RADICALS OF POLYNOMIAL RINGS

S. A. AMITSUR

Introduction. Let R be a ring and let $R[x]$ be the ring of all polynomials in a commutative indeterminate x over R. Let $J(R)$ denote the Jacobson radical **(5)** of the ring R and let $L(R)$ be the lower radical **(4)** of R. The main object of the present note is to determine the radicals $J(R[x])$ and $L(R[x])$. The Jacobson radical $J(R[x])$ is shown to be a polynomial ring $N[x]$ over a nil ideal N of R and the lower radical $L(R[x])$ is the polynomial ring $L(R)[x]$. A partial result of the first case and a parallel solution to the second case have been obtained also independently and by different methods by N. H. McCoy (simultaneously with the author).

The present method of attacking these problems can be applied to many other radicals arising from π-properties **(1)** of rings. Let $\pi(R)$ denote the π-radical of a ring R. $J(R)$ is an example of a radical satisfying $\pi(R[x]) = P[x]$ where $P = \pi(R[x]) \cap R$, and $L(R)$ represents a class of radicals satisfying $\pi(R[x]) = \pi(R)[x]$. The results obtained can be easily extended to polynomials in any number of variables.

It is shown that $J(R[x]) = N[x]$ where N is a nil ideal in R. Snapper, who studied the Jacobson radical of polynomial rings over commutative rings R, has shown **(7)** that N is the *maximal* nil ideal in R. The extension of this result to arbitrary rings seems to be very difficult. Though we verify this fact for algebras over non-denumerable fields, the general problem of determining the ideal N remains open.

1. The Jacobson radical

LEMMA 1J: *Let $N = J(R[x]) \cap R$, then $J(R[x]) \neq 0$ implies $N \neq 0$.*

The proof of this Lemma, which is a keystone in the extension of the results on the Jacobson radical to arbitrary radicals, seems to be rather elementary if R is an algebra over an infinite field, or if R is of characteristic zero; but the proof is far more complicated in the general case.

Recall that the Jacobson radical is a radical of the type dealt in **(1)**. In particular, it follows by Corollary 1.1 of **(1)** that $J(R[x])$ remains invariant under the automorphisms of $R[x]$. For example, consider the automorphism[1]: $f(x) \rightarrow f(x+1)$ of $R[x]$, or more generally the automorphism: $f(x) \rightarrow f(x+\lambda)$, where λ is an endomorphism of the additive group of R satisfying $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for $a, b \in R$.

Put $J = J(R[x])$. If the Lemma is not true, then we have a case where $J \neq 0$ but $J \cap R = N = 0$. Let $f(x)$ be a non zero polynomial of minimum

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[1]One does not need to assume that R contains a unit.

degree belonging to J. By the previous remarks it follows also that $f(x+1) \in J$. Hence $f_0(x) = f(x+1) - f(x) \in J$ since the degree of $f_0(x)$ is less than that of $f(x)$. The minimality of the latter implies that $f_0(x) = 0$. Thus $f(x+1) = f(x)$.

If R is of characteristic zero, then one readily verifies that $f(x+1) = f(x)$ can hold only if $f(x) = a \in R$. Thus $0 \neq a \in J \cap R$ which is a contradiction. Another immediate contradiction is readily obtained if R is assumed to be an algebra over an infinite field F. Indeed, the preceding arguments can be applied as well to the automorphisms: $g(x) \rightarrow g(x + \lambda)$, $\lambda \in F$, of $R[x]$. This yields that $f(x+\lambda) = f(x)$ for all $\lambda \in F$. Since F is infinite, the last relation implies that $f(x) = a \in R$, and thus $0 \neq a \in N$ which contradicts the assumption $N = 0$.

