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ANALYTIC STRUCTURE OF SCHLÄFLI FUNCTION

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§ 1. Introduction

In this note it is shown that *Schläfli function can be simply expressed in terms of hyperlogarithmic functions, namely iterated integrals of forms with logarithmic poles in the sense of K. T. Chen* (Theorem 1). It is also discussed *the relation between Schläfli function and hypergeometric ones of Mellin-Sato type* (Theorem 2). From a combinatorial point of view the structure of hyperlogarithmic functions seem very interesting just as the dilog $\int_0^x \log(1-x)/x dx$ (so-called Abel-Rogers function) has played a crucial part in Gelfand-Gabriev-Losik's formula of 1st Pontrjagin classes. See also [3].

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§ 2. Gauss-Bonnet theorem

Let S^n be a n dimensional unit sphere in $\boldsymbol{R}^{n+1}$ with the standard metric and $S_1, S_2, \cdots, S_{n+1}$ be $(n+1)$ hyperplanes in $\boldsymbol{R}^{n+1}$ through the origin which are in general position. Let

$$S_j : f_j = 0 \tag{2.1}$$

where $f_j = \sum_{\nu=1}^{n+1} u_{j\nu} x_\nu$ with $\sum_{\nu=1}^{n+1} u_{j\nu}^2 = 1$. The set of all points of S^n satisfying the inequalities

$$f_1 \geqq 0, \cdots, f_{n+1} \geqq 0 \tag{2.2}$$

form a n dimensional spherical simplex denoted by Δ. We denote by

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$\langle i, j\rangle$ the dihedral angle between S_i and S_j subtended by Δ. Then Δ is uniquely determined up to the motion of congruences by the $n(n+1)/2$ quantities $-\cos\langle i, j\rangle = a_{ij}$ so that the volume V of Δ can be regarded as an analytic function of the variables a_{ij} of the $n(n+1)/2$ dimensional complex affine space $\mathfrak{X}$, which is defined by *Schläfli's integral* on Δ:

$$V = \int_{\Delta} \sum_{j=1}^{n+1} (-1)^{j-1} \cdot x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1} \tag{2.3}$$

and which can also be expressed as

$$V = \frac{1}{2^{n/2-1} \cdot \Gamma(n/2+1)} \int_{f_1 \geqq 0, \cdots, f_{n+1} \geqq 0} e^{-1/2(x_1^2+\cdots+x_{n+1}^2)} dx_1 \wedge \cdots \wedge dx_{n+1} . \tag{2.3$'$}$$

Let $\Delta(\varepsilon_1 i_1, \varepsilon_2 i_2, \cdots, \varepsilon_p i_p)$ or $V(\varepsilon_1 i_1, \cdots, \varepsilon_p i_p)$ $(1 \leqq p \leqq n+1, \varepsilon_j = \pm 1)$ denote the chains in S^n defined by the inequalities

$$\varepsilon_1 f_{i_1} \geqq 0, \cdots, \varepsilon_p f_{i_p} \geqq 0 \tag{2.4}$$

or the volumes of them respectively. Clearly we have

$$\begin{cases} V(\varepsilon_1 i_1, \cdots, \varepsilon_p i_p) = V(-\varepsilon_1 i_1, \cdots, -\varepsilon_p i_p) , \\ V(\varepsilon_1 i_1, \cdots, \varepsilon_p i_p) + V(\varepsilon_1 i_1, \cdots, \varepsilon_{p-1} i_{p-1}, -\varepsilon_p i_p) \\ \qquad = V(\varepsilon_1 i_1, \cdots, \varepsilon_{p-1} i_{p-1}) \qquad \text{and} \\ V(\varepsilon_1 i_1) = 1/2\,|S^n| \end{cases} \tag{2.5}$$

where $|S^n|$ denotes the volume of S^n equal to $2\pi^{n/2}/\Gamma(n/2)$.

The following Gauss-Bonnet theorem is well-known ([8], [11]).

PROPOSITION 1. *For odd n*

$$\{(n-1)/2\}\,|S^n| = \sum_{\nu=2}^{n-1} \sum_{i_1 < \cdots < i_\nu} (-1)^\nu V(i_1, i_2, \cdots, i_\nu) \tag{2.6}$$

and for even n

$$\{(n-1)/2\}\,|S^n| = \sum_{\nu=2}^{n} \sum_{i_1 < \cdots < i_\nu} (-1)^\nu V(i_1, \cdots, i_\nu) - 2V(1, 2, \cdots, n+1) . \tag{2.7}$$

This Proposition simply follows from the following combinatorial lemma.

LEMMA 1. *Let μ be a finitely additive measure on a space X and $U_1, U_2, \cdots, U_m$ be a finite number of measurable subsets of X. Then we have*

$$(2.8) \qquad \mu\Big(\bigcap_j (X - U_j)\Big) = \mu(X) + \sum_{\nu=1}^{m} (-1)^\nu \mu(U_{i_1} \cap \cdots \cap U_{i_\nu})$$

if $\mu(X) < \infty$.

According to this Proposition all the volumes $V(\varepsilon_1 i_1, \cdots, \varepsilon_p i_p)$ are expressed as linear combinations of $|S^n|$, $V(i, j)$, $V(i_1, i_2, i_3, i_4)$, $\cdots$, $V(i_1, \cdots, i_{2\nu})$ where $2\nu - 1 = n$ or $n - 1$ according as n is odd or even.

