (we refer to [9, 13] for background on hyperplane arrangements). Thus $\mathcal{A}^0$ consists of the linear hyperplanes which are the kernels of the forms in $\mathcal{A}$ and $\mathcal{A}^m$ is a deformation of $\mathcal{A}^0$, in the sense of [1, 10].

The characteristic polynomial [9, Section 2.3] [13, Section 1.3] of $\mathcal{A}^m$, denoted $\chi_{\mathcal{A}}(q, m)$, is a fundamental combinatorial and topological invariant which can be expressed as

$$\chi_{\mathcal{A}}(q, m) = \sum_{i=0}^{d} c_i(m) q^i. \tag{1.1}$$

The coefficient $c_i(m)$ is equal to the sum of the values $\mu(y)$ of the Möbius function on the intersection poset of $\mathcal{A}^m$ (see Subsection 1.1 for definitions), taken over all elements y in this poset of dimension i. Alternatively, $(-1)^{d-i} c_i(m)$ can be defined as the rank of the $(d - i)$-th singular cohomology group of the complement of the union of the complexified hyperplanes of $\mathcal{A}^m$ in the d-dimensional complex vector space $V \otimes_{\mathbb{Q}} \mathbb{C}$ (see [8]).

We will be concerned with the behavior of $\chi_{\mathcal{A}}(q, m)$ as a function of m. Let $\mathbb{N} := \{0, 1, \dots\}$ and recall that a function $f \colon \mathbb{N} \to \mathbb{R}$ is called a *quasi-polynomial* with period N if there exist polynomials $f_1, f_2, \dots, f_N \colon \mathbb{N} \to \mathbb{R}$ such that $f(m) = f_i(m)$ for all $m \in \mathbb{N}$ with $m \equiv i \pmod N$. The degree of f is the maximum of the degrees of the f_i. Our main result is the following theorem.

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Theorem 1.1 *Under the previous assumptions on $\mathcal{A}$, the coefficient $c_i(m)$ of q^i in $\chi_{\mathcal{A}}(q,m)$ is a quasi-polynomial in m of degree at most $d-i$. Moreover, the degree of $c_0(m)$ is equal to d and*

$$\chi_{\mathcal{A}}(q,-m) = (-1)^d \chi_{\mathcal{A}}(-q, m-1). \tag{1.2}$$

In particular we have $\chi_{\mathcal{A}}(q,-1) = (-1)^d \chi_{\mathcal{A}}(-q)$, where $\chi_{\mathcal{A}}(q)$ is the characteristic polynomial of $\mathcal{A}^0$. Let $\mathcal{A}^m_{\mathbb{R}}$ denote the arrangement of affine hyperplanes in the real d-dimensional vector space $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ defined by the same equations defining the hyperplanes of $\mathcal{A}^m$. Let $r_{\mathcal{A}}(m) = (-1)^d \chi_{\mathcal{A}}(-1,m)$ and $b_{\mathcal{A}}(m) = (-1)^d \chi_{\mathcal{A}}(1,m)$ so that, for $m \in \mathbb{N}$, $r_{\mathcal{A}}(m)$ and $b_{\mathcal{A}}(m)$ count the number of regions and bounded regions, respectively, into which $V_{\mathbb{R}}$ is dissected by the hyperplanes of $\mathcal{A}^m_{\mathbb{R}}$ [13, Section 2.2] [14].

Corollary 1.2 *Under the previous assumptions on $\mathcal{A}$, the function $r_{\mathcal{A}}(m)$ is a quasi-polynomial in m of degree d, and for all positive integers m, $(-1)^d r_{\mathcal{A}}(-m)$ is equal to the number $b_{\mathcal{A}}(m-1)$ of bounded regions of $\mathcal{A}^{m-1}_{\mathbb{R}}$.*

Theorem 1.1 and its corollary belong to a family of results demonstrating some kind of combinatorial reciprocity law; see [11] for a systematic treatment of such phenomena. Not surprisingly, the proof given in Section 2 is a simple application of the main results of Ehrhart theory [12, Section 4.6]. More specifically, equation (1.2) will follow from the reciprocity theorem [12, Theorem 4.6.26] for the Ehrhart quasi-polynomial of a rational polytope. An expression for the coefficient of the leading term m^d of either $c_0(m)$ or $r_{\mathcal{A}}(m)$ is also derived in that section. Some examples, including the motivating example in which $\mathcal{A}^0_{\mathbb{R}}$ is the arrangement of reflecting hyperplanes of a Weyl group, and remarks are discussed in Section 3. In the remainder of this section we give some background on characteristic and Ehrhart (quasi-)polynomials needed in Section 2. We will denote by $\#S$ or $|S|$ the cardinality of a finite set S.

