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THE TWO-SIDED FACTORIZATION OF ORDINARY DIFFERENTIAL OPERATORS

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1. Introduction. Throughout this paper we shall use I to denote a given interval, not necessarily bounded, of real numbers and C^n to denote the real valued n times continuously differentiable functions on I and C^0 will be abbreviated to C. By a *differential operator of order n* we shall mean a linear function $L: C^n \to C$ of the form

(1.1) $$Ly = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \ldots + p_1 y' + p_0 y, \; y \in C^n,$$

where $p_n(x) \neq 0$ for $x \in I$ and $p_j \in C^j$, $0 \leqq j \leqq n$. The function p_n is called the *leading coefficient* of L.

It is well known (see, for example, [**2**, pp. 73–74]) that a differential operator L of order n uniquely determines both a differential operator L^* of order n (the *adjoint* of L) and a bilinear form $[\cdot,\cdot]_L$ (the *Lagrange bracket*) so that if D denotes differentiation, we have for $u, v \in C^n$,

(1.2) $$vLu - uL^*v = D([u, v]_L).$$

If L is given by (1.1) then L^* and $[\cdot,\cdot]_L$ are given explicitly by

$$L^*v = \sum_{j=0}^{n} (-1)^j (p_j v)^{(j)}, \quad v \in C^n,$$

and

$$[u, v]_L = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i (v p_k)^{(i)} u^{(k-1-i)}, \quad u, v \in C^n.$$

If $u, v \in C^n$ and $Lu = L^*v = 0$ on I, it follows from (1.2) that $[u, v]_L$ is constant on I. When this constant is zero, we say that u and v are *conjugate* solutions respectively of $Ly = 0$ and $L^*y = 0$; see [**11**, p. 343].

If L is a differential operator of order n and $y \in C^n$, we say that y *satisfies the homogeneous equation of L* or that y is a *solution* of this equation if $Ly = 0$ on I. A set of solutions $\{y_1, \ldots, y_n\}$ of $Ly = 0$ is said to be a *fundamental set* if it is a basis for the linear space of all possible solutions. This linear space is an n dimensional subspace of C^n.

For $y_1, \ldots, y_n \in C^{n-1}$, the *Wronskian* of $y_1, \ldots, y_n$ is defined by

$$W(y_1, \ldots, y_n) = \det(y_j^{(i-1)})_{1 \leqq i,j \leqq n}.$$

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In [**8**] (see also [**9**]) Pólya proved that if, for a differential operator of order n, the equation $Ly = 0$ has solutions $y_1, \ldots, y_n$ so that $W(y_1, \ldots, y_k)(x) \neq 0$ for $x \in I$ and $1 \leqq k \leqq n$, then L can be factorized into a product of first order operators. The converse statement is also true, [**2**, p. 91]. In [**10**], Zettl established that there are differential operators R, Q so that $L = RQ$ if, and only if, there are solutions $y_1, \ldots, y_q$ of $Ly = 0$, where q is the order of Q, so that $W(y_1, \ldots, y_q)(x) \neq 0$ for $x \in I$. For the self-adjoint case, that is when $L = L^*$, Heinz [**4**, Satz 3], gave conditions involving conjugate solutions of $Ly = 0$ which are necessary and sufficient for the factorization $L = R^*Q = Q^*R$. Hill and Nillsen in [**5**] have given a different proof of Heinz's result in the case $Q = R$ using a method which also characterizes the case in which L can be written in the form $L = Q^*DQ$. Further for $L = Q^*Q$ or $L = Q^*DQ$, Theorems 4.1 and 4.3 of [**5**] clarify the relation between the solutions of $Ly = 0$ and those of $Qy = 0$.

The main concern of the present paper is to investigate when the differential operator L may be written as $L = R^*VS$ and in constructing solutions of the equations $Sy = 0$, $Vy = 0$, $V^*y = 0$ and $Ry = 0$ from known solutions of $Ly = 0$ and $L^*y = 0$. Moreover if these known solutions are conjugate then the solutions constructed for $Vy = 0$ and $V^*y = 0$ will be conjugate with respect to V. Iteration of the result will lead to factorizations of the form $L = R_1^* \ldots R_r^* VS_r \ldots S_1$. Our main results then, can be regarded as generalizations of the work of Heinz [**4**] mentioned above and are also closely related to those of Zettl, [**10**]. Theorem 4.3 may also be regarded as an extension of the classical technique of the reduction of the order of a differential operator (see [**6**, p. 121] and [**5**, Section 5]). The methods of proof depend heavily on the fundamental results on factorization of differential operators developed in [**11**]. For basic facts about ordinary differential operators we refer the reader to [**1**] and [**6**].