§ 3. Application of Schläfli's formula

We denote by $D\begin{pmatrix} i_1 i_2 \cdots i_p \\ j_1 j_2 \cdots j_p \end{pmatrix}$ the subdeterminant of the symmetric matrix A

$$(3.1) \qquad A = \begin{bmatrix} 1 & a_{12} \cdots & & a_{1,n+1} \\ a_{21} & 1 \cdots & & a_{2,n+1} \\ \vdots & \ddots & & \vdots \\ & & 1 & a_{n,n+1} \\ a_{n+1,1} & \cdots & a_{n+1,n} & 1 \end{bmatrix}$$

consisting of $i_1, \cdots, i_p$ th. lines and $j_1, \cdots, j_p$ th. columns. In particular we shall abreviate $D\begin{pmatrix} i_1 i_2 \cdots i_p \\ i_1 i_2 \cdots i_p \end{pmatrix}$ by $D(i_1, \cdots, i_p)$. The matrix A defines a spherical simplex $\varDelta$ if and only if A is positive definite. In such a case Hadamard's inequality implies

$$(3.2) \qquad D(i_1, \cdots, i_p) \geqq D(j_1, \cdots, j_q)$$

if $(i_1, \cdots, i_p) \subset (j_1, \cdots, j_q)$. We denote by E the identity matrix where $\langle i, j \rangle$ are all equal to $\pi/2$.

NOTATION. We denote by I a subset of indices $\{i_1, i_2, \cdots, i_p\}$ of $\{1, 2, \cdots, n+1\}$ different from each other and by I its length p.

Let $\varDelta^*(i_1, \cdots, i_p)$ be a $(n-p)$ dimensional subsimplex of $\varDelta$ contained in the intersection $S_{i_1 i_2 \cdots i_p}$ of S^n and the hyperplanes $f_{i_1} = 0, \cdots, f_{i_p} = 0$. We denote by $V^*(i_1, \cdots, i_p)$ the $(n-p)$ dimensional volume of $\varDelta^*(i_1, \cdots, i_p)$. Then Schläfli's fundamental equality can be stated as follows:

SCHLÄFLI'S FORMULA.

$$(3.3) \qquad dV = \sum_{i<j} V^*(i, j) d\langle i, j \rangle$$

Proof. [15] or [10] p. 337–p. 340.

This also implies the following:

$$dV^*(I) = \sum_{j_1<j_2, I\cap(j_1,j_2)=\emptyset} V^*(I,(j_1,j_2))d\left\langle {I \atop j_1, j_2} \right\rangle \tag{3.4}$$

where $\left\langle {I \atop j_1, j_2} \right\rangle$ denotes the dihedral angle between S_{j_1} and S_{j_2} subtended by $\varDelta^*(I)$ in the $(n-p)$ dimensional sphere S_I.

From now on *we shall assume n equal to odd $2\nu-1$.* Let T be a lower triangular matrix:

$$T = \begin{pmatrix} 1 & & & 0 \\ t_{21} & t_{22} & & \\ \vdots & \vdots & \ddots & \\ t_{n+1,1} & t_{n+1,2} & \cdots & t_{n+1,n+1} \end{pmatrix} \tag{3.5}$$

such that $t_{22}>0,\cdots,t_{n+1,n+1}>0$ and $T\cdot{}^tT = A$. T is uniquely determined by A and we have

$$\langle 1,2\rangle = 1/2i \log\left(\frac{-t_{21}+it_{22}}{-t_{21}-it_{22}}\right). \tag{3.6}$$

The lower triangular matrix T_{12} corresponding to $\varDelta^*(1,2)$ is equal to

$$\begin{pmatrix} 1 & & & 0 \\ \lambda_4 t_{43} & \lambda_4 t_{44} & & \\ \vdots & & \ddots & \\ \lambda_{n+1}t_{n+1,3} & \lambda_{n+1}t_{n+1,4} & \cdots & \lambda_{n+1}t_{n+1,n+1} \end{pmatrix} \tag{3.7}$$

where λ_j denotes $1/\sqrt{1+t_{j3}^2+\cdots+t_{jj}^2}$ for $4\leqq j\leqq n+1$. Therefore by induction we have

$$\left\langle {12 \atop 34} \right\rangle = 1/2i \log\left(\frac{-t_{43}+it_{44}}{-t_{43}-it_{44}}\right) \tag{3.8}$$

or more generally

$$\left\langle {12\cdots 2\mu-3\ \ 2\mu-2 \atop 2\mu-1, 2\mu} \right\rangle = 1/2i \log\left(\frac{-t_{2\mu,2\mu-1}+it_{2\mu,2\mu}}{-t_{2\mu,2\mu-1}-it_{2\mu,2\mu}}\right) \tag{3.9}$$

for $0\leqq\mu\leqq\nu-1$. On the other hand a simple calculation shows that $t_{i,i-1}/t_{i,i}$ is equal to

$$D\binom{12\cdots i-2,i-1}{12\cdots i-2,i}\Big/\sqrt{D(1,2,\cdots,i-2,i-1)D(1,2,\cdots,i-2,i-1,i)}$$

so that (3.9) is equal to

(3.10)

$$1/2i\log\left[\frac{-D\binom{12\cdots 2\mu-2\ 2\mu-1}{12\cdots 2\mu-2\ 2\mu}+i\sqrt{D(1,2,\cdots,2\mu-2)D(1,2,\cdots,2\mu)}}{-D\binom{12\cdots 2\mu-2\ 2\mu-1}{12\cdots 2\mu-2\ 2\mu}-i\sqrt{D(1,2,\cdots,2\mu-2)D(1,2,\cdots,2\mu)}}\right].$$