1.1 Arrangements of Hyperplanes

Let V be a d-dimensional vector space over a field $\mathbb{K}$. An *arrangement of hyperplanes* in V is a finite collection $\mathcal{H}$ of affine subspaces of V of codimension one (we will allow this collection to be a multiset). The *intersection poset* of $\mathcal{H}$ is the set $L_{\mathcal{H}} = \{\bigcap \mathcal{F} \colon \mathcal{F} \subseteq \mathcal{H}\}$ of all intersections of subcollections of $\mathcal{H}$, partially ordered by reverse inclusion. It has a unique minimal element $\widehat{0} = V$, corresponding to the subcollection $\mathcal{F} = \varnothing$. The *characteristic polynomial* of $\mathcal{H}$ is defined by

$$\chi_{\mathcal{H}}(q) = \sum_{x \in L_{\mathcal{H}}} \mu(x) q^{\dim x},$$

where μ stands for the Möbius function on $L_{\mathcal{H}}$ defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \widehat{0}, \\ -\sum_{y < x} \mu(y) & \text{otherwise.} \end{cases}$$

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Equivalently [9, Lemma 2.55] we have

$$\chi_{\mathcal{H}}(q) = \sum_{\mathcal{G} \subseteq \mathcal{H}} (-1)^{\#\mathcal{G}} q^{\dim(\cap \mathcal{G})}, \tag{1.3}$$

where the sum is over all $\mathcal{G} \subseteq \mathcal{H}$ with $\bigcap \mathcal{G} \neq \varnothing$.

In the case $\mathbb{K} = \mathbb{R}$, the connected components of the space obtained from V by removing the hyperplanes of $\mathcal{H}$ are called *regions* of $\mathcal{H}$. A region is *bounded* if it is a bounded subset of V with respect to a usual Euclidean metric.

1.2 Ehrhart Quasi-Polynomials

A convex polytope $P \subseteq \mathbb{R}^n$ is said to be a *rational* or *integral* polytope if all its vertices have rational or integral coordinates, respectively. If P is rational and P° is its relative interior, then the functions defined for nonnegative integers m by the formulas

$$i(P, m) = \#(mP \cap \mathbb{Z}^n), \quad \bar{i}(P, m) = \#(mP^\circ \cap \mathbb{Z}^n)$$

are quasi-polynomials in m of degree $d = \dim(P)$, related by the Ehrhart reciprocity theorem [12, Theorem 4.6.26]

$$i(P, -m) = (-1)^d \bar{i}(P, m). \tag{1.4}$$

The function $i(P, m)$ is called the *Ehrhart quasi-polynomial* of P. The coefficient of the leading term m^d in either $i(P, m)$ or $\bar{i}(P, m)$ is a constant equal to the normalized d-dimensional volume of P (meaning the d-dimensional volume of P normalized with respect to the lattice $V_P \cap \mathbb{Z}^n$, where V_P is the parallel translate of the affine span of P in $\mathbb{R}^n$ through the origin). If P is an integral polytope then $i(P, m)$ is a polynomial in m of degree d, called the *Ehrhart polynomial* of P.

2 Proof of Theorem 1.1

In this section we prove Theorem 1.1 and Corollary 1.2 and derive a formula for the coefficient of the leading term m^d of $r_{\mathcal{A}}(m)$. In what follows $\mathcal{A}$ is as in the beginning of Section 1. We use the notation $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ and $[a, b]_{\mathbb{Z}} = [a, b] \cap \mathbb{Z}$ for $a, b \in \mathbb{Z}$ with $a \leq b$.