2. Preliminary results. The purpose of this section is to mention some results to be used in the sequel. The first lemma can be proved easily from (1.2) and the uniqueness of L^* and $[\cdot,\cdot]_L$.

LEMMA 2.1. *Let M, N be differential operators. Then*

$$(MN)^* = N^*M^*.$$

The proof of the next result follows that in [**11**, p. 344].

LEMMA 2.2. *Let L, R, Q be differential operators, so that $L = RQ$. Then Q is uniquely determined by L and R and R is uniquely determined by L and Q.*

THEOREM 2.3. *Let L be given by* (1.1) *where $p_n(x) \neq 0$ for $x \in I$ and*

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$p_j \in C^j$ for $0 \leqq j \leqq n$. Let $y_1, \ldots, y_n$ be a fundamental set of solutions of $Ly = 0$. Then $W(y_1, \ldots, y_n)(x) \neq 0$ for $x \in I$ and the functions

$$(2.1) \qquad y_i^* = (-1)^{n+i} W(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)/p_n W(y_1, \ldots, y_n), \quad 1 \leqq i \leqq n,$$

form a fundamental set of solutions of $L^*y = 0$. Moreover if $L^*y = 0$ and $[y_i, y]_L = 0$ for $1 \leqq i \leqq k$ then y is a linear combination of $y_{k+1}^*, \ldots, y_n^*$.

Proof. The claim can be readily deduced from [**7**, pp. 33–38] and [**11**, p. 348].

The final result of this section is due to Zettl, [**11**, Theorem 1 and Lemma 6]. The more general form in which it is stated can be proved in the manner indicated in [**5**, Theorem 2.5].

THEOREM 2.4. *Let L be given by (1.1) where $p_n(x) > 0$ for $x \in I$. Let α, β be real numbers with $\alpha + \beta = 1$. Then there are differential operators R, Q which have respectively leading coefficients p_n^α and p_n^β, and orders $n - k$ and k so that $L = RQ$ if, and only if, there are solutions $y_1, \ldots, y_k$ of $Ly = 0$ so that $W(y_1, \ldots, y_k)(x) \neq 0$ for $x \in I$. When this is the case*

$$Qy = p_n^\beta W(y_1, \ldots, y_k, y)/W(y_1, \ldots, y_k), \text{ for } y \in C^n.$$

Also if y_i^*, $1 \leqq i \leqq k$, are given by (2.1) with k and p_n^β in place of n and p_n respectively, we have

$$R^*y = \sum_{i=1}^{k} (-1)^n [y_i, y]_L y_i^*, \quad \text{for } y \in C^n.$$

In this case $R^*y = 0$ if, and only if, $[y_i, y]_L = 0$ for $1 \leqq i \leqq k$.

3. A Wronskian of Wronskians. The purpose of this section is to prove an identity involving a Wronskian of Wronskians (see Lemma 3.5 below) which will be used later.

$\mathbf{R}^n$ will denote real n-dimensional Euclidean space whose elements we shall think of as $n \times 1$ column matrices. If $x_1, \ldots, x_k \in \mathbf{R}^n$, $(x_1 \ldots x_k)$ will denote the $n \times k$ matrix whose columns are $x_1, \ldots, x_k$ in that order. For $1 \leqq k \leqq n$ the projection map $\pi_k : \mathbf{R}^n \rightarrow \mathbf{R}^k$ is given by saying that for $x \in \mathbf{R}^n$, $\pi_k(x)$ is that element of $\mathbf{R}^k$ obtained by taking the first k coördinates of x.

LEMMA 3.1. *Let $x_1, \ldots, x_n \in \mathbf{R}^n$ and let $1 \leqq k < n$. If*

$$(3.1) \qquad \det(\pi_{k+1}(x_1) \ldots \pi_{k+1}(x_k)\pi_{k+1}(x_j)) = 0 \quad \text{for } k+1 \leqq j \leqq n,$$

then either

$$(3.2) \qquad \det(x_1 \ldots x_n) = 0,$$

or

$$(3.3) \qquad \det(\pi_k(x_1) \ldots \pi_k(x_k)) = 0.$$

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Proof. Since (3.1) holds, we can find for each $j = k+1, \dots, n$, numbers α_{ij}, β_j, $1 \leqq i \leqq k$ so that

$$\sum_{i=1}^{k} \alpha_{ij} \pi_{k+1}(x_i) + \beta_j \pi_{k+1}(x_j) = 0, \tag{3.4}$$

and

$$(\alpha_{1j}, \dots, \alpha_{kj}, \beta_j) \neq (0, \dots, 0, 0). \tag{3.5}$$

If $\beta_j = 0$ for some j, we deduce from (3.4) that

$$\sum_{i=1}^{k} \alpha_{ij} \pi_k(x_i) = 0,$$

so that (3.3) holds.