NOTATION. If I and J are two subsets of indices $I=(i_1,\cdots,i_p)$ and $J=(i_1,i_2,\cdots,i_p,i_{p+1},i_{p+2})$, then we denote by $\omega\binom{I}{J}$ the 1-form defined by

$$1/2i\,d\log\left[\frac{-D\binom{i_1i_2\cdots i_pi_{p+1}}{i_1i_2\cdots i_pi_{p+2}}+i\sqrt{D(I)D(J)}}{-D\binom{i_1i_2\cdots i_pi_{p+1}}{i_1i_2\cdots i_pi_{p+2}}-i\sqrt{D(I)D(J)}}\right]. \tag{3.11}$$

When A is equal to E, namely $\langle i,j\rangle$ are all equal to $\pi/2$, V is reduced to $(1/2)^{n+1}\cdot|S^n|$. We denote by $\mathfrak{X}_{i_1i_2\cdots i_p}$ the divisor defined by the equation $D(i_1,i_2,\cdots,i_p)=0$ in $\mathfrak{X}$. Let $\hat{\mathscr{X}}$ be a $2^{(2^n-1)}$-covering of $\mathscr{X}$ ramified over $\mathfrak{X}_{i_1i_2\cdots i_{2\mu}}$ $(1\leqq\mu\leqq\nu)$, uniformizing all the functions $\sqrt{D(i_1,i_2,\cdots,i_{2\mu})}$, and π be the natural projection from $\hat{\mathscr{X}}$ onto $\mathfrak{X}$. If p is even, the form $\omega\binom{I}{J}$ of (3.11) is well-defined 1-form on $\hat{\mathscr{X}}$ *which has logarithmic poles along* $\pi^{-1}(\mathfrak{X}_{i_1i_2\cdots i_pi_{p+1}})$ *or* $\pi^{-1}(\mathfrak{X}_{i_1i_2\cdots i_pi_{p+2}})$ in view of Jacobi's identity:

$$\begin{aligned}&D(i_1\cdots i_p)D(i_1\cdots i_pi_{p+1}i_{p+2})\\&\qquad=D(i_1\cdots i_pi_{p+1})D(i_1\cdots i_pi_{p+2})-D\binom{i_1\cdots i_pi_{p+1}}{i_1\cdots i_pi_{p+2}}^2.\end{aligned} \tag{3.12}$$

DEFINITION. Let $\Omega(M;p,q)$ be the space of continuous paths from a point p to a point q in a differentiable manifold M, and $\omega_1,\omega_2,\cdots,\omega_m$ a finite number of differential 1-forms on M. Let γ be a path of $\Omega(M;p,q)$ namely a differentiable function $\varphi\colon[0,1]\to M$ such that $\varphi(0)=p$ and $\varphi(1)=q$. Let $f_j(t)dt$ be the pull-back of each ω_j by φ. According to K. T. Chen (see [4]) we consider the following integral

$$\int_0^1 f_1(t_1)dt_1\int_0^{t_1}f_2(t_2)dt_2\cdots\int_0^{t_{m-1}}f_m(t_m)dt_m \tag{3.13}$$

which will be called *"iterated integral of order m"* and denoted by

$$\int_{r} \omega_1 \circ \omega_2 \circ \cdots \circ \omega_m . \tag{3.14}$$

Now by (3.3), (3.4), (3.10) and (3.11) we can conclude the following:

THEOREM 1. *For odd n, V is expressed in terms of iterated integrals of forms of logarithmic poles $\omega\binom{I}{J}$ on $\hat{\mathscr{X}}$:*

$$V = \sum_{(I_0, I_1, \cdots, I_\nu)} \sum_{\sigma=0}^{\nu} \int_E^A \omega\binom{I_0}{I_1} \circ \omega\binom{I_1}{I_2} \circ \cdots \circ \omega\binom{I_{\sigma-1}}{I_\sigma} \cdot \frac{|S^{n-2\sigma}|}{2^{n+1-2\sigma}} \tag{3.15}$$

where we put $|S^{-1}| = 1$ and $(I_0, I_1, \cdots, I_\nu)$ run through all families of subsets of indices such that (i) $|I_0| = 0, |I_1| = 2, \cdots, |I_\nu| = 2\nu$ *and* (ii) $I_0 = \emptyset \subset I_1 \subset I_2 \subset \cdots \subset I_\nu$. *The above iterated integrals are done on each path from E to A in $\hat{\mathscr{X}}$.*