Proof of Theorem 1.1 and Corollary 1.2 Using formula (1.3), we get

$$\chi_{\mathcal{A}^m}(q) = \sum_{\mathcal{G} \subseteq \mathcal{A}^m} (-1)^{\#\mathcal{G}} q^{\dim(\bigcap \mathcal{G})}, \tag{2.1}$$

where the sum is over all $\mathcal{G} \subseteq \mathcal{A}^m$ with $\bigcap \mathcal{G} \neq \varnothing$. Clearly for this to happen $\mathcal{G}$ must contain at most one hyperplane of the form $\alpha(x) = k$ for each $\alpha \in \mathcal{A}$. In other words we must have $\mathcal{G} = \mathcal{F}_b$ for some $\mathcal{F} \subseteq \mathcal{A}$ and map $b \colon \mathcal{F} \to [-m, m]_{\mathbb{Z}}$ sending α to b_α, where $\mathcal{F}_b$ consists of the hyperplanes $\alpha(x) = b_\alpha$ for $\alpha \in \mathcal{F}$. Let us denote by $\dim \mathcal{F}$

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the dimension of the linear span of $\mathcal{F}$ in V^* and observe that $\dim(\bigcap \mathcal{F}_b) = d - \dim \mathcal{F}$ whenever $\bigcap \mathcal{F}_b$ is nonempty. From the previous observations and (2.1) we get

$$\begin{aligned}\chi_{\mathcal{A}}(q,m) &= \sum_{\mathcal{F}\subseteq\mathcal{A}} \sum_{\substack{b\colon \mathcal{F}\to[-m,m]_{\mathbb{Z}} \\ \bigcap \mathcal{F}_b \neq \varnothing}} (-1)^{\#\mathcal{F}_b} q^{\dim(\bigcap \mathcal{F}_b)} \\ &= \sum_{\mathcal{F}\subseteq\mathcal{A}} (-1)^{\#\mathcal{F}} q^{d-\dim \mathcal{F}} \#\{b\colon \mathcal{F}\to[-m,m]_{\mathbb{Z}}, \bigcap \mathcal{F}_b \neq \varnothing\}.\end{aligned}$$

Let us write $\mathcal{F} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $b_i = b_{\alpha_i}$, so that b can be identified with a column vector in $\mathbb{Q}^n$. Then $\bigcap \mathcal{F}_b$ is nonempty if and only if the linear system $\alpha_i(x) = b_i$, $1 \leq i \leq n$, has a solution in $\mathbb{Q}^d$ or, equivalently, if and only if b lies in the image $\operatorname{Im} T_{\mathcal{F}}$ of the linear transformation $T_{\mathcal{F}}\colon \mathbb{Q}^d \to \mathbb{Q}^n$ mapping $x \in \mathbb{Q}^d$ to the column vector in $\mathbb{Q}^n$ with coordinates $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x)$. It follows that

$$\begin{aligned}\#\{b\colon \mathcal{F}\to[-m,m]_{\mathbb{Z}}, \cap\mathcal{F}_b \neq \varnothing\} &= \#\operatorname{Im} T_{\mathcal{F}} \cap ([-m,m]_{\mathbb{Z}})^n \\ &= \#\operatorname{Im} T_{\mathcal{F}} \cap [-m,m]^n \cap \mathbb{Z}^n \\ &= \#(m\,(\operatorname{Im} T_{\mathcal{F}} \cap [-1,1]^n) \cap \mathbb{Z}^n) \\ &= \#(mP_{\mathcal{F}} \cap \mathbb{Z}^n),\end{aligned}$$

where $P_{\mathcal{F}} = (\operatorname{Im} T_{\mathcal{F}} \otimes_{\mathbb{Q}} \mathbb{R}) \cap [-1,1]^n$. Clearly $P_{\mathcal{F}}$ is a rational convex polytope of dimension $\dim(\operatorname{Im} T_{\mathcal{F}}) = \dim \mathcal{F}$ and $\#\,(mP_{\mathcal{F}} \cap \mathbb{Z}^n) = i(P_{\mathcal{F}}, m)$ is the Ehrhart quasi-polynomial of $P_{\mathcal{F}}$. From the above we conclude that

$$\chi_{\mathcal{A}}(q,m) = \sum_{\mathcal{F}\subseteq\mathcal{A}} (-1)^{\#\mathcal{F}} q^{d-\dim \mathcal{F}} i(P_{\mathcal{F}}, m). \tag{2.2}$$