Suppose then that $\beta_j \neq 0$ for $k+1 \leqq j \leqq n$. Let

$$w_j = \beta_j^{-1} \sum_{i=1}^{k} \alpha_{ij} x_i + x_j.$$

Then

$$\det(x_1 \dots x_n) = \det(x_1 \dots x_k w_{k+1} \dots w_n),$$

and, if γ_{ij} denotes the (i, j)-th entry in $(x_1 \dots x_k w_{k+1} \dots w_n)$, we see from (3.4) that $\gamma_{ij} = 0$ for $1 \leqq i \leqq k+1 \leqq j \leqq n$. Also $A = (\pi_{k+1}(x_1) \dots \pi_{k+1}(x_k))$ is a $(k+1) \times k$ matrix so we may select $\lambda_1, \dots, \lambda_{k+1} \in \mathbf{R}$, not all zero, with the property that if $r_1, \dots, r_{k+1}$ are the rows of A, then

$$\sum_{i=1}^{k+1} \lambda_i r_i = 0.$$

Hence, from our above remark, it follows that the first $k+1$ rows of $(x_1 \dots x_k w_{k+1} \dots w_n)$ are linearly dependent and so (3.2) follows.

LEMMA 3.2. *Let $y_1, \dots, y_n \in C^{n-1}$ and let $1 \leqq k < n$. Then if $x \in I$ and*

$$W(y_1, \dots, y_k, y_j)(x) = 0, \quad \textit{for } k+1 \leqq j \leqq n,$$

then either $W(y_1, \dots, y_k)(x) = 0$ or $W(y_1, \dots, y_n)(x) = 0$.

Proof. The result is immediate on applying Lemma 3.1 to $x_1, \dots, x_n$ where

$$x_i = \begin{bmatrix} y_i(x) \\ y_i'(x) \\ \cdot \\ \cdot \\ \cdot \\ y_i^{(n-1)}(x) \end{bmatrix} \in \mathbf{R}^n.$$

A proof of the next lemma may be found in [**2**, pp. 87–88].

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LEMMA 3.3. *Let* $y_1, \ldots, y_n, y \in C^{n-1}$. *Then the following hold on* I:

$$(3.6)\qquad W(yy_1, \ldots, yy_n) = y^n W(y_1, \ldots, y_n).$$

If $W(y_1, \ldots, y_{n-1})(x) \neq 0$ *and* $W(y_1, \ldots, y_n)(x) \neq 0$ *for* $x \in I$ *and* $y_1, \ldots, y_n, y \in C^n$ *then*

$$(3.7)\qquad \left[\frac{W(y_1, \ldots, y_{n-1}, y)}{W(y_1, \ldots, y_{n-1}, y_n)}\right]' = \frac{W(y_1, \ldots, y_{n-1})W(y_1, \ldots, y_n, y)}{[W(y_1, \ldots, y_n)]^2}.$$

The following result can be readily established using the definition of the Wronskian, the expansion of a determinant along a column and the properties of the sign of a permutation.

LEMMA 3.4. *The following hold on* I. *If* $y_1, \ldots, y_{r-1}, y_{r+1}, \ldots, y^n \in C^{n-1}$, *then*

$$(3.8)\qquad W(y_1, \ldots, y_{r-1}, 1, y_{r+1}, \ldots, y_n) = (-1)^{r+1} W(y_1', \ldots, y_{r-1}', y_{r+1}', \ldots, y_n').$$

If $y_1, \ldots, y_n \in C^{n-1}$ *and* σ *is a permutation of* $\{1, 2, \ldots, n\}$ *with sign denoted by* sign σ, *then*

$$(3.9)\qquad W(y_{\sigma(1)}, \ldots, y_{\sigma(n)}) = (\operatorname{sign} \sigma) W(y_1, \ldots, y_n).$$

LEMMA 3.5. *Let* $y_1, \ldots, y_n \in C^{n-1}$ *and let* $1 \leqq k < n$. *Then*

$$(3.10)\qquad W(W(y_1, \ldots, y_k, y_{k+1}), W(y_1, \ldots, y_k, y_{k+2}), \ldots, W(y_1, \ldots, y_k, y_n)) = W(y_1, \ldots, y_k)^{n-k-1} W(y_1, \ldots, y_n).$$