Remark. The right hand side of (3.15) depends only on homotopy classes of paths provided A is fixed. In fact Chen's formula of the exterior differentiation of iterated integrals show (see Proposition 4.1.2 in [4])

$$\begin{aligned} &d \sum_{(I_0, I_1, \cdots, I_\sigma)} \int \omega\binom{I_0}{I_1} \circ \omega\binom{I_1}{I_2} \circ \cdots \circ \omega\binom{I_{\sigma-1}}{I_\sigma} \\ &= \sum_{(I_0, I_1, \cdots, I_\sigma)} \int \sum_{\tau=1}^{\sigma-1} (-1)^\tau \omega\binom{I_0}{I_1} \circ \cdots \circ \omega\binom{I_{\tau-1}}{I_\tau} \wedge \omega\binom{I_\tau}{I_{\tau+1}} \circ \cdots \circ \omega\binom{I_{\sigma-1}}{I_\sigma} \end{aligned} \tag{3.16}$$

where $(I_0, I_1, \cdots, I_{\sigma-1})$ run through all the subsets of indices such that $|I_0| = 0, |I_1| = 2, \cdots, I_\sigma = 2\sigma$ and $I_0 \subset I_1 \subset \cdots \subset I_{\sigma-1} \subset I_\sigma, I_\sigma$ being fixed. This vanishes in view of the following identities:

$$\sum_{\substack{I \subset K \subset J \\ |K| = |I| + 2}} \omega\binom{I}{K} \wedge \omega\binom{K}{J} = 0 \tag{3.17}$$

for any subsets of indices I and J such that $|I| + 4 = |J|$, which can be proved by a direct calculation.

COROLLARY OF THEOREM 1. *The monodromy of the many valued function V on $\hat{\mathscr{X}}$ is contained in a unipotent subgroup of upper triangular matrices.*

Proof. This follows from a general theory of iterated integrals (see [4] p. 222). In our situation the variation of V along an arbitrary loop on $\hat{\mathscr{X}} - \bigcup_{\substack{i_1<\cdots<i_{2\mu} \\ 1\leqslant\mu\leqslant\nu}} \pi^{-1}(\mathfrak{X}_{i_1\cdots i_{2\mu}})$ can be written as a linear combination of the iterated integrals

$$\sum_{I_1,\cdots,I_{\sigma-1}} \omega\binom{I_0}{I_1} \circ \omega\binom{I_1}{I_2} \circ \cdots \circ \omega\binom{I_{\sigma-1}}{I_\sigma} \tag{3.18}$$

which is closed on Ω because of (3.16). This fact can also be proved in a direct way by using a generalized Picard-Lefschetz formula due to F. Pham.

According to H. Poincaré and Lappo-Danilevski we shall call "*hyper-logarithmic functions of order m*" functions of iterated integrals of mth order of forms with logarithmic poles, so that *V is a hyperlogarithmic function of order ν on $\hat{\mathscr{X}}$.*

The volume of a double-rectangular tetrahedron was investigated by H. S. M. Coxeter [6]. By his notations we have $\langle 1,3\rangle = \langle 1,4\rangle = \langle 2,4\rangle = 0$, $\langle 1,2\rangle = \pi/2 - \alpha$, $\langle 2,3\rangle = \beta$ and $\langle 3,4\rangle = \pi/2 - \gamma$. Then V is written as follows:

$$\begin{aligned} V - |S^3|/16 = &-\int d\alpha\cdot 1/2i \log\left(\frac{-\sin\gamma\cos\alpha + i\sqrt{D}}{-\sin\gamma\cos\alpha - i\sqrt{D}}\right) \\ &+\int d\beta\cdot 1/2i \log\left(\frac{-\sin\alpha\cos\beta\sin\gamma + i\sin\beta\sqrt{D}}{-\sin\alpha\cos\beta\sin\gamma - i\sin\beta\sqrt{D}}\right) \\ &-\int d\gamma\cdot 1/2i \log\left(\frac{-\sin\alpha\cos\gamma + i\sqrt{D}}{-\sin\alpha\cos\gamma - i\sqrt{D}}\right) \end{aligned} \tag{3.19}$$

which gives the same formula as (4.11) in [6], where D means $D(1,2,3,4) = \cos^2\alpha\cdot\cos^2\gamma - \cos^2\beta$.

§ 4. Power series expansion of V

The integral (1.3) can also be expressed as follows:

$$V = (n+1)\int_{\substack{f_1\geqq 0,\cdots,f_{n+1}\geqq 0 \\ 1\geqq x_1^2+\cdots+x_{n+1}^2}} dx_1\wedge\cdots\wedge dx_{n+1}\,. \tag{4.1}$$

By change of variables the right hand side is transformed into

$$(n+1)/D\int_{\substack{1\geqq Q(y_1,\cdots,y_{n+1}) \\ y_1\geqq 0,\cdots,y_{n+1}\geqq 0}} dy_1\wedge\cdots\wedge dy_{n+1} \tag{4.2}$$

where Q denotes the quadratic polynomial $\sum_{j=1}^{n+1} y_j^2 + \sum_{i\neq j} b_{ij}y_iy_j$ with $b_{ij} = b_{ji}$ and D denotes $D(1, 2, \cdots, n+1)$. b_{ij} are determined by the relation:

$$B = K^{-1}\cdot A^{-1}\cdot K^{-1}\,, \tag{4.3}$$

where B denotes the matrix

$$\begin{bmatrix} 1 & b_{12} & \cdots & b_{1,n+1} \\ b_{21} & 1 & \cdots & b_{2,n+1} \\ \vdots & & \ddots & \vdots \\ b_{n+1,1} & b_{n+1,2} & \cdots & 1 \end{bmatrix} \tag{4.4}$$

and K denotes the diagonal matrix with positive elements Diag $[\rho_1, \cdots, \rho_{n+i}]$, ρ_i equal to

$$\sqrt{\frac{D(1, \cdots, i-1, i+1, \cdots, n+1)}{D(1, 2, \cdots\cdots\cdots\cdots\cdots\cdots, n+1)}}\,.$$

It is easily seen that *the correspondence* (4.3) *is birational on* $\hat{\mathfrak{X}}$, *leaving fixed the divisors* $\bigcup_{1\leqslant\mu\leqslant\nu}\ \bigcup_{i_1<i_2<\cdots<i_{2\mu}} \pi^{-1}(\mathfrak{X}_{i_1i_2\cdots i_{2\mu}})$ or $\bigcup_{1\leqslant\mu\leqslant\nu}\ \bigcup_{i_1i_2<\cdots<i_{2\mu-1}} \pi^{-1}(\mathfrak{X}_{i_1i_2\cdots i_{2\mu-1}})$.