In order to obtain a contradiction in the general case we have to make some additional remarks: Let p be a prime number and let R_p be the set of all elements of R which are of characteristic p. Note that R is an ideal in R and, therefore $R_p[x]$ is an ideal in $R[x]$. We may assume that $f(x) \in R_p[x]$. Indeed, let $f(x) = a_0x^n + \ldots + a_n$. Since $N = 0$ it follows that $n \geqslant 1$. Hence, since $f(x+1) - f(x) = na_0x^{n-1} + \ldots = 0$ we obtain that $na_0 = 0$. Let m be the minimal integer satisfying $ma_0 = 0$ and let p be a prime dividing m. Thus $(m/p)a_0 \neq 0$ and clearly $(m/p)a_0 \in R_0$. We may replace $f(x)$ by the polynomial $(m/p)f(x)$ which belongs also to J and which is of the same degree as $f(x)$. Namely, we suppose that the highest coefficient of $f(x)$ belongs to R_p. Now $pf(x) \in J$ and its degree is smaller than the degree of $f(x)$, hence the minimality of the latter yields $pf(x) = 0$, that is, $f(x) \in R_p[x]$.

Next we show that if a polynomial $g(x) \in R_p[x]$ satisfies $g(x+1) = g(x)$ then $g(x) = h(x^p - x)$ is a polynomial in $x^p - x$ with coefficients in R_p. The proof is carried out by induction on the degree of $g(x)$. First, let $g(x)$ be a polynomial of degree $k < p$. Since $g(x+1) = g(x)$ it follows that $g(x+\nu) = g(x)$ for all integers ν. Now,

$$g(x+\nu) = g(\nu) + xg_1(\nu) + \ldots + x^m g_m(\nu) = b_0 + xb_1 + \ldots + x^m b_m = g(x).$$

Hence, $g(\nu) = b_0$ for all integers ν. Clearly, R_p is an algebra over the finite field $GF[p]$ of p elements. Thus we obtained that $g(x) - b_0$ vanishes for p elements of the field $GF[p]$. Since the degree of $g(x) - b_0$ is less than p, it follows that $g(x) - b_0 = 0$, that is, $g(x) = b_0 \in R_p$.

Let $g(x)$ be a polynomial of arbitrary degree,[2] then $g(x) = h(x)(x^p - x) + k(x)$, with the degree of $k(x) < p$. Hence, $g(x+1) = h(x+1)(x^p - x) + k(x+1)$ and since $g(x+1) = g(x)$ we obtain

$$[h(x+1) - h(x)](x^p - x) = k(x) - k(x+1).$$

The degree of the right-hand side of the last equality is less than p and the degree of the left-hand side, if not zero, is $\geqslant p$. It follows, therefore, that

[2]This holds in the ring $R^*{}_p[x]$, where R^* is obtained by adjoining a unit to R, but, clearly, $g(x)$ and $k(x)$ belong to $R_p[x]$.

$k(x) = k(x + 1)$ and $h(x + 1) = h(x)$. By the previous case we know that $k(x) = k_0 \in R_p$ and by induction it follows that $h(x) = h_0(x^p - x)$. Thus $g(x) = h_0(x^p - x)(x^p - x) + k_0$ is a polynomial in $R_p[x^p - x]$.

The last preparatory remark we need before completing the proof of the Lemma is to the effect that if a polynomial $h(x^p - x)$ belongs to the Jacobson radical of $R_p[x]$, then it belongs also to the Jacobson radical of $R_p[x^p - x]$. Indeed, let $k(x) \in h(x^p - x)\, R_p[x^p - x]$, then $k(x + 1) = k(x)$. Clearly $k(x)$ belongs to the Jacobson radical of $R_p[x]$ hence its quasi-inverse $k'(x)$ is uniquely determined. The quasi-inverse of $k(x + 1)$ is readily seen to be $k'(x + 1)$; hence $k(x) = k(x + 1)$ implies that $k'(x + 1) = k'(x)$. Consequently $k'(x) \in R_p[x^p - x]$. This proves that the right ideal $h(x^p - x)$ $R_p[x^p - x]$ is quasi-regular in $R_p[x^p - x]$. Thus $h(x^p - x)$ belongs to the Jacobson radical of $R_p[x^p - x]$.