Proof. We first introduce some simplifying notation. Let

$$W_k = W(y_1, \ldots, y_k)$$

and

$$W_{k,n} = W(W(y_1, \ldots, y_k, y_{k+1}), \ldots, W(y_1, \ldots, y_k, y_n)).$$

For $k + 1 \leqq j_i \leqq n$, $i = 1, \ldots, r$ where $r \leqq n - k$ we set

$$W_k(j_1, \ldots, j_r) = W(y_1, \ldots, y_k, y_{j_1}, \ldots, y_{j_r}).$$

If $k = n - 1$ both sides of (3.10) equal $W(y_1, \ldots, y_n)$ so that the result holds. Assume $1 \leqq k < n - 1$ and let $x \in I$ be given. We seek to construct permutations $\tau_0, \tau_1, \ldots, \tau_{n-k-1}$ of $\{k + 1, \ldots, n\}$ so that the following hold:

$$(3.11)\qquad \tau_r(k + i) = \tau_p(k + i), \qquad \text{for } 1 \leqq i \leqq r \leqq p \leqq n - k - 1,$$

$$(3.12)\qquad \tau_p(k + p + 1) < \tau_p(k + p + 2) < \ldots < \tau_p(n), \qquad \text{for } 0 \leqq p < n - k - 1,$$

$$(3.13)\qquad W_k(\tau_p(k + 1), \ldots, \tau_p(k + p))(x) \neq 0, \qquad \text{for } 1 \leqq p \leqq n - k - 1,$$

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and

$$(3.14)\quad W_{k,n}(x)[W_k(\tau_p(k+1),\dots,\tau_p(k+p))](x)^{n-k-p-1} = (\text{sign } \tau_p)W_k(x)^{n-k-1}W[W_k(\tau_p(k+1),\dots,\tau_p(k+p), \tau_p(k+p+1)), \dots, W_k(\tau_p(k+1),\dots,\tau_p(k+p),\tau_p(n))](x),$$
$$\text{for } 1 \leqq p \leqq n-k-1.$$

Condition (3.12) forces us to take τ_0 equal to the identity permutation; (3.11), (3.13) and (3.14) are vacuous for $p = 0$. Suppose that such permutations $\tau_0, \dots, \tau_q$, $q < n - k - 1$, have been found. We shall show that either (3.10) holds at x with both sides equal to zero or that there is a permutation τ_{q+1} so that (3.11)–(3.14) now hold for $\tau_0, \dots, \tau_{q+1}$.

Consider the numbers

$$(3.15)\quad W_k(\tau_q(k+1),\dots,\tau_q(k+q),\tau_q(k+i))(x),$$
$$i = q+1,\dots,n-k.$$

If all these are zero we may apply Lemma 3.2 to assert that either

$$W_k(\tau_q(k+1),\dots,\tau_q(k+q))(x) = 0$$

or that

$$W_k(\tau_q(k+1),\dots,\tau_q(n))(x) = 0.$$

For $q = 0$, with the first alternative interpreted as $W_k(x) = 0$, we see that both sides of (3.10) are zero. For $q > 0$, condition (3.13) forces the second alternative to hold and then $W(y_1,\dots,y_n)(x) = 0$, so that again both sides of (3.10) are zero, where we have used (3.13) and (3.14) with q in place of p.

Consider, then, the case when not all the numbers in (3.15) are zero. We select i_0, $q + 1 \leqq i_0 \leqq n - k$ so that

$$W_k(\tau_q(k+1),\dots,\tau_q(k+q),\tau_q(k+i_0))(x) \neq 0.$$

(For $q = 0$ this is to be interpreted as $W_k(k + i_0)(x) \neq 0$.) The permutation τ_{q+1} of $\{k+1,\dots,n\}$ is then given by

$$\tau_{q+1}(k+i) = \tau_q(k+i), \text{ for } 1 \leqq i \leqq q,$$

$$\tau_{q+1}(k+q+1) = \tau_q(k+i_0), \text{ and}$$

$$\tau_{q+1}(k+q+2) < \tau_{q+1}(k+q+3) < \dots < \tau_{q+1}(n).$$

It is easy to check that (3.11), (3.12) and (3.13) hold for $\tau_0, \dots, \tau_{q+1}$ and further that

$$(3.16)\quad \text{sign } \tau_{q+1} = (-1)^{i_0+1}(\text{sign } \tau_q),$$

$$(3.17)\quad \tau_q(k+i) = \tau_{q+1}(k+i+1), \text{ for } q+1 \leqq i \leqq i_0 - 1,$$

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and

$$(3.18) \quad \tau_q(k+i) = \tau_{q+1}(k+i), \text{ for } i_0+1 \leqq i \leqq n-k.$$