Now we are going to prove the following theorem:

THEOREM 2. *As a function of the variables b_{ij}, V has a convergent power series expansion at the origin*:

$$2^{n+1}\cdot DV/(n+1) = \sum_{\sigma_{ij}\geqq 0} \frac{\prod_{i<j}(-2b_{ij})^{\sigma_{ij}}}{\prod_{i<j}\sigma_{ij}\,!} \cdot \frac{\prod_{k=1}^{n+1}\Gamma\left(\dfrac{\sigma_{1,k}+\cdots+\sigma_{k-1,k}+\sigma_{k,k+1}+\cdots+\sigma_{k,n+1}}{2}\right)}{\Gamma\left(\dfrac{n+1}{2}+1\right)} \tag{4.5}$$

which is a so-called generalized hypergeometric series. For this kind of functions see Appendix.

To prove Theorem 2 we want to prove a slightly more general theorem by making use of a technic introduced in [1].

THEOREM 2′. *The integral*

$$\varphi = \int_{\substack{1\geqq Q\\ y_1\geqq 0,\cdots,y_{n+1}\geqq 0}} (1-Q)^{\lambda_0} y_1^{\lambda_1}\cdot y_2^{\lambda_2}\cdots y_{n+1}^{\lambda_{n+1}} dy_1\wedge\cdots\wedge dy_{n+1} \tag{4.6}$$

for $\lambda_0 \geqq 0, \lambda_1 \geqq 0, \cdots, \lambda_{n+1} \geqq 0$ *has a convergent power series expansion near the origin*:

$$2^{n+1}\varphi = \frac{\prod_{i<j}(-2b_{ij})^{\sigma_{ij}}}{\prod_{i<j}\sigma_{ij}!}$$

$$\text{(4.7)} \qquad \frac{\prod_{k=1}^{n+1}\Gamma\left(\frac{\sigma_{1k}+\cdots+\sigma_{k-1,k}+\sigma_{k,k+1}+\cdots+\sigma_{k,n+1}+\lambda_k+1}{2}\right)\Gamma(\lambda_0+1)}{\Gamma\left(\frac{\lambda_1+\cdots+\lambda_{n+1}+n+1}{2}+\lambda_0+1\right)}$$

To prove (4.7) we need

LEMMA 2. *If* $\lambda_0, \lambda_1, \cdots, \lambda_{n+1}$ *are all sufficiently large,*

$$\text{(4.8)} \qquad \begin{aligned} &\int_{\substack{1\geqq Q,\\ y_1\geqq 0,\cdots,y_{n+1}\geqq 0}} y_1^{\lambda_1}\cdot y_2^{\lambda_2}\cdots y_{n+1}^{\lambda_{n+1}}(1-Q)^{\lambda_0-1}dy_1\wedge\cdots\wedge dy_{n+1} \\ &\qquad = (n+1+2\lambda_0+\lambda_1+\cdots+\lambda_{n+1})/2\lambda_0 \\ &\qquad\quad \cdot\int y_1^{\lambda_1}\cdot y_2^{\lambda_2}\cdots y_{n+1}^{\lambda_{n+1}}(1-Q)^{\lambda_0}dy_1\wedge\cdots\wedge dy_{n+1}\,. \end{aligned}$$

Proof. We have by exterior differentiation

$$\begin{aligned} &d\Big(-\frac{(1-Q)^{\lambda_0}y_1^{\lambda_1}\cdots y_{n+1}^{\lambda_{n+1}}}{2\lambda_0}\sum(-1)^{j-1}y_j dy_1\wedge\cdots \\ &\qquad \wedge dy_{j-1}\wedge dy_{j+1}\wedge\cdots\wedge dy_{n+1}\Big) \\ &\quad = \left\{\frac{1}{(1-Q)} - \frac{1}{2\lambda_0}\left(\sum_1^{n+1}\lambda_j+2\lambda_0+n+1\right)\right\}(1-Q)^{\lambda_0}y_1^{\lambda_1}\cdots y_{n+1}^{\lambda_{n+1}} \\ &\qquad \cdot dy_1\wedge\cdots\wedge dy_{n+1}\,. \end{aligned}$$

Integrating both sides we get Lemma 2.