Equivalently we have

$$c_i(m) = \sum_{\substack{\mathcal{F}\subseteq\mathcal{A} \\ \dim \mathcal{F} = d-i}} (-1)^{\#\mathcal{F}} i(P_{\mathcal{F}}, m) \tag{2.3}$$

for $0 \leq i \leq d$, where the $c_i(m)$ are as in (1.1). Since $i(P_{\mathcal{F}}, m)$ is a quasi-polynomial in m of degree $\dim \mathcal{F}$, it follows from (2.3) that $c_i(m)$ is a quasi-polynomial in m of degree at most $d - i$ and that $r_{\mathcal{A}}(m) = \sum_{i=0}^{d} (-1)^{d-i} c_i(m)$ is a quasi-polynomial in m of degree at most d. Moreover we have $r_{\mathcal{A}}(m) \geq (2m+2)^d$ for $m \geq 0$ since $\mathcal{A}$ contains d linearly independent forms and the corresponding hyperplanes of $\mathcal{A}^m_{\mathbb{R}}$ dissect $V_{\mathbb{R}}$ into $(2m+2)^d$ regions. It follows that the degree of $r_{\mathcal{A}}(m)$ is no less than d, which implies that the degrees of $r_{\mathcal{A}}(m)$ and $c_0(m)$ are, in fact, equal to d.

It remains to prove the reciprocity relation (1.2). For $\mathcal{F} \subseteq \mathcal{A}$ with $\#\mathcal{F} = n$ let $W_{\mathcal{F}}$ be the real linear subspace $\operatorname{Im} T_{\mathcal{F}} \otimes_{\mathbb{Q}} \mathbb{R}$ of $\mathbb{R}^n$, so that $P_{\mathcal{F}} = W_{\mathcal{F}} \cap [-1,1]^n$. We have

$$\begin{aligned}mP^{\circ}_{\mathcal{F}} \cap \mathbb{Z}^n &= (W_{\mathcal{F}} \cap [-m,m]^n)^{\circ} \cap \mathbb{Z}^n \\ &= W_{\mathcal{F}} \cap [-(m-1), m-1]^n \cap \mathbb{Z}^n \\ &= (m-1)P_{\mathcal{F}} \cap \mathbb{Z}^n\end{aligned}$$

and hence $\bar{i}(P_{\mathcal{F}}, m) = i(P_{\mathcal{F}}, m-1)$. The Ehrhart reciprocity theorem (1.4) implies that

$$i(P_{\mathcal{F}}, -m) = (-1)^{\dim \mathcal{F}} i(P_{\mathcal{F}}, m-1). \tag{2.4}$$

Equation (1.2) follows from (2.2) and (2.4). ■

The following corollary is an immediate consequence of the case $i = 0$ of (2.3), the equation $r_{\mathcal{A}}(m) = \sum_{i=0}^{d} (-1)^{d-i} c_i(m)$, and the fact that the degree of $c_i(m)$ is less than d for $1 \le i \le d$.

Corollary 2.1 *The coefficient of the leading term m^d in $r_{\mathcal{A}}(m)$ is equal to the expression*

$$\sum_{\substack{\mathcal{F} \subseteq \mathcal{A} \\ \dim \mathcal{F} = d}} (-1)^{\#\mathcal{F} - d} \operatorname{vol}_d(P_{\mathcal{F}}),$$

where $P_{\mathcal{F}}$ is as in the proof of Theorem 1.1 *and* $\operatorname{vol}_d(P_{\mathcal{F}})$ *is the normalized d-dimensional volume of $P_{\mathcal{F}}$.* ■

The coefficient of the leading term m^d in $r_{\mathcal{A}}(m)$ can also be described as the limit $\lim_{m \to \infty} r_{\mathcal{A}}(m)/m^d$.

3 Examples and Remarks

In this section we list a few examples, questions, and remarks.