We must also check that (3.14) holds for $p = q + 1$. Using (3.6) and (3.8) we obtain

$$\begin{aligned}
&W[W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(k+q+1)), \ldots, \\
&\quad W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(n))](x) \\
&\quad = [W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(k+i_0))(x)]^{n-k-q} \\
&\quad \times W\left[\frac{W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(k+q+1))}{W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(k+i_0))}, \ldots,\right. \\
&\qquad \left. 1, \ldots, \frac{W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(n))}{W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(k+i_0))}\right](x) \\
&\quad = (-1)^{i_0+1}[W_k(\tau_q(k+1), \ldots, \tau_q(k+q))(x)]^{n-k-q-1} \\
&\quad \times [W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(k+i_0))(x)]^{-n+k+q+2} \\
&\quad \times W[W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(k+i_0), \tau_q(k+q+1)) \\
&\qquad , \ldots, W_k(\tau_q(k+1), \ldots, \tau_q(k+q), \tau_q(k+i_0), \tau_q(n))](x).
\end{aligned}$$

We use properties (3.16), (3.17) and (3.18) and rewrite (3.14) (with $p = q$) using the above calculations. After some tedious manipulations we arrive at (3.14) with $p = q + 1$.

This inductive step may be carried out $n - k - 1$ times unless for some $q < n - k - 1$ all the numbers in (3.15) are zero. In this latter case we have seen that (3.10) holds at x. In the former we let $p = n - k - 1$ in (3.14) to obtain

$$\begin{aligned}
W_{k,n}(x) &= (\operatorname{sign} \tau_{n-k-1})(W_k(x))^{n-k-1} W_k(\tau_{n-k-1}(k+1), \ldots, \\
&\qquad\qquad \tau_{n-k-1}(n))(x) \\
&= (\operatorname{sign} \tau_{n-k-1})^2 (W_k(x))^{n-k-1} W_k(y_{k+1}, \ldots, y_n)(x)
\end{aligned}$$

by (3.9). This establishes the result at x which was an arbitrary point of I, and so the lemma is proved.

This identity was originally proved by Frobenius [**3**, p. 247] by a method requiring the additional hypothesis that for each $x \in I$ none of $W(y_1)(x)$, $W(y_1, y_2)(x), \ldots, W(y_1, \ldots, y_k)(x)$ is zero. Our proof avoids this assumption and this feature will be important for subsequent applications.

4. Factorization of differential operators. The first result of this section is a generalization of Satz 3 in [**4**] and Theorem 3.1 in [**5**].

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THEOREM 4.1. *Let L be a differential operator of order $n \geqq 2$ where L is given by* (1.1) *and $p_n(x) > 0$ for $x \in I$. The following conditions on L are equivalent.*

(4.1) *There are differential operators, R, S and V of orders r, s and $n - r - s$ respectively so that $L = R^*VS$.*

(4.2) *There are solutions $y_1, \ldots, y_s$ of $Ly = 0$ on I and $z_1, \ldots, z_r$ of $L^*y = 0$ on I so that for $x \in I$, $W(y_1, \ldots, y_s)(x) \neq 0$, $W(z_1, \ldots, z_r)(x) \neq 0$ and $[y_i, z_j]_L = 0$ for $1 \leqq i \leqq s$, $1 \leqq j \leqq r$.*

When these conditions hold $r + s \leqq n$ and R, S may be taken to be given by

(4.3) $$Ry = W(z_1, \ldots, z_r, y)/W(z_1, \ldots, z_r), \quad \text{for } y \in C^n,$$

and

(4.4) $$Sy = W(y_1, \ldots, y_s, y)/W(y_1, \ldots, y_s), \quad \text{for } y \in C^n.$$

Assume further that $y_{s+1}, \ldots, y_q$ are also solutions of $Ly = 0$ and that $z_{r+1}, \ldots, z_p$ are solutions of $L^*y = 0$ so that $[y_i, z_j]_L = 0$ when $1 \leqq i \leqq q$, $1 \leqq j \leqq r$ or when $1 \leqq i \leqq s$, $1 \leqq j \leqq p$. Then the following hold.

(4.5) *The functions*

$$W(y_1, \ldots, y_s, y_j)/W(y_1, \ldots, y_s), \quad s + 1 \leqq j \leqq q,$$

are solutions of $Vy = 0$ and are linearly independent provided that $W(y_1, \ldots, y_q)$ is not identically zero on I. If this is the case then $r + q \leqq n$ and if $n = r + q$ these functions form a fundamental set of solutions of $Vy = 0$ and the functions

$$W(y_1, \ldots, y_s, y_{s+1}, \ldots, y_{i-1}, y_{i+1}, \ldots, y_q)/p_nW(y_1, \ldots, y_q),$$
$$s + 1 \leqq i \leqq q,$$

form a fundamental set of solutions of $V^*y = 0$.