We turn now to the proof of the Lemma. Since $f(x) \in J \cap R_p[x]$ and $R_p[x]$ is an ideal in $R[x]$, it follows that $f(x) \in J(R_p[x]) = R_p[x] \cap J$. It was shown that $f(x + 1) = f(x)$, hence $f(x) \in R_p[x^p - x]$. Thus by the previous remarks, it follows that $f(x) = g(x^p - x)$ and $g(x^p - x)$ belongs to the Jacobson radical $J(R_p[x^p - x])$. The mapping $h(x) \rightarrow h(x^p - x)$ determines an automorphism between $R_p[x]$ and $R_p[x^p - x]$. It follows now, by Theorem 1.7 of **(1)**, that $J(R_p[x])$ is the image of $J(R_p[x^p - x])$. In particular, it follows that since $g(x^p - x) = f(x) \in J(R_p[x^p - x])$, $g(x) \in J(R_p[x])$. But $g(x)$ is of lower degree[3] than $f(x)$; hence, $g(x) \in J(R_p[x]) = J \cap R_p[x]$ implies that $g(x) \in J$, which contradicts the minimality of $f(x)$. This completes the proof of the Lemma.

LEMMA 2J: $J(R[x]) = N[x]$, *where* $N = J(R[x]) \cap R$.

Indeed, since $N \subseteq J = J(R[x])$, it follows that $N[x]\, R[x] \subseteq NR[x] \subseteq J$. Hence $N[x] \subseteq J$. Consider the homomorphism: $R[x] \rightarrow R[x]/N[x]$. The kernel of this homomorphism is $N[x] \subseteq J$. It follows, therefore, by Theorem 1.7 of **(1)** that $J(R[x]/N[x]) = J/N[x]$. Let $\bar{R} = R/N$, then $R[x]/N[x] \cong \bar{R}[x]$. Now:

$$\begin{aligned} J(\bar{R}[x]) \cap \bar{R} &\cong J/N[x] \cap (R, N[x])/N[x] = (J \cap (R, N[x]))/N[x] \\ &= (J \cap R, N[x])/N[x] = (N, N[x])/N[x] = \bar{0} \end{aligned}$$

since $J \supseteq N[x]$. Hence, Lemma 1 implies that $\bar{0} = J(\bar{R}[x]) = J/N[x]$. Consequently, $J = N[x]$, as required.

It remains now to determine the structure of the ideal N.

LEMMA 3J: *N is a nil ideal in R.*

Clearly, N is an ideal in R. Let $r \in N \subseteq J$, then $r.rx = r^2x \in J$. Let $q(x)$ be the quasi-inverse of r^2x, that is, $q(x) + r^2x + q(x)r^2x = 0$. In other words

$$q(x) = -r^2x - q(x)r^2x.$$

[3]This is true since the degree of $f(x)$ is $\geqslant 1$.

Substitute $q(x)$ on the right by the whole expression of the right-hand side of this equality. Repeating this process yields

$$q(x) = -r^2x + (r^2x)^3 + \ldots + (-1)^n(r^2x)^n + (-1)^{n+1}(r^2x)^{n+1} + (-1)^{n+1}q(x)(r^2x)^{n+1}.$$

Choose n to be greater than the degree of $q(x)$. Equating the coefficient of x^n on both sides yields that $r^{2n} = 0$. This proves that N is a nil ideal.

