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ELATION GENERALIZED QUADRANGLES FOR WHICH THE NUMBER OF LINES ON A POINT IS THE SUCCESSOR OF A PRIME

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Abstract

We show that an elation generalized quadrangle that has $p + 1$ lines on each point, for some prime p, is classical or arises from a flock of a quadratic cone (that is, is a *flock quadrangle*).

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1. Introduction

A *generalized quadrangle* is an incidence structure of points and lines such that, if P is a point and ℓ is a line not incident with P, then there is a unique line through P that meets ℓ in a point. From this property, if there is a line containing at least three points or if there is a point on at least three lines, then one can see that there are constants s and t such that each line is incident with $t + 1$ points, and each point is incident with $s + 1$ lines. Such a generalized quadrangle is said to have *order* (s, t), and hence its point–line dual is a generalized quadrangle of order (t, s). Of the known generalized quadrangles, most admit a group of elations (see Section 2 for a definition) and are called *elation generalized quadrangles*. In this paper, we will be interested in elation generalized quadrangles where the parameter t is prime.

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If $\mathcal{S}$ is an elation generalized quadrangle of order (p, t), for some prime p, then the elation group G is a p-group (see [Fro88, Lemma 6], and note that s and t are interchanged!). In this situation, we have, by a deep result of Bloemen, Thas and Van Maldeghem [BTVM96], that $\mathcal{S}$ is isomorphic to one of the classical generalized quadrangles $\mathrm{W}(p)$, $\mathrm{Q}(4, p)$, or $\mathrm{Q}(5, p)$. The same is not true if we interchange points and lines. Suppose that $\mathcal{S}$ is an elation generalized quadrangle of order (s, p) (where p is a prime). Again, by a result of Frohardt [Fro88, Lemma 6], we have that the elation group is a p-group; however, there exist candidates for $\mathcal{S}$ that are not classical but are known as *flock quadrangles*. These elation generalized quadrangles are obtained from a flock of $\mathrm{PG}(3, p)$ (a partition of the points of a quadratic cone of $\mathrm{PG}(3, p)$, minus its vertex, into conics) and they have order (p^2, p). Such a quadrangle is classical if and only if the flock is linear; and there do exist nonlinear flocks for p a prime at least 5 (see [PT84, Section 10.6]). In this paper, we prove the following result that is complementary to that of Bloemen, Thas and Van Maldeghem.

THEOREM 1.1. *If p is a prime, then an elation generalized quadrangle of order (s, p) is classical or a flock quadrangle.*

Note that the above theorem does not hold when p is replaced by a prime power since the duals of the Tits quadrangles $T_3(O)$ arising from the Tits ovoids are elation generalized quadrangles of order (q^2, q) (for $q = 2^h$ and h an odd number at least 3) that are not flock quadrangles, and the Roman elation generalized quadrangles of Payne are of order (q^2, q) (for $q = 3^h$, $h > 2$) but are not flock quadrangles.

The proof of Theorem 1.1 relies on the following result concerning Kantor families for groups of order p^5 (see Section 2 for a definition of Kantor families).

THEOREM 1.2. *If p is an odd prime and G is a finite p-group of order p^5 that admits a Kantor family of order (p^2, p), then G is an extraspecial group of exponent p.*

In Sections 2 and 3, we briefly revise the basic background theory and definitions needed for this paper. Kantor families for groups of order p^5 are then investigated in Section 4, and Theorem 1.2 is proved in Section 5. Finally in Section 6 we prove Theorem 1.1.

Though our group theoretic notation is standard, we briefly review it for the sake of a reader whose interest lies more in geometry than in group theory. If a is a group element of order p and $\alpha \in \mathbb{F}_p$ then, identifying α with an element in $\{0, \ldots, p-1\}$, we may write a^α. If a and b are group elements, then we define their *commutator* as $[a, b] = a^{-1}b^{-1}ab$. The properties of group commutators that we need in this paper are listed, for instance, in [Rob96, Section 5.1.5]. The *centre* of a group G consists of those elements $z \in G$ that satisfy $[g, z] = 1$ for all $g \in G$. If H, K are subgroups of a group G, then the *commutator subgroup* $[H, K]$ is generated by all commutators $[a, b]$ where $a \in H$ and $b \in K$. The *derived subgroup* G' of G is defined as $[G, G]$. The symbol $\gamma_i(G)$ denotes the ith term of the *lower central series* of G; that is $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and, for $i \geqslant 3$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$. The *nilpotency class* of a p-group is the smallest c such that $\gamma_{c+1}(G) = 1$. The *Frattini subgroup* $\Phi(G)$ of a finite group