(4.6) *The functions*

$$W(z_1, \ldots, z_r, z_j)/W(z_1, \ldots, z_r), \; r + 1 \leqq j \leqq p,$$

are solutions of $V^*y = 0$ and are linearly independent provided that $W(z_1, \ldots, z_p)$ is not identically zero on I. If this is the case then $s + p \leqq n$ and if $n = s + p$ these functions form a fundamental set of solutions of $V^*y = 0$ and the functions

$$W(z_1, \ldots, z_r, z_{r+1}, \ldots, z_{i-1}, z_{i+1}, \ldots, z_p)/p_nW(z_1, \ldots, z_r),$$
$$r + 1 \leqq i \leqq p,$$

form a fundamental set of solutions of $Vy = 0$.

Now assume further that for $x \in I$, $W(y_1, \ldots, y_q)(x) \neq 0$,

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$W(z_1, \ldots, z_p)(x) \neq 0$ and that $[y_i, z_j]_L = 0$ for $1 \leqq i \leqq q$, $1 \leqq j \leqq p$. Then the following hold.

(4.7) *$p + q \leqq n$ and the functions*

$$W(y_1, \ldots, y_s, y_j)/W(y_1, \ldots, y_s),\ s + 1 \leqq j \leqq q,$$

and

$$W(z_1, \ldots, z_r, z_j)/W(z_1, \ldots, z_r), \quad r + 1 \leqq j \leqq p,$$

are mutually conjugate solutions with respect to V of $Vy = 0$ and $V^*y = 0$ respectively.

(4.8) *V can be factorized as $V = G^*WH$ where G and H have leading coefficients 1 and are of orders $p - r$ and $q - s$ respectively. The functions in (4.7) form fundamental sets of solutions for $Hy = 0$ and $Gy = 0$ respectively.*

Proof. Let (4.1) hold. By Lemma 2.1 there is no loss of generality in assuming that R and S both have leading coefficient 1. Further, $L^* = S^*V^*R$ and by Theorem 2.4 applied to L and L^* we deduce that there are solutions $y_1, \ldots, y_s$ of $Ly = 0$ and $z_1, \ldots, z_r$ of $L^*y = 0$ so that, for $x \in I$, $W(y_1, \ldots, y_s)(x) \neq 0$, $W(z_1, \ldots, z_r)(x) \neq 0$, R is given by (4.3) and S by (4.4). Theorem 2.4 also yields

$$(4.9) \qquad V^*Ry = (R^*V)^*y = \sum_{i=1}^{s} (-1)^n [y_i, y]_L y_i^*, \quad \text{for } y \in C^n,$$

where y_i^* is given by (2.1) with s in place of n and $p_n = 1$. Since $Rz_j = 0$ for $1 \leqq j \leqq r$ and since $y_1^*, \ldots, y_s^*$ are linearly independent by Theorem 2.3 we deduce that $[y_i, z_j]_L = 0$ for $1 \leqq i \leqq s$, $1 \leqq j \leqq r$.

Conversely, let (4.2) hold. By Theorem 2.4 we may write $L = WS$ where S is given by (4.4) and

$$W^*y = \sum_{i=1}^{s} (-1)^n [y_i, y]_L y_i^*, \quad \text{for } y \in C^n,$$

where y_i^* is obtained from (2.1) *mutatis mutandis*. As $[y_i, z_j]_L = 0$, we deduce that $W^*z_j = 0$ for $1 \leqq j \leqq r$ so that by Theorem 2.4, $W^* = V^*R$ where R is given by (4.3). Since $W = R^*V$ we now have $L = R^*VS$ and (4.1) holds.

The fact that, in this case, $r + s \leqq n$ follows immediately from the last statement in Theorem 2.3. The claims that $r + q \leqq n$, $s + p \leqq n$ and $p + q \leqq n$ in (4.5), (4.6) and (4.7) respectively follow in the same way.

To prove (4.6), observe from (4.9) that Rz_j, $r + 1 \leqq j \leqq p$, are solutions of $V^*y = 0$. This proves the first part of (4.6). Also if Lemmas 3.3 and 3.5 are invoked we see that

$$W(Rz_{r+1}, \ldots, Rz_p) = W(z_1, \ldots, z_p)/W(z_1, \ldots, z_r),$$

which proves the statement about linear independence. If $n = s + p$,

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then $p - r = n - r - s$ so that $Rz_{r+1}, \ldots, Rz_p$ form a fundamental set of solutions of $V^*y = 0$ as V^* has order $n - r - s$. If Lemmas 3.3 and 3.5 are used to calculate

$$W(Rz_{r+1}, \ldots, Rz_{i-1}, Rz_{i+1}, \ldots, Rz_p)/W(Rz_{r+1}, \ldots, Rz_p), \qquad \text{for } r + 1 \leqq i \leqq p,$$

the remainder of (4.6) follows from Theorem 2.3.

Now (4.5) can be proved analogously by applying Theorem 2.4 to $L^* = S^*V^*R$.