Levitzki's locally nilpotent radical $s\sigma(R)$ of a ring R is defined **(2**, p. 130**)** as the maximal ideal of R with the property that its finitely generated subrings are nilpotent. One readily observes that the polynomial ring $s\sigma(R)[x]$ is a nil ideal and, therefore, it also is quasi-regular. Consequently $s\sigma(R)[x] \subseteq J$, and thus $s\sigma(R) \subseteq N$. Summarizing the results obtained, we have

THEOREM 1. *$J(R[x]) = N[x]$ where $N = J(R[x]) \cap R$ is a nil ideal containing the locally nilpotent radical $s\sigma(R)$ of R.*

Remark. If R is commutative, or more generally satisfies a polynomial identity, then it is known **(6)** that the nil ideals of R are locally nilpoint ideals. Thus in this case $N \subseteq s\sigma(R)$, and therefore, $J(R[x])$ is a nil ideal and N is the maximal nil ideal of R.

We restrict ourselves now to the case where R is an algebra over an infinite field F. An ideal I in an algebra R is called an LBI-ideal **(3)** if I is a nil ideal and every finitely generated *submodule* of I is of bounded index. One readily observes that if $f(x) \in I[x]$, where I is an LBI-ideal, then $f(x)$ is nilpotent and its index is bounded by the index of the submodule of I generated by the coefficient of $f(x)$. Thus, $I[x] \subseteq J$. The maximal LBI-ideal, $LBI(R)$, of R is known **(3)** as the LBI-radical of R. Hence, the preceding arguments yield, in view of the fact that $LBI(R) \supseteq s\sigma(R)$, that:

COROLLARY. *$N \supseteq LBI(R) \supseteq s\sigma(R)$:*

It was shown in **(3)** that if R is an algebra over a *non denumerable* field F, then every nil ideal in R is an LBI-ideal. Consequently, for such algebras $LBI(R) \supseteq N$, which in particular implies that $N[x]$ is a nil ideal. Since the nil ideals are quasi-regular, it follows that:

THEOREM 2. *If R is an algebra over a non denumerable field F, then the Jacobson radical $J(R[x]) = N[x]$ is the maximal nil ideal of $R[x]$, and N is the maximal nil ideal of R.*

One conjectures that in all cases $J(R[x])$ is the maximal nil ideal of R. This would follow immediately if one could supply a positive answer to the still-open problem of Levitzki which requires to show that every nil ring is locally nilpotent, since in that case $N = s\sigma(R)$ will hold for every ring.

2. The lower radical. Let $L(R)$ denote the lower radical of the ring R. From the results of **(2**, Corollary 2.2**)**, we know that the lower radical arises from a property L of rings. Recall that a ring R is an L-ring if every non zero homomorphic image of R contains non zero nilpotent ideals. The property L satisfies the same requirements of **(1)** as the property of quasi-regularity: namely, $L(=\sigma^*$ in the notations of **(1)**) is an SRZ-property of rings. We have also in this case:

LEMMA 1L: *Let* $L = (R[x]) \cap R$; *then* $L(R[x]) \neq 0$ *implies* $L \neq 0$.

LEMMA 2L: $L(R[x]) = L[x]$.

The proof of the two Lemmas follows in parallel lines the proof of Lemma 1J and Lemma 2J, except that at one place in the proof of Lemma 1J we have used the definition of quasi-regularity and not the general requirements of an SR property. The proof of this point for the lower radical is what remains to complete the proof of the present two Lemmas. That is: we have only to show that "if $f(x) \in R_p[x^p - x]$ belongs to the lower radical of $R_p[x]$, then it belongs also to the lower radical of $R_p[x^p - x]$." Indeed, the ideal generated by $f(x)$ in $R_p[x^p - x]$ is a subring of the ideal generated by $f(x)$ in $R_p[x]$. The latter is an L-ideal, since $f(x) \in L(R_p[x])$. By Corollary 2.2 of **(2)**, it follows that subrings of L-rings are L-rings; hence the ideal generated by $f(x)$ in $R_p[x^p - x]$ is also an L-ideal. Consequently, $f(x) \in L(R_p[x^p - x])$. This completes the proof of Lemma 1L and, therefore, also of Lemma 2L.

In parallel to Lemma 3J, one has to characterize the ideal L. In the present case we can show that $L = L(R)$.