G is the intersection of all the maximal subgroups. If G is a finite p-group, then $\Phi(G) = G'G^p$ and $\log_p |G : \Phi(G)|$ is the size of a minimal set of generators for G. The basic properties of the Frattini subgroup of a p-group can be found, for instance, in [Rob96, Section 5.3]. The *exponent* of a finite group G is the smallest positive n such that $g^n = 1$ for all $g \in G$.

2. Generalized quadrangles and Kantor families

2.1. The basics A (finite) generalized quadrangle is an incidence structure of points $\mathcal{P}$, lines $\mathcal{L}$, together with a symmetric point–line incidence relation satisfying the following axioms:

(i) each point lies on $t + 1$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line;
(ii) each line contains $s + 1$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point; and
(iii) if P is a point and ℓ is a line not incident with P, then there is a unique point on ℓ collinear with P.

We say that our generalized quadrangle has order (s, t) (or order s if $s = t$), and the point–line dual of a generalized quadrangle of order (s, t) is again a generalized quadrangle but of order (t, s). Higman's inequality states that the parameters s and t bound one another; that is, if $s, t > 1$ then $t \leqslant s^2$ and, dually, $s \leqslant t^2$. A collineation θ of $\mathcal{S}$ is an *elation* about the point P if it is either the identity collineation, or it fixes each line incident with P and fixes no point not collinear with P. If there is a group G of elations of $\mathcal{S}$ about the point P such that G acts regularly on the points not collinear with P, then we say that $\mathcal{S}$ is an *elation generalized quadrangle* with elation group G and *base point* P. Necessarily, G has order s^2t.

The *classical generalized quadrangles* $W(q)$, $Q(4, q)$, $H(3, q^2)$, $Q(5, q)$ and $H(4, q^2)$ are elation generalized quadrangles and arise as polar spaces of rank 2. The first of these is the incidence structure of all points of $PG(3, q)$ and totally isotropic lines with respect to a null polarity, and is a generalized quadrangle of order q. The point–line dual of $W(q)$ is $Q(4, q)$, the parabolic quadric of $PG(4, q)$, and is therefore a generalized quadrangle of order q (see [PT84, 3.2.1]). The incidence structure of all points and lines of a nonsingular Hermitian variety in $PG(3, q^2)$, which forms the generalized quadrangle $H(3, q^2)$ of order (q^2, q), has as its point–line dual the elliptic quadric $Q(5, q)$ in $PG(5, q)$, which is a generalized quadrangle of order (q, q^2) (see [PT84, 3.2.3]). The remaining classical generalized quadrangle, $H(4, q^2)$, is the incidence structure of all points and lines of a nonsingular Hermitian variety in $PG(4, q^2)$, and is of order (q^2, q^3) (see [PT84, 3.1.1]).

2.2. Kantor families Now standard in the theory of elation generalized quadrangles are the equivalent objects known commonly as *4-gonal families* or *Kantor families* (after their inventor). Let G be a group of order s^2t and suppose there exist two families of subgroups $\mathcal{F} = \{A_0, \dots, A_t\}$ and $\mathcal{F}^* = \{A_0^*, \dots, A_t^*\}$ of G such that:

(a) every element of $\mathcal{F}$ has order s and every element of $\mathcal{F}^*$ has order st;
(b) $A_i \leqslant A_i^*$ for all i;
(c) $A_i \cap A_j^* = 1$ for $i \neq j$ (the 'tangency condition'); and
(d) $A_i A_j \cap A_k = 1$ for distinct i,j,k (the 'triple condition').