To prove (4.8) first observe that since (4.1) and (4.2) are equivalent we may write $L = P^*WQ$ where 1 is the leading coefficient of both P and Q and y_i, $1 \leqq i \leqq q$, z_i, $1 \leqq i \leqq p$ are respectively fundamental sets of solutions of $Qy = 0$ and $Py = 0$. Since (4.2) holds, Theorem 2.4 can be applied to find operators G, H of leading coefficient 1 so that $Q = HS$ and $P = GR$. Thus

$$L = R^*WS = P^*WQ = R^*(G^*WH)S,$$

so that $V = G^*WH$ by Lemma 2.2. Also $H(Sy_j) = 0$ for $s + 1 \leqq j \leqq q$ and $G(Rz_j) = 0$ for $r + 1 \leqq j \leqq p$. H has order $q - s$ and the Sy_j, $s + 1 \leqq j \leqq q$ are linearly independent by (4.4) and (4.5), so we see that $Sy_{s+1}, \ldots, Sy_q$ form a fundamental set of solutions of $Hy = 0$. The corresponding statement for G is proved in like manner and so (4.8) is established.

Finally we use Theorem 2.4 again to obtain for $y \in C^{n-r-s}$,

$$W^*Gy = (G^*W)^*y = \sum_{i=s+1}^{q} (-1)^{n-r-s}[Sy_i, y]_V(Sy_i)^*$$

where $(Sy_i)^*$ is obtained from (2.1) *mutatis mutandis*. Since $G(Rz_i) = 0$ for $r + 1 \leqq i \leqq p$ and since the $(Sy_i)^*$, $s + 1 \leqq i \leqq q$, are independent, we deduce that $[Sy_i, Rz_j]_V = 0$ for $s + 1 \leqq i \leqq q$, $r + 1 \leqq j \leqq p$. By (4.3) and (4.4), this proves (4.7).

We should remark that the requirements that $W(y_1, \ldots, y_q)$, $W(z_1, \ldots, z_p)$ be not identically zero in (4.5) and (4.6) could be replaced equivalently by demanding that $y_1, \ldots, y_q$ and $z_1, \ldots, z_p$ be linearly independent respectively, for in each case the functions in question are solutions of a common differential equation (see [**6**, p. 118]).

COROLLARY 4.2. *Consider the case where $L = Q^*DQ$ where Q is a differential operator of order r. Then there is a fundamental set $y_1, \ldots, y_r$ of solutions of $Qy = 0$ so that $W(y_1, \ldots, y_r)(x) \neq 0$ for $x \in I$ and $[y_i, y_j]_L = 0$ for $i, j = 1, \ldots, n$. Also there is a one dimensional vector space $\mathscr{V}$ of functions so that if $f \in \mathscr{V}$,*

$$Lf = 0, [y_i, f]_L = 0, \quad \textit{for } 1 \leqq i \leqq n$$

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and $W(y_1, \ldots, y_r, f)/W(y_1, \ldots, y_r)$ *is constant on* I. $\mathscr{V}$ *cannot be chosen to have dimension greater than one.*

Proof. From [**5**, Theorem 4.1] it follows that if $\mathscr{V}$ is the set of functions so that $Lf = 0$ and $[y_i, f]_L = 0$ for $1 \leqq i \leqq r$, then $\mathscr{V}$ is one dimensional. The result we have just proved shows that for $f \in \mathscr{V}$

$$D[W(y_1, \ldots, y_r, f)/W(y_1, \ldots, y_r)] = 0,$$

and so the result is proved.

THEOREM 4.3. *Let L be a differential operator of order $n \geqq 4$ and let L be given by* (1.1) *where $p_n(x) > 0$ for $x \in I$. Let integers $s_0 = 0 < s_1 < s_2 < \ldots < s_p$ and $r_0 = 0 < r_1 < r_2 < \ldots < r_p$ be given. Then the following conditions are equivalent.*

(4.10) *There are differential operators $S_1, \ldots, S_p$, $V_1, \ldots, V_p$ and $R_1, \ldots, R_p$ so that S_j has order $s_j - s_{j-1}$, V_j has order $n - r_j - s_j$ and R_j has order $r_j - r_{j-1}$, and*

$$L = R_1^* \ldots R_j^* V_j S_j \ldots S_1, \quad \text{for } 1 \leqq j \leqq p.$$

(4.11) *There are solutions $y_1, \ldots, y_{s_p}$ of $Ly = 0$ and solutions $z_1, \ldots, z_{r_p}$ of $L^*y = 0$ so that for $x \in I$ and $1 \leqq i, j \leqq p$,*

$$W(y_1, \ldots, y_{s_j})(x) \neq 0,\ W(z_1, \ldots, z_{r_j})(x) \neq 0 \text{ and}$$

$$[y_i, z_j]_L = 0; \text{ for } 1 \leqq i \leqq s_p,\ 1 \leqq j \leqq r_p.$$

When these conditions are satisfied the operators in (4.10) *may be chosen so that the following hold.*