THEOREM 3. $L(R[x]) = L(R)[x]$.

Indeed, since $L = L(R[x]) \cap R \subseteq L(R[x])$ and L is an ideal in R, it follows by Corollary 2.2 of **(2)** that L is an L-ideal in R. Hence $L \subseteq L(R)$. The converse $L(R) \subseteq L$ will follow immediately from the following:

LEMMA 3L. *If S is an L-ring then $S[x]$ is also an L-ring.*

Indeed, let $S[x] \to \overline{S[x]}$ be a homomorphism of $S[x]$ onto a ring $\overline{S[x]}$. This homomorphism induces a homomorphism of S onto a ring $\bar{S} \subseteq \overline{S[x]}$. If $\bar{x}$ denotes the image of x, then clearly $\overline{S[x]} = \bar{S}[\bar{x}]$. Thus if $\overline{S[x]} \neq 0$ then $\bar{S} \neq 0$. Since S is an L-ring, $\bar{S}$ contains a non zero nilpotent ideal $\bar{Q}$. Consequently, $\bar{Q}[\bar{x}]$ is a nilpotent ideal of $\overline{S[x]}$, which proves that $S[x]$ is also a ring.

To complete the proof of Theorem 3, we note that Lemma 3L implies that $L(R)[x]$ is an L-ideal. Hence $L(R)[x] \subseteq L(R[x]) = L[x]$. Thus $L(R) \subseteq L$.

3. Infinite sets of indeterminates. Let $R[x_\alpha]$ be the ring of all polynomials in a set of α indeterminates $\{x_i\}$ where α is any cardinal number. A simple induction procedure, or a proof similar to that of Lemma 2J, yields

THEOREM 4 (a). $J(R[x_\alpha]) = N_\alpha[x_\alpha]$ *where* $N_\alpha = J(R[x_\alpha]) \cap R$ *is a nil ideal and* $N_\beta \supseteq N_\alpha \supseteq s\sigma(R)$ *for all* $\beta \leqslant \alpha$.

(b) $L(R[x_\alpha]) = L(R)[x_\alpha]$.

Furthermore, we have

THEOREM 5. *Let α be an infinite cardinal, then $J(R[x_\alpha]) = N_\alpha[x_\alpha]$ is the maximal nil ideal of $R[x_\alpha]$ and $N_\alpha = N_\beta$ for all $\beta \geqslant \alpha$. If R is an algebra over an infinite field, then $N_\alpha = LBI(R)$.*

Let x_1 be an indeterminate of the set $\{x_i\}$ and let $\{x_i'\}$ denote the rest of the indeterminates. Since α is not finite, the sets $\{x_i\}$ and $\{x_i'\}$ have the same cardinal number. Hence $J(R[x_i']) = N_\alpha[x_i]$ and $J(R[x_i]) = N_\alpha[x_i]$. Clearly $R[x_i] = R'[x_1]$ where $R' = R[x_i']$. It follows now by Theorem 1, that

$$J(R'[x_1]) = N'[x_1], \quad N' = R' \cap J(R'[x_1]).$$

Since $J(R'[x_1]) = J(R[x_i]) = N_\alpha[x_i]$, it follows that $N' = N_\alpha[x_i']$. By Theorem 1 it follows that N' is a nil ideal. Since $\{x_i\}$ and $\{x_i'\}$ are of the same cardinal number, one obtains $N_\alpha[x_i'] \cong N_\alpha[x_i]$. Consequently, $N_\alpha[x_i]$ is a nil ideal; thus $J(R[x_\alpha])$ is a nil ideal and, therefore, it is the maximal nil ideal of $R[x_\alpha]$.