Then the triple $(G, \mathcal{F}, \mathcal{F}^*)$ is called a *Kantor family*, but we will also say that $(\mathcal{F}, \mathcal{F}^*)$ is a *Kantor family for* G. The pair (s, t) is said to be the *order* of $(\mathcal{F}, \mathcal{F}^*)$. From a Kantor family as described above, we can define a point–line incidence structure as follows.

Points	Lines
elements g of G	right cosets $A_i g$
right cosets $A_i^* g$	symbols $[A_i]$
a symbol ∞	

Note that $A_i \in \mathcal{F}$, $A_i^* \in \mathcal{F}^*$, $g \in G$. Incidence comes in four flavours (points on the left, lines on the right):

$$\begin{array}{rcll} g & \sim & A_i g & \\ A_i^* g & \sim & [A_i] & \\ A_i^* g & \sim & A_i h, & \text{where } A_i h \subseteq A_i^* g, \\ \infty & \sim & [A_i]. & \end{array}$$

It turns out that this incidence structure is an elation generalized quadrangle of order (s, t) with base point ∞ and elation group G. Remarkably, all elation generalized quadrangles arise this way [PT84, Section 8.2], and we obtain a so-called *translation generalized quadrangle* when G is abelian [PT84, 8.2.3].

3. Flock quadrangles and special groups

3.1. Flock generalized quadrangles A q*-clan* is a set of 2×2 matrices over GF(q), of size q, the difference of any two being anisotropic. Payne introduced q-clans in [Pay85], and used them to construct elation generalized quadrangles of order (q^2, q). A *flock* of the quadratic cone $\mathcal{C}$ with vertex v in PG$(3, q)$ is a partition of the points of $\mathcal{C}\backslash\{v\}$ into conics. Thas [Tha87] showed that a flock gives rise to an elation generalized quadrangle of order (q^2, q), which we call a *flock quadrangle*. The flocks of PG$(3, q)$ have been classified by Law and Penttila [LP03] for q at most 29. A *BLT-set of lines* of W(q) is a set $\mathcal{L}$ of $q+1$ lines of W(q) such that no line of W(q) is concurrent with more than two lines of $\mathcal{L}$. In [BLT90], it was shown that, for q odd, a flock of a quadratic cone in PG$(3, q)$ gives rise to a BLT-set of lines of W(q). Also for q odd, Knarr [Kna92] gave a direct geometric construction of an elation generalized quadrangle from a BLT-set of lines of W(q). The ingredients of the Knarr construction are as follows:

(i) a symplectic polarity ρ of PG$(5, q)$;
(ii) a point P of PG$(5, q)$;
(iii) a 3-space inducing a W(q) contained in $P^{\perp}$, but not containing P; and
(iv) a BLT-set of lines $\mathcal{L}$ of W(q).

For each element ℓ_i of $\mathcal{L}$, let π_i be the plane spanned by ℓ_i and P. Then we construct an elation generalized quadrangle as follows.

Points	Lines
points of PG$(5, q)$ not in P^{ρ}	totally isotropic planes not contained in P^{ρ} and meeting some π_i in a line
lines of PG$(5, q)$ not incident with P but contained in some π_i	the planes π_i
the point P	

Incidence is inherited from that of PG$(5, q)$.

Kantor [Kan91, Lemma] showed that a Kantor family of the flock elation group that is constructed from a q-clan gives rise to a BLT-set of lines of W(q). We show in Section 6 that, for q prime, any Kantor family of a flock elation group gives rise to a BLT-set of lines of W(q), and the resulting flock quadrangle obtained by the Knarr construction is isomorphic to the elation generalized quadrangle arising from the given Kantor family.

3.2. Special and extraspecial groups A finite p-group G is *special* if its centre, its derived subgroup and its Frattini subgroup coincide. Moreover, we say that a special group is *extraspecial* if its centre is cyclic of prime order. The exponent of a special group is either p or p^2. Further, the order of an extraspecial group is of the form p^{2m+1}, where m is a positive integer. For each such m there are, up to isomorphism, precisely two extraspecial groups of order p^{2m+1}, one with exponent p, and another with exponent p^2 (see [Asc00, Section 8]). The elation groups of the flock quadrangles of order (p^2, p) are extraspecial of exponent p (see [Pay89]).