(4.12) $V_j = R_{j+1}^* V_{j+1} S_{j+1}$, *for $0 \leqq j \leqq p - 1$ where $V_0 = L$.*

(4.13) *$y_1, \ldots, y_{s_j}$ is a fundamental set of solutions of $(S_j \ldots S_1)y = 0$ and $z_1, \ldots, z_{r_j}$ is a fundamental set of solutions of $(R_j \ldots R_1)y = 0$ for $1 \leqq j \leqq p$.*

(4.14) *The $s_p - s_j$ functions*

$$W(y_1, \ldots, y_{s_j}, y_k)/W(y_1, \ldots, y_{s_j}),\ s_j + 1 \leqq k \leqq s_p,$$

are linearly independent solutions of $V_jy = 0$, $1 \leqq j \leqq p - 1$ and the corresponding $s_{j+1} - s_j$ functions obtained as above by restricting k to the range $s_j + 1 \leqq k \leqq s_{j+1}$ form a fundamental set of solutions of $S_{j+1}y = 0$ for $1 \leqq j \leqq p - 1$.

(4.15) *The $r_p - r_j$ functions*

$$W(z_1, \ldots, z_{r_j}, z_k)/W(z_1, \ldots, z_{r_j}),\ r_j + 1 \leqq k \leqq r_p,$$

are $r_p - r_j$ linearly independent solutions of $V_j^*y = 0$, $1 \leqq j \leqq p - 1$ and the corresponding $r_{j+1} - r_j$ functions obtained as above by restricting k to

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the range $r_j + 1 \leqq k \leqq r_{j+1}$ *form a fundamental set of solutions of* $R_{j+1}y = 0$ *for* $1 \leqq j \leqq p - 1$.

(4.16) *The functions in* (4.14) *for* $s_j + 1 \leqq k \leqq s_p$ *and the functions in* (4.15) *for* $r_j + 1 \leqq k \leqq r_p$ *are mutually conjugate solutions with respect to* V_j *of* $V_jy = 0$ *and* $V_j^*y = 0$ *respectively for* $1 \leqq j \leqq p - 1$.

Proof. Let (4.11) hold. By Theorem 4.1 there are operators M_j, N_j, V_j for $1 \leqq j \leqq p$ so that $L = M_j^* V_j N_j$, M_j is given by (4.3) with r_j in place of r, N_j is given by (4.4) with s_j in place of s and V_j has order $n - r_j - s_j$. Theorem 2.4 shows that if $1 \leqq j \leqq p - 1$, we may write $M_{j+1} = R_{j+1}M_j$ and $N_{j+1} = S_{j+1}N_j$ where R_{j+1} has order $r_{j+1} - r_j$ and S_{j+1} has order $s_{j+1} - s_j$. Thus

$$L = M_j^* V_j N_j = M_{j+1}^* V_{j+1} N_{j+1} = M_j^*(R_{j+1}^* V_{j+1} S_{j+1}) N_j \qquad \text{for } 1 \leqq j \leqq p - 1,$$

so by Lemma 2.2 we deduce that $V_j = R_{j+1}^* V_{j+1} S_{j+1}$. If we define $R_1 = M_1$ and $S_1 = N_1$ then (4.12) is proved.

Also from the above we have $M_j = R_j \ldots R_2 R_1$ and $N_j = S_j \ldots S_2 S_1$ and now (4.13) is immediate, as also is (4.10). Since $L = M_j^* V_j N_j$, (4.14) follows from (4.5) and (4.15) from (4.6). Finally (4.16) is an easy consequence of (4.7).

This shows that (4.11) implies all other conditions in the theorem. That (4.10) implies (4.11) comes from the equivalence of the two conditions (4.1) and (4.2) and is straightforward. This completes the proof.

In Section 5 of [**5**] results have been obtained concerning the successive reduction of the order of L by 2. This corresponds to the situation in Theorem 4.3 of having $r_j = s_j = j$ and $L^* = \pm L$. Thus the factorization in (4.10) of L can be regarded as a process which successively reduces the order of L by $r_1 + s_1, r_2 + s_2, \ldots, r_p + s_p$. However whereas the argument used in [**5**] depends upon properties of the Lagrange bracket, the procedure here is based more directly on Theorem 2.4. It is possible that the Lagrange bracket plays a significant rôle in these more general factorization results.

Remark. This work was done while the second author was visiting The University of Calgary.

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