Proof of Theorem 2. For sufficiently large $\lambda_0, \lambda_1, \cdots, \lambda_{n+1}$ we have

$$\text{(4.9)} \qquad \begin{aligned} (-1)^\sigma\frac{\partial^\sigma\varphi}{\partial b_{i_1i_2}\cdots\partial b_{i_{2\sigma-1}i_{2\sigma}}} &= 2^\sigma\lambda_0(\lambda_0-1)\cdots(\lambda_0-\sigma+1) \\ &= \int_{\substack{1\geqq Q\\ y_1\geqq 0,\cdots,y_{n+1}\geqq 0}} (1-Q)^{\lambda_0-\sigma}\cdot y_1^{\lambda_1}\cdots y_{n+1}^{\lambda_{n+1}}\cdot y_{i_1}\cdot y_{i_2} \\ &\qquad \cdots y_{i_{2\sigma-1}}\cdot y_{i_{2\sigma}}dy_1\wedge\cdots\wedge dy_{n+1}\,. \end{aligned}$$

According to Lemma 2 the right hand side is equal to

$$\begin{aligned}(4.10)\quad &\prod_{k=1}^{\sigma}\Big(\sum_{1}^{n+1}\lambda_j + 2\lambda_0 + n + 1 + 2\sigma - 2(k-1)\Big)\\ &\int_{\substack{1\geq Q,\\ y_1\geq 0,\cdots,y_{n+1}\geq 0}} (1-Q)^{\lambda_0} y_1^{\lambda_1}\cdots y_{n+1}^{\lambda_{n+1}} y_{i_1}y_{i_2}\cdots y_{i_{2\sigma-1}}y_{i_{2\sigma}} dy_1\wedge\cdots\wedge dy_{n+1}.\end{aligned}$$

When b_{ij} are all zero, then φ is reduced to

$$(4.11)\qquad \frac{\prod_{k=1}^{n+1}\Gamma\Big(\frac{\lambda_k+1}{2}\Big)\Gamma(\lambda_0+1)}{2^n\Gamma\Big(\frac{\lambda_1+\cdots+\lambda_{n+1}+n+1}{2}+\lambda_0+1\Big)}.$$

Theorem 2′ follows from (4.8) ~ (4.11), because the convergence of the power series (4.7) is obvious. The proof is complete.

Now we want to express V as power series expansion of the variables t_{ij} similar to (3.5) so that

$$(4.12)\qquad \begin{cases} f_1 = x_1\,,\\ f_2 = t_{21}\cdot x_1 + x_2\\ \cdots\\ f_{n+1} = t_{n+1,1}x_1 + \cdots + t_{n+1,n}x_n + x_{n+1}\,.\end{cases}$$

We consider the integral $V(\lambda_1, \lambda_2, \cdots, \lambda_{n+1})$:

$$(4.13)\qquad \int_{f_1\geq 0,\cdots,f_{n+1}\geq 0} f_1^{\lambda_1}\cdot f_2^{\lambda_2}\cdots f_{n+1}^{\lambda_{n+1}} dx_1\wedge dx_2\wedge\cdots\wedge dx_{n+1}\,.$$

Then for large $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ we have

$$\begin{aligned}&\frac{\partial^\sigma V(\lambda_1,\lambda_2,\cdots,\lambda_{n+1})}{(\partial t_{21})^{\sigma_{21}}\cdots(\partial t_{ij})^{\sigma_{ij}}\cdots(\partial t_{n+1,n})^{\sigma_{n+1,n}}}\\ &\quad = \int_{f_1\geq 0,\cdots,f_{n+1}\geq 0} f_1^{\lambda_1}\cdot f_2^{\lambda_2-\sigma_{21}}\cdots f_{n+1}^{\lambda_{n+1}-\sigma_{n+1,1}-\cdots-\sigma_{n+1,n}}\cdot x_1^{\sigma_{21}+\cdots+\sigma_{n+1,1}}\\ &\qquad \cdot x_2^{\sigma_{32}+\cdots+\sigma_{n+1,2}}\cdots x_n^{\sigma_{n+1,n}} dx_1\wedge\cdots\wedge dx_{n+1}\\ &\qquad \cdot\prod_{i=1}^{n+1}\lambda_i(\lambda_i-1)\cdots(\lambda_i-\sigma_{i,1}-\cdots-\sigma_{i,i-1})\,.\end{aligned}$$

For all $t_{ij}=0$, the above is reduced to

$$\frac{\prod_{k=1}^{n+1}\Gamma\Big(\frac{\lambda_k-\sigma_{k1}-\cdots-\sigma_{k,k-1}+\sigma_{k+1,k}+\cdots+\sigma_{n+1,k}+1}{2}\Big)}{2^{n+1}\Gamma\Big(\frac{\lambda_1+\lambda_2+\cdots+\lambda_{n+1}+n+1}{2}\Big)}$$

$$\prod_{i=1}^{n+1} \lambda_i(\lambda_i - 1) \cdots (\lambda_i - \sigma_{i,1} - \cdots - \sigma_{i,i-1} + 1)$$

so that (4.13) is equal to

$$\sum \frac{\prod \Gamma\left(\frac{\lambda_k - \sigma_{k1} - \cdots - \sigma_{k,k-1} + \sigma_{k+1,k} + \cdots + \sigma_{n+1,k} + 1}{2}\right)}{2^{n+1}\Gamma\left(\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_{n+1} + n + 1}{2}\right)\prod_{i>j} \sigma_{ij}!}$$

$$\prod \frac{\Gamma(\lambda_i + 1)}{\Gamma(\lambda_i - \sigma_{i,1} - \cdots - \sigma_{i,i-1} + 1)} .$$

In particular if $\lambda_1 = \cdots = \lambda_{n+1} = 0$ we have the volume V:

THEOREM 2′.