Here we recall a few facts about extraspecial p-groups that can be readily found in [Asc00, Section 8]. The quotient group $E/Z(E)$ is an elementary abelian p-group forming a vector space V over GF(p). Moreover, the map from V^2 to $Z(E)$ defined by

$$\langle Z(E)x, Z(E)y\rangle = [x, y]$$

defines an alternating form on V. Thus for $m = 2$, we obtain the generalized quadrangle W(p), where the totally isotropic subspaces correspond to abelian subgroups of E properly containing $Z(E)$.

4. Kantor families for p-groups of order p^5

Recall that the elation group of a generalized quadrangle of order (p^2, p), p prime, has order p^5. Thus we provide in this section some powerful tools that will enable us to prove Theorem 1.2.

LEMMA 4.1. *Let $(G, \mathcal{F}, \mathcal{F}^*)$ be a Kantor family giving rise to an elation generalized quadrangle $\mathcal{S}$ of order (s, t). Suppose that H is a subgroup of G of order t^3 such that, for all $A \in \mathcal{F}$ and $A^* \in \mathcal{F}^*$,*

$$|A^* \cap H| \geqslant t^2 \quad \text{and} \quad |A \cap H| \geqslant t.$$

Then

$$(\{A \cap H \mid A \in \mathcal{F}\}, \{A^* \cap H \mid A^* \in \mathcal{F}^*\})$$

is a Kantor family for H giving rise to an elation generalized quadrangle of order t.

PROOF. Suppose that A and B are a pair of distinct elements of $\mathcal{F}$, and let A^* and B^* be the respective elements of $\mathcal{F}^*$ such that $A \leqslant A^*$ and $B \leqslant B^*$. Since A and B^* intersect trivially, we have that

$$|A^*H| \geqslant |A^*(B \cap H)| = \frac{|A^*||B \cap H|}{|A^* \cap B \cap H|} \geqslant st^2.$$

Therefore

$$|A^* \cap H| = |A^*||H|/|A^*H| \leqslant t^2,$$

and so A^* and H intersect in t^2 elements, for all $A^* \in \mathcal{F}^*$. Similarly,

$$|AH| \geqslant |A(B^* \cap H)| = |A||B^* \cap H| \geqslant st^2,$$

and so $|A \cap H| = t$, for all $A \in \mathcal{F}$. The 'triple' and 'tangency' conditions follow from those in $(G, \mathcal{F}, \mathcal{F}^*)$. □

THEOREM 4.2. *Let p be an odd prime. A generalized quadrangle of order (p^2, p) with an elation subquadrangle of order p is isomorphic to* H$(3, p^2)$. *Moreover, the subquadrangle here is isomorphic to* W(p) *and so is not a translation generalized quadrangle.*

PROOF. Let $\mathcal{S}$ be a generalized quadrangle of order (p^2, p) with an elation subquadrangle $\mathcal{S}'$ of order p. By [BTVM96], an elation generalized quadrangle of order p is isomorphic to either W(p) or Q$(4, p)$. Now every line of our given generalized quadrangle of order (p^2, p) induces a spread of the subquadrangle; but Q$(4, p)$ has no spreads for p odd (see [PT84, 3.4.1(i)]). Therefore, $\mathcal{S}'$ is isomorphic to W(p). It was proved by Brown [Bro02], and independently by Brouns, Thas and Van Maldeghem [BTVM02], that if a generalized quadrangle $\mathcal{S}$ of order (q, q^2) has a subquadrangle $\mathcal{S}'$ isomorphic to $Q(4, q)$, and if in $\mathcal{S}'$ each ovoid $\mathcal{O}_X$ consisting of all of the points collinear with a given point X of $\mathcal{S}\backslash\mathcal{S}'$ is an elliptic quadric, then $\mathcal{S}$ is isomorphic to $Q(5, q)$. By a result of Ball, Govaerts and Storme [BGS06], if p is a prime then every ovoid of $Q(4, p)$ is an elliptic quadric. Therefore, by dualizing, we have that $\mathcal{S}$ is isomorphic to $H(3, p^2)$. □

The reason why we have pointed out that the subquadrangle is not a translation generalized quadrangle will become apparent in Section 5. We obtain the following consequence of Theorem 4.2.