Example 3.1 If $V = \mathbb{Q}$ and $\mathcal{A}$ consists of two forms $\alpha_1, \alpha_2 \colon V \to \mathbb{Q}$ with $\alpha_1(x) = x$ and $\alpha_2(x) = 2x$ for $x \in V$, then $\mathcal{A}^m$ consists of the affine hyperplanes (points) in V defined by the equations $x = k$ and $x = k/2$ for $k \in [-m, m]_{\mathbb{Z}}$. One can check that

$$\chi_{\mathcal{A}}(q, m) = \begin{cases} q - 3m - 1, & \text{if } m \text{ is even} \\ q - 3m - 2, & \text{if } m \text{ is odd} \end{cases}$$

and that (1.2) holds. Moreover we have

$$r_{\mathcal{A}}(m) = \begin{cases} 3m + 2, & \text{if } m \text{ is even} \\ 3m + 3, & \text{if } m \text{ is odd.} \end{cases}$$

Note that $\operatorname{vol}_d(P_{\mathcal{F}})$ takes the values 2, 2, and 1 when $\mathcal{F} = \{\alpha_1\}, \{\alpha_2\}$ and $\{\alpha_1, \alpha_2\}$, respectively.

Example 3.2 If $V = \mathbb{Q}^d$ and $\mathcal{A}$ consists of the coordinate functions $\alpha_i(x) = x_i$ for $1 \le i \le d$, then $\mathcal{A}^m$ consists of the affine hyperplanes in V given by the equations $x_i = k$ with $1 \le i \le d$, $k \in [-m, m]_{\mathbb{Z}}$ and $\chi_{\mathcal{A}}(q, m) = (q - 2m - 1)^d$, which is a polynomial in q and m satisfying (1.2).

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Example 3.3 Let Φ be a finite, irreducible, crystallographic root system spanning the Euclidean space $\mathbb{R}^d$, endowed with the standard inner product $(\,\cdot\,,\,\cdot\,)$ (we refer to [4,5,7] for background on root systems). Fix a positive system Φ^+ and let Q_Φ and W be the coroot lattice and Weyl group, respectively, corresponding to Φ. Let also $\mathcal{A}_\Phi^m$ denote the m-th *generalized Catalan arrangement* associated to Φ [1,2,10], consisting of the affine hyperplanes in $\mathbb{R}^d$ defined by the equations $(\alpha, x) = k$ for $\alpha \in \Phi^+$ and $k \in [-m, m]_\mathbb{Z}$ (so that $\mathcal{A}_\Phi^0$ is the real reflection arrangement associated to Φ). If V is the $\mathbb{Q}$-span of Q_Φ then there exists a finite collection $\mathcal{A}$ of linear forms on V (one for each root in Φ^+) such that, in the notation of Section 1, $\mathcal{A}_\mathbb{R}^m$ coincides with $\mathcal{A}_\Phi^m$. In [2, Theorem 1.2] a uniform proof was given of the formula

$$\chi_\mathcal{A}(q, m) = \prod_{i=1}^{d} (q - mh - e_i) \tag{3.1}$$

for the characteristic polynomial of $\mathcal{A}_\Phi^m$, where $e_1, e_2, \ldots, e_d$ are the exponents and h is the Coxeter number of Φ. Thus the reciprocity law (1.2) in this case is equivalent to the well-known fact [5, Section V.6.2] [7, Lemma 3.16] that the numbers $h - e_i$ are a permutation of the e_i. As was already deduced in [2, Corollary 1.3], it follows from (3.1) that

$$r_\mathcal{A}(m) = \prod_{i=1}^{d} (mh + e_i + 1) \quad \text{and} \quad b_\mathcal{A}(m) = \prod_{i=1}^{d} (mh + e_i - 1)$$

are polynomials in m of degree d (a fact which was the main motivation behind this paper). Setting $N(\Phi, m) = \frac{1}{|W|}\, r_\mathcal{A}(m)$ and $N^+(\Phi, m) = \frac{1}{|W|}\, b_\mathcal{A}(m)$, as in [3,6], our Corollary 1.2 implies that

$$(-1)^d N(\Phi, -m) = N^+(\Phi, m - 1).$$

It was suggested in [6, Remark 12.5] that this equality, first observed in [6, (2.12)], may be an instance of Ehrhart reciprocity. This was confirmed in [3, Section 7] using an approach which is different from the one followed in this paper. Finally we note that Corollary 2.1 specializes to the curious identity