$$V = \underset{\lambda_j \to 0}{\text{l.i.m.}}\ V(\lambda_1, \cdots, \lambda_{n+1})$$

$$(4.14) \qquad = \sum_{\sigma_{ij} \geqq 0} \frac{\prod_{k=1}^{n+1} \Gamma\left(\frac{-\sigma_{k,1} - \cdots - \sigma_{k,k-1} + \sigma_{k+1,k} + \cdots + \sigma_{n+1,k} + 1}{2}\right)}{2^{n+1}\Gamma\left(\frac{n+1}{2}\right) \prod_{i>j} \sigma_{ij}! \prod_{i=2}^{n+1} \Gamma(-\sigma_{i1} - \cdots - \sigma_{ii-1} + 1)}$$

$$\cdot \prod_{i>j} (t_{ij})^{\sigma_{ij}}$$

where the quotients

$$\frac{\Gamma\left(\frac{-\sigma_{k1} - \cdots - \sigma_{k,k-1} + \sigma_{k+1,k} + \cdots + \sigma_{n+1,k} + 1}{2}\right)}{\Gamma(-\sigma_{k1} - \cdots - \sigma_{k,k-1} + 1)}$$

have definite values and the right hand side is well-defined. This is also a hypergeometric function.

§ 5. Hyperbolic case

Let H be the hyperbolic space form defined by

$$(5.1) \qquad \begin{cases} -x_1^2 - \cdots - x_n^2 + x_{n+1}^2 = 1 \\ x_{n+1} > 0 \end{cases}$$

with the standard metric. In view of (2.3)′ the analytic continuation V_θ of V along the path $\{\varphi_\theta\}$ (see (2.1))

$$(5.2) \qquad \varphi_\theta : \begin{cases} u_{j,n+1} \to u'_{j,n+1} = u_{j,n+1} \cdot e^{\sqrt{-1}\theta} \\ u_{jk} \quad\ \to u'_{jk} = u_{jk} \end{cases}$$

from $\theta=0$ to $\theta=\frac{\pi}{2}$ $(1\leqq j\leqq n+1, 1\leqq k\leqq n)$ can be written as follows:

$$V_{\pi/2}/(\sqrt{-1})^n=\frac{1}{2^{n/2-1}\Gamma\left(\frac{n}{2}+1\right)} \tag{5.3}$$
$$\cdot\int_{f_1\geqq 0,\cdots,f_{n+1}\geqq 0} e^{-\frac{1}{2}(-x_1^2-\cdots-x_n^2+x_{n+1}^2)}dx_1\wedge\cdots\wedge dx_n\wedge dx_{n+1}\,.$$

The second hand side is equal to the volume V' of the simplex Δ' in H defined by $f_1\geqq 0,\cdots,f_{n+1}\geqq 0$ and with the faces $H_i: f_i=0$. The dihedral angle between H_i and H_j subtended by Δ' is equal to (see (2.1))

$$-\cos\langle i,j\rangle'=\frac{\sum_{\nu=1}^n u_{i\nu}u_{j\nu}-u_{in+1}u_{jn+1}}{\sqrt{\sum_{\nu=1}^n u_{i\nu}^2-u_{in+1}^2}\sqrt{\sum_{\nu=1}^n u_{j\nu}^2-u_{jn+1}^2}} \tag{5.4}$$

so that Schläfli formula has the form:

$$dV'=-\sum_{i<j}V'(i,j)d\langle i,j\rangle' \tag{5.5}$$

where $V'(i,j)$ means the volume of the $(n-2)$ dimensional subsimplex $\Delta^*(i,j)'$ defined by $f_i=f_j=0$.

We denote by A' the matrix corresponding to $\langle i,j\rangle'$:

$$A'=\begin{pmatrix} 1 & -\cos\langle 12\rangle' & \cdots & & -\cos\langle 1,n+1\rangle' \\ -\cos\langle 21\rangle' & 1 & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & & & 1 & -\cos\langle n,n+1\rangle' \\ -\cos\langle n+1,1\rangle' & \cdots & & -\cos\langle n+1,n\rangle' & 1 \end{pmatrix}. \tag{5.6}$$

Then Theorem 1 implies the following:

THEOREM 1'. *For odd n we have*

$$(\sqrt{-1})^nV'=\sum_{(I_0,I_1,\cdots,I_\nu)}\sum_{\sigma=0}^{\nu}\int_E^{A'}\omega\binom{I_0}{I_1}\circ\omega\binom{I_1}{I_2}\circ\cdots\circ\omega\binom{I_{\sigma-1}}{I_\sigma}\frac{|S^{n-2\sigma}|}{2^{n+1-2\sigma}} \tag{5.7}$$

where the integrals are done on each path from E to A' in $\hat{\mathfrak{X}}$.

COROLLARY. *Let $\bar{V}'$ be the volume of a hyperbolic simplex $\bar{\Delta}'$ corresponding to a fixed point $\bar{A}'\in\hat{\mathfrak{X}}$. Then $V'-\bar{V}'$ is equal to a linear combination of the iterated integrals along a path from $\bar{A}'$ to A':*

(5.8) $$\int_{\bar{A}'}^{A'} \omega\binom{I_0}{I_1} \circ \omega\binom{I_1}{I_2} \circ \cdots \circ \omega\binom{I_{\sigma-1}}{I_\sigma} \qquad (0 \leqq \sigma \leqq \nu)$$

where $I_0, I_1, \cdots, I_\nu$ *run through all the subsets of indices such that* (i) $|I_0| = 0, |I_1| = 2, \cdots, |I_\nu| = 2\nu$ *and* (ii) $I_0 = \emptyset \subset I_1 \subset \cdots \subset I_\nu$.