LEMMA 4.3. *Let p be a prime and let $(G, \mathcal{F}, \mathcal{F}^*)$ be a Kantor family giving rise to an elation generalized quadrangle $\mathcal{S}$ of order (p^2, p). Suppose that H is a subgroup of G of order p^3 with the property that, for all $A^* \in \mathcal{F}^*$, we have $|A^* \cap H| \geqslant p^2$. Then $\mathcal{S}$ is isomorphic to* $H(3, p^2)$.

PROOF. Let $A \in \mathcal{F}$ and $A^* \in \mathcal{F}^*$ such that $A \leqslant A^*$. The condition $|A^* \cap H| \geqslant p^2$ implies that $A^* H \neq G$. This gives $AH \neq G$, and so $|A \cap H| \geqslant p$. Now it follows from Lemma 4.1 that H gives rise to an elation subquadrangle $\mathcal{S}'$ of order p. The remainder follows from Theorem 4.2. □

For p odd, $W(p)$ is not a translation generalized quadrangle, which implies in the previous lemma that H is nonabelian. The next result gives more information about Kantor families for groups of order p^5.

LEMMA 4.4. *Suppose that G is a group with order p^5 and let $(\mathcal{F}, \mathcal{F}^*)$ be a Kantor family of order (p^2, p) for G. Then the following hold.*

(i) *None of the members of $\mathcal{F}$ is normal in G. In particular, G is nonabelian.*
(ii) *If G is not extraspecial and H is a subgroup of G of order p^3, then there is a subgroup U of G such that $|U| = p^3$ and $HU = G$.*
(iii) *The group G is not generated by two elements.*
(iv) *The nilpotency class of G is two.*
(v) *The subgroup G' is elementary abelian.*

PROOF. If G is an extraspecial group with order p^5, then properties (i), (iii), (iv) and (v) are valid for G, and so we may assume, for the entire proof, that G is not extraspecial.

(i) Assume by contradiction that $A \in \mathcal{F}$ is normal, and choose distinct B, $C \in \mathcal{F} \setminus \{A\}$. Then AB is a subgroup of G with order p^4 and so $AB \cap C = 1$ is impossible, violating the triple condition.

(ii) Let H be a subgroup of G with order p^3. Since the elation group of $H(3, p^2)$ is extraspecial with exponent p, Lemma 4.3 implies that there is $A^* \in \mathcal{F}^*$ such that $|H \cap A^*| = p$, and so $HA^* = G$.

(iii) Since $G/\Phi(G)$ is not cyclic, $|\Phi(G)| \leqslant p^3$. Further, $\Phi(G)U = G$ implies that $U = G$, and hence it follows from part (ii) that $\Phi(G) \neq p^3$. Therefore we obtain that $|\Phi(G)| \leqslant p^2$, and so a minimal generating set of G has at least three elements.

(iv) A group of order p^5 has nilpotency class at most four. If the nilpotency class of G is four, then $|G'| = |\Phi(G)| = p^3$, which is a contradiction by the previous paragraph. We claim that the nilpotency class of G is not three. Suppose by contradiction that it is three. In this case, as G is not generated by two elements, $G/G' \cong C_p \times C_p \times C_p$ and $|G' : \gamma_3(G)| = |\gamma_3(G)| = p$. Choose $a, b \in G$ such that $\langle [a, b]\gamma_3(G)\rangle = G'/\gamma_3(G)$. Let $c_1 \in G$ such that $\langle aG', bG', c_1G'\rangle = G/G'$. Then there are $\alpha, \beta \in \mathbb{F}_p$ such that $[a, c_1] \equiv [a, b]^\alpha \pmod{\gamma_3(G)}$ and $[b, c_1] \equiv [a, b]^\beta \pmod{\gamma_3(G)}$. Set $c = c_1 a^\beta b^{-\alpha}$. Then $\langle aG', bG', cG'\rangle = G/G'$ and $[a, c] \equiv [b, c] \equiv 1 \pmod{\gamma_3(G)}$; that is $[a, c], [b, c] \in \gamma_3(G)$. By the Hall–Witt identity, $[a, b, c] = [c, b, a][a, c, b] = 1$. As $\gamma_3(G) = \langle [a, b, a], [a, b, b], [a, b, c]\rangle$, this implies that either $[a, b, a] \neq 1$ or $[a, b, b] \neq 1$. Hence the subgroup $\langle a, b\rangle$ has nilpotency class three and order p^4 (see also [Sch03, Corollary 2.2(i)]).