Let $\{x_i\}$ be a set of indeterminates of cardinality $\alpha \geqslant N_0$ and let $\{y_j\}$ be a finite set of new indeterminates. Since the cardinality of the set $\{x_i\}$ and $\{x_i, y_j\}$ is α, we have $J(R[x_i, y_j]) = N_\alpha[x_i, y_j]$, $J(R[x_i]) = N_\alpha[x_i]$ where $N_\alpha = R \cap J(R[x_i]) = R \cap J(R[x_i, y_j])$. By the preceding proof it follows that $N_\alpha[x_i, y_j]$ is a nil ideal. Hence, the ring of all polynomials over $N_\alpha[x_i]$ in any number (finite or non finite) of indeterminates is a nil ring. Clearly, the non finite case can be reduced to the finite case which has just been proved.

Now let $\beta \geqslant \alpha$ and let $\{x_i\}$ be a set of indeterminates of cardinality α and $\{x_i, z_j\}$ a set of indeterminates of cardinality β. By the previous remark it follows that $N_\alpha[x_i, z_j]$ is a nil ideal, hence $N_\alpha[x_i, z_j] \subseteq J(R[x_i, z_j])$. On the other hand $J(R[x_i, z_j]) = N_\beta[x_i, z_j]$; hence, $N_\alpha \subseteq N_\beta$. Since $\beta \geqslant \alpha$, it follows by Theorem 4 that $N_\beta \subseteq N_\alpha$. Thus $N_\beta = N_\alpha$.

Let R be an algebra over an infinite field F and let $a_1, \ldots, a_n$ be a finite set of elements of N_0. Since α is an infinite ordinal, we have a finite set of indeterminates $x_1, \ldots, x_n \in \{x_i\}$ and thus, $a_1x_1 + \ldots + a_nx_n \in N_\alpha[x_\alpha]$. It follows by the previous result that $(a_1x_1 + \ldots + a_nx_n)^m = 0$ for some integer m. This immediately implies that the module generated by the set $(a_1, \ldots, a_n)$ contains nil elements of index $\leqslant m$. Consequently, $N_\alpha \subseteq LBI(R)$, and the fact that $N_\alpha \supseteq LBI(R)$ completes the proof of the theorem.

4. π-radicals. We follow in this section the notation of **(1)** and **(2)**.

The similarity between the proofs of Lemma 1J, 2J and Lemmas 1L, 2L exhibits the generality of the methods used. The only place where the explicit definitions of the quasi-regularity and the L-property were involved was in proving that if $f(x) \in R_p[x^p - x]$ belongs to the radical considered of $R_p[x]$, then it belongs also to the same type of radical of $R_p[x^p - x]$. The proof of this fact for the L-property uses only the fact that a subring of an L-ring is an L-ring. This condition for arbitrary properties π was denoted in **(1)** as (D_s). Thus we have:

LEMMA 4. *If π is an RZ-property satisfying (D_s) then* $\pi(R_p[x]) \cap R_p[x^p - x] \subseteq \pi(R_p[x^p - x])$.

The method used in proving Lemmas 1L, 2L and Lemmas 1J, 2J, yields also

THEOREM 1π. *If π is an RZ-property and R is an algebra over an infinite field or of characteristic zero, or π satisfies the condition that $\pi(R_p[x]) \cap R_p[x^p - x] \subseteq \pi(R_p[x^p - x])$ then: $\pi(R[x]) \neq 0$ implies that $\pi(R[x]) \cap R \neq 0$.*

THEOREM 2π. *If π and R are the same as in the preceding Lemma, then $\pi(R[x]) = P[x]$ where $P = \pi(R[x]) \cap R$.*

One readily verifies also, as in the proof of Theorem 3, that:

THEOREM 3π. *If π and R are as above and if π satisfies the condition that a polynomial ring $S[x]$ over a π-ring S is also a π-ring then $\pi(R[x]) = \pi(R)[x]$.*

Properties satisfying the conditions of Theorem 1π are readily seen to be nillity, locally finiteness and locally nilpotency. The latter satisfies also Theorem 3π.

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Hebrew University,
Jerusalem, Israel