$$h^d = \sum_F (-1)^{\#F - d} \operatorname{vol}_d(P_F), \tag{3.2}$$

where in the sum on the right hand-side F runs through all subsets $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of Φ^+ spanning $\mathbb{R}^d$, P_F is the intersection of the cube $[-1, 1]^n$ with the image of the linear transformation $T_F \colon \mathbb{R}^d \to \mathbb{R}^n$ mapping $x \in \mathbb{R}^d$ to the column vector in $\mathbb{R}^n$ with coordinates $(\alpha_1, x), (\alpha_2, x), \ldots, (\alpha_n, x)$ and $\operatorname{vol}_d(P_F)$ is the normalized d-dimensional volume of P_F. If Φ has type A_d in the Cartan–Killing classification, then (3.2) translates to the equation

$$(d + 1)^d = \sum_G (-1)^{e(G) - d} \operatorname{vol}_d(Q_G),$$

where in the sum on the right hand-side G runs through all connected simple graphs on the vertex set $\{1, 2, \ldots, d + 1\}$, $e(G)$ is the number of edges of G, and Q_G is the

d-dimensional polytope in $\mathbb{R}^d$ defined in the following way. Let τ be a spanning tree of G with edges labelled in a one to one fashion with the variables $x_1, x_2, \dots, x_d$. For any edge e of G which is not an edge of T let R_e be the region of $\mathbb{R}^d$ defined by the inequalities $-1 \le x_{i_1} + x_{i_2} + \cdots + x_{i_k} \le 1$, where $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ are the labels of the edges (other than e) of the fundamental cycle of the graph obtained from T by adding the edge e. The polytope Q_G is the intersection of the cube $[-1, 1]^d$ and the regions R_e.

Remark 3.4. It is well known [13, Corollary 3.5] that the coefficients of the characteristic polynomial of a hyperplane arrangement strictly alternate in sign. As a consequence, in the notation of (1.1), we have $(-1)^{d-i} c_i(m) > 0$ for all $m \in \mathbb{N}$ and $0 \le i \le d$. We do not know of an example of a collection $\mathcal{A}$ of forms for a which a negative number appears among the coefficients of the quasi-polynomials $(-1)^{d-i} c_i(m)$.

Remark 3.5. If the matrix defined by the forms in $\mathcal{A}$ with respect to some basis of V is integral and totally unimodular, meaning that all its minors are $-1, 0$ or 1, then the polytopes $P_{\mathcal{F}}$ in the proof of Theorem 1.1 are integral and, as a consequence, the functions $c_i(m)$ and $r_{\mathcal{A}}(m)$ are polynomials in m. This assumption on $\mathcal{A}$ is satisfied in the case of graphical arrangements, that is, when $\mathcal{A}$ consists of the forms $x_i - x_j$ on $\mathbb{Q}^r$, where $1 \le i < j \le r$, corresponding to the edges $\{i, j\}$ of a simple graph G on the vertex set $\{1, 2, \dots, r\}$. The degree of the polynomial $r_G(m) := r_{\mathcal{A}}(m)$ is equal to the dimension of the linear span of $\mathcal{A}$, in other words to the rank of the cycle matroid of G.

Remark 3.6. Let $\mathcal{A}$ and $\mathcal{H}$ be finite collections of linear forms on a d-dimensional $\mathbb{Q}$-vector space V spanning V^*. Using the notation of Section 1, let $\mathcal{H}_m$ denote the union of $\mathcal{A}^m_{\mathbb{R}}$ with the linear arrangement $\mathcal{H}^0_{\mathbb{R}}$. It follows from Theorem 1.1, the Deletion-Restriction theorem [9, Theorem 2.56], and induction on the cardinality of $\mathcal{H}$ that the function $r(\mathcal{H}_m)$ is a quasi-polynomial in m of degree d. Given a region R of $\mathcal{H}^0_{\mathbb{R}}$, let $r_R(m)$ denote the number of regions of $\mathcal{H}_m$ which are contained in R, so that

$$r(\mathcal{H}_m) = \sum_R r_R(m)$$

where R runs through the set of all regions of $\mathcal{H}^0_{\mathbb{R}}$. Is the function $r_R(m)$ always a quasi-polynomial in m?

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