Let $H = \langle c, G'\rangle$. Clearly, $|H| = p^3$ and $G/H = \langle aH, bH\rangle$. Let U be a subgroup of G such that $HU = G$, and so $HU/H = G/H = \langle aH, bH\rangle$. This shows that there are $h_1, h_2 \in H$ such that $ah_1, bh_2 \in U$. Since $[a, h_1], [a, h_2] \in \gamma_3(G)$ and $[a, b, h_1] = [a, b, h_2] = 1$, we obtain that $\langle [ah_1, bh_2]\gamma_3(G)\rangle = G'/\gamma_3(G)$ and either $[ah_1, bh_2, ah_1] \neq 1$ or $[ah_1, bh_2, bh_2] \neq 1$. Thus U contains G' and U is a group of order at least p^4. This, however, is a contradiction, by part (ii). Therefore the nilpotency class of G is not three. Since, by part (i), the nilpotency class of G is not one, we obtain that the class of G must be two.

(v) By part (iv), we only need to show that the exponent of G' is p. By [Rob96, 5.2.5], the quotient $G'/\gamma_3(G) = G'$, as an abelian group, is an epimorphic image of the tensor product $(G/G') \otimes_{\mathbb{Z}} (G/G')$, which implies that the exponent of $G'/\gamma_3(G) = G'$ divides the exponent of G/G'. As G is not generated by two elements, the size, and hence the exponent, of G' is at most p^2. However, if this exponent is p^2, then $G/G' \cong (C_p)^3$, which is impossible. □

The next lemma describes the case when either G' or $\Phi(G)$ is small.

LEMMA 4.5. *Suppose that G is a group with order p^5 and let $(\mathcal{F}, \mathcal{F}^*)$ be a Kantor family for G.*

(i) *If $|G'| = p$, then all members of $\mathcal{F} \cup \mathcal{F}^*$ are abelian.*
(ii) *If $|\Phi(G)| = p$, then all members of $\mathcal{F} \cup \mathcal{F}^*$ are elementary abelian. Moreover, if p is odd, then, in this case, G has exponent p.*
(iii) *If p is odd and G is extraspecial, then G has exponent p and all members of $\mathcal{F} \cup \mathcal{F}^*$ are elementary abelian.*

PROOF. (i) Let us first assume that $|G'| = p$. It suffices to prove, for all $A^* \in \mathcal{F}^*$, that A^* is abelian. We argue by contradiction and assume that $A^* \in \mathcal{F}^*$ is not abelian. In this case the derived subgroup $(A^*)'$ of A^* is nontrivial, and, as $(A^*)' \leqslant G'$, we obtain that $(A^*)' = G'$. Let $A \in \mathcal{F}$ such that $A \leqslant A^*$. Then A is a maximal subgroup of A^*, and so $(A^*)' = G' \leqslant A$. Thus A is normal in G, which is impossible by Lemma 4.4(i). Therefore A^* is abelian, as claimed.

(ii) The assertion that the members of the Kantor family are elementary abelian can be proved by substituting $\Phi(A^*)$ in the place of $(A^*)'$ and $\Phi(G)$ in the place of G' in the previous paragraph. Let p be an odd prime. In this case, as $|G'| = p$, the elements of G with order p form a subgroup $\Omega(G)$ of G. Let $A \in \mathcal{F}$ and $B^* \in \mathcal{F}^*$ such that $A \cap B^* = 1$. In this case $AB^* = G$ and $A, B^* \leqslant \Omega(G)$. Therefore $G = \Omega(G)$, which amounts to saying that G has exponent p.

(iii) This part follows immediately from part (ii). □

The following lemma is a generalization of [Kan91, Lemma].

