MAXIMUM GRAPHS NON-HAMILTONIAN-CONNECTED FROM A VERTEX

by G. R. T. HENDRY

(Received 30 September, 1982)

A path (cycle) in a graph G is called a *hamiltonian path* (*cycle*) of G if it contains every vertex of G. A graph is *hamiltonian* if it contains a hamiltonian cycle. A graph G is *hamiltonian-connected* if it contains a u-v hamiltonian path for each pair u, v of distinct vertices of G. A graph G is *hamiltonian-connected from a vertex* v of G if G contains a v-w hamiltonian path for each vertex $w \neq v$. Considering only graphs of order at least 3, the class of graphs hamiltonian-connected from a vertex properly contains the class of hamiltonian-connected graphs and is properly contained in the class of hamiltonian graphs.

Our present object is to determine the maximum possible number of edges in a graph of order $p \geqslant 3$ which is not hamiltonian-connected from a vertex and to describe the graphs which attain this maximum. The proof of Theorem 1 leans heavily on the following 3 results (Theorems A and B are due to Ore and Bondy and Theorem C is due to Chartrand and Nordhaus).

THEOREM A. *If G is a non-hamiltonian graph of order $p \geqslant 3$, then* (i) $|E(G)| \leqslant \frac{1}{2}(p^2 - 3p + 4)$ *and* (ii) *if equality holds in* (i) *then G is $K_2 + \bar{K}_3$ or, for $p \geqslant 3$, $K_2 . K_{p-1}$ (i.e. the graph of order p having 2 blocks: K_2 and K_{p-1}).*

THEOREM B. *If G is a non-hamiltonian-connected graph of order $p \geqslant 4$ then* (i) $|E(G)| \leqslant \frac{1}{2}(p^2 - 3p + 6)$ *and* (ii) *if equality holds in* (i) *then G is $K_6 - K_3$ or, for $p \geqslant 4$, $K_p - K_{1,p-3}$.*

THEOREM C. *Let G be a hamiltonian graph of order p. If there exists a vertex u such that*

$$\deg(u) + \deg(v) \geqslant p + 1$$

for each vertex v not adjacent to u, then G is hamiltonian-connected from at least 2 *vertices.*

THEOREM 1. *If G is a graph of order $p \geqslant 3$ which is not hamiltonian-connected from a vertex then* (i) $|E(G)| \leqslant \frac{1}{2}(p^2 - 3p + 4)$ *and* (ii) *if equality holds in* (i) *then G is C_4, $K_2 + \bar{K}_3$ or, for $p \geqslant 3$, $K_2 . K_{p-1}$.*

Proof. If $p = 3$ the result is obvious; therefore assume $p \geqslant 4$.

(i) Since G is not hamiltonian-connected from a vertex, it is not hamiltonian-connected and so, by Theorem B(i), $|E(G)| \leqslant \frac{1}{2}(p^2 - 3p + 6)$. If $|E(G)| = \frac{1}{2}(p^2 - 3p + 6)$ then, by Theorem B(ii), G is either $K_6 - K_3$ or, for $p \geqslant 4$, $K_p - K_{1,p-3}$. But neither of these is possible since $K_6 - K_3$ is hamiltonian-connected from any of its vertices of degree 3 and $K_p - K_{1,p-3}$ is hamiltonian-connected from the vertex at the centre of the $K_{1,p-3}$. Therefore, $|E(G)| \leqslant \frac{1}{2}(p^2 - 3p + 4)$.

Glasgow Math. J. **25** (1984) 97–98.

(ii) Now suppose that G has exactly $\frac{1}{2}(p^2-3p+4)$ edges. If G is not hamiltonian then, by Theorem A(ii), G is either $K_2+\bar{K}_3$ or, for $p \geq 4$, $K_2 . K_{p-1}$. From now on suppose that G is hamiltonian. If $p=4$ then clearly G must be C_4. Therefore, suppose $p \geq 5$ and that $v_1, v_2, \ldots, v_p, v_1$ is a hamiltonian cycle of G. If G has a vertex, say v_1, of degree $p-1$ then G is hamiltonian-connected from both v_2 and v_p which is a contradiction. However, G must have a vertex of degree $p-2$ since, if it doesn't, then it has at most $\frac{1}{2}(p^2-3p) < \frac{1}{2}(p^2-3p+4)$ edges. Suppose, without loss of generality, that $\deg(v_1)=p-2$ and that v_j is the vertex not adjacent to v_1. If $\deg(v_j) \geq 3$ then it follows from Theorem C that G is hamiltonian-connected from at least 2 vertices which is a contradiction. Therefore, $\deg(v_j)=2$ and we have

$$\begin{aligned}\tfrac{1}{2}(p^2-3p+4) &= |E(G)| \\ &= \tfrac{1}{2}\sum_i \deg(v_i) \\ &\leq \tfrac{1}{2}(2+(p-1)(p-2)) \\ &= \tfrac{1}{2}(p^2-3p+4).\end{aligned}$$

Equality must hold throughout so that every vertex of G, except v_j, has degree $p-2$. In particular, $\deg(v_{j-1})=p-2$ and if v_k is the single vertex not adjacent to v_{j-1} then, since $k \neq j$, $\deg(v_k)=p-2$. If $2p-4=\deg(v_{j-1})+\deg(v_k) \geq p+1$ then, by Theorem C, G is hamiltonian-connected from at least 2 vertices. Therefore, $2p-4<p+1$, i.e. $p<5$. This is a contradiction which completes the proof.

REFERENCES

1. J. A. Bondy, Variations on the Hamiltonian theme, *Canad. Math. Bull.* **15** (1972), 57–62.

2. G. Chartrand and E. A. Nordhaus, Graphs Hamiltonian-connected from a vertex, *The theory and applications of graphs*, edited by G. Chartrand et al. (John Wiley, 1981), 189–201.

3. O. Ore, Hamilton connected graphs, *J. Math. Pures Appl.* **42** (1963), 21–27.

4. O. Ore, Note on Hamilton circuits, *Amer. Math. Monthly* **67** (1960), 55.

UNIVERSITY OF ABERDEEN

Present address:
12 CLIFF TERRACE
BUCKIE
BANFFSHIRE
SCOTLAND
AB5 1LX