Proof. This easily follows from Proposition 1.5.1 in [4].

Appendix. Hypergeometric functions of Mellin-Sato type

We reproduce here briefly Sato's result in [14].

Let G be the group of m-product of $C^* = C - (0)$ and X be its dual, Hom (G, C^*) which is isomorphic to Z^m. We denote by X_C its complexification. Let $\{\chi_1, \cdots, \chi_m\}$ be a basis of X so that any ω of X_C can be written as $\omega = \sum_{j=1}^m s_j\chi_j$ with $(s_1, \cdots, s_m) \in C^m$.

NOTATION. For a rational function $f(\nu)$ on Z we denote by $\prod_{\nu=0}^{e-1} f(\nu)$ the product:

(6.1) $$\begin{cases} \prod_{\nu=0}^{e-1} f(\nu) & e \geqq 1 \\ 1 \Big/ \prod_{\nu=e}^{-1} f(\nu) & e < 0 \\ 1 & e = 0 \end{cases}$$

Under this situation Sato's fundamental theorem says

THEOREM A, 1. *Each class in the cohomology* $H^1(X, C(x))$ *can be represented by a so-called "b-function"*

(6.2) $$b_\chi(\omega) = \prod_{\kappa=1}^{k} \left\{ \prod_{\nu=0}^{e_\kappa(\chi)-1} (e_\kappa(\omega) + \alpha_\kappa + \nu) \right\}$$

where α_κ *denotes a constant and* e_κ *a suitable* Q*-valued linear function on* Z^m.

Let X_C^* be the dual of X_C so that X_C^* is isomorphic to the Lie algebra corresponding to G. For any point τ of X_C^* we put $e^\tau = t$, where $t = (t_1, \cdots, t_m) \in G$. We denote by t^χ the pairing $e^{\langle \chi, \tau \rangle}$.

DEFINITION. Arbitrary function u on X_C^* satisfied by the following system of (pseudo) differential equations

(6.3) $$b_\chi\left(t_1\frac{\partial}{\partial t_1}, \cdots, t_m\frac{\partial}{\partial t_m}\right)u = t^{-\chi}\cdot u$$

for any $\chi \in X$, is called "*hypergeometric function of Mellin-Sato type*". This system is maximally overdetermined on X_C^*.

LEMMA A, 1. *The Mellin transform of a generalized Γ-function* $\hat{u}(\omega)$

(6.4) $$\begin{cases} \hat{u}(\omega) = \prod_{\kappa=1}^{k} \Gamma(e_\kappa(\omega) + \alpha_\kappa) \\ u(t) = \int \hat{u}(\omega)t^\omega d\omega_1 \cdots d\omega_m \end{cases}$$

is a hypergeometric function of M-S type if it exists.

Proof. Easy.

THEOREM 2′ *$D\cdot V$ is a Mellin transform of* $\hat{V}$

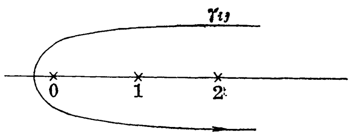
(6.5) $$\hat{V} = \prod_{k=1}^{n+1} \Gamma\left(\frac{s_{1k} + \cdots + s_{k-1,k} + s_{k,k+1} + \cdots + s_{k,n+1} + 1}{2}\right) \cdot \prod_{i<j} \Gamma(-s_{ij})$$

namely we have the following integral representation:

(6.6) $$\frac{2^n\Gamma\left(\frac{n+1}{2} + 1\right)DV}{n+1} = \left(\frac{1}{2\pi i}\right)^{n(n+1)/2} \int_\gamma \hat{V}(s) \prod_{i<j} (-2a_{ij})^{s_{ij}} \prod_{1\leq i<j\leq n+1} ds_{ij}$$

where γ denotes a chain of $n(n+1)/2$ dimension which is the product of paths γ_{ij} defined on each s_{ij}-plane as in the figure:

s_{ij}-plane



Proof. The integral on each γ_{ij} is equal to the sum of all residues on $s_{ij} = 0, 1, 2, 3, \cdots$ which gives the power series expansion (4.5).

It is easy to see that $D\cdot V$ is a hypergeometric function of M–S type in the variables a_{ij}^2.

Finally some problems unknown to the author are raised here.

PROBLEM 1. To determine all meromorphic 1-forms on $\hat{\mathscr{X}}$ with logarithmic poles along $\bigcup_{1\leqslant\mu\leqslant\nu}\bigcup_{(i_1 i_2\cdots i_{2\mu-1})}\pi^{-1}(\mathfrak{X}_{i_1 i_2\cdots i_{2\mu-1}})$ and the infinity. It is seen by residue calculus that $2^{n-3}\cdot n(n+1)$ such 1-forms of the type (3.11) are linearly independent over $\boldsymbol{C}$. For the further properties of logarithmic poles see [7] and [9].

PROBLEM 2. What kind of functions are the inverse of hyperlogarithmic functions? They could be a generalization of exponential functions which satisfy some kind of addition formula and are related to A. N. Parsin's generalized Jacobian variety (see [12] and [13]).

PROBLEM 3. To determine the order of the maximally overdetermined system of (pseudo-) differential equations (6.3).

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