LEMMA 4.6. *Let p be an odd prime and let $(\mathcal{F}, \mathcal{F}^*)$ be a Kantor family for an extraspecial group E of order p^5. Then the image of $\mathcal{F}^*$ in $E/Z(E)$ corresponds to a BLT-set of lines of* $\mathrm{W}(p)$.

PROOF. First note that, by Lemma 4.5(iii), all the members of $\mathcal{F}^*$ are abelian and hence each $A^* \in \mathcal{F}^*$ induces an abelian subgroup of $E/Z(E)$, and so a totally isotropic line of the associated $\mathrm{W}(p)$ geometry. Therefore, every member of $\mathcal{F}^*$ contains $Z(E)$. Suppose by way of contradiction that there is a line of $\mathrm{W}(p)$ concurrent with three elements of $\mathcal{L} = \{A^*/Z(E) : A^* \in \mathcal{F}^*\}$. Then there exists an abelian subgroup H of E of order p^3, and three elements A^*, B^*, C^* of $\mathcal{F}^*$ such that H intersects each of these elements in a subgroup of order p^2 properly containing $Z(E)$ (note that H contains $Z(E)$). Let A, B, C be the unique elements of $\mathcal{F}$ contained in A^*, B^*, C^* respectively. Now $(H \cap B)Z(E)$ is contained in B^* and so $A \cap (H \cap B)Z(E) = 1$. Also, we have that $|H \cap A| = p$ as $p^2 = |H \cap A^*| = |(H \cap A)Z(E)| = |H \cap A||Z(E)|$ (similarly, $|H \cap B| = p$). Thus

$$\begin{aligned} |(H \cap A)(H \cap B)Z(E)| &= \frac{|H \cap A||(H \cap B)Z(E)|}{|H \cap A \cap (H \cap B)Z(E)|} \\ &= |H \cap A||(H \cap B)Z(E)| \\ &= |H \cap A||H \cap B||Z(E)| \\ &\geqslant p^3, \end{aligned}$$

and so one can see that $H = (H \cap A)(H \cap B)Z(E)$. So

$$C^* \cap H = (C \cap (H \cap A)(H \cap B))Z(E)$$

and, by the condition $AB \cap C = 1$, we have that $C^* \cap H = Z(E)$, giving us the desired contradiction. Therefore, $\mathcal{L}$ is a BLT-set of lines of $\mathrm{W}(p)$. □

5. Proof of Theorem 1.2

In this section we prove Theorem 1.2. By Lemma 4.5(iii), an extraspecial group with order p^5 and exponent p^2 does not admit a Kantor family with order (p^2, p). Hence we may assume, for a proof by contradiction, that:

G is a group of order p^5 and $(\mathcal{F}, \mathcal{F}^*)$ is a Kantor family for G with order (p^2, p).

Our aim is to derive a contradiction. First note that Lemma 4.4 implies that one of the following must hold:

(I) $G/G' \cong C_p \times C_p \times C_{p^2}$ and $G' \cong C_p$;
(II) $G/G' \cong (C_p)^3$ and $G' \cong (C_p)^2$; or
(III) $G/G' \cong (C_p)^4$ and $G' \cong C_p$.

We show, case by case, that none of the above possibilities can occur. We let Z denote the centre of G.

Case (I). Using the argument in the proof of Lemma 4.4(iv), we can choose generators a, b, c of G such that $G' = \langle [a, b]\rangle$ and $c \in Z$. It also follows that $Z = \langle z, \Phi(G)\rangle$, and so $|Z| = p^3$. By Lemma 4.5(ii), all members of $\mathcal{F}^*$ must be abelian and so [Hac96, Theorem 3.2 and Lemma 2.2] imply that the subgroups $A \cap Z$ and $A^* \cap Z$ with $A \in \mathcal{F}$ and $A^* \in \mathcal{F}^*$ form a Kantor family for Z with order p. This, however, contradicts Theorem 4.2, since the subquadrangle here is not a translation generalized quadrangle (note that Z is abelian). Hence case (I) cannot occur.

Case (II). First we claim that it is possible to choose the generators x, y and z of G such that $G' = \langle [x, y], [x, z]\rangle$ and $[y, z] = 1$. Let x, y, z be generators of G. Then $G' = \langle [x, y], [x, z], [y, z]\rangle$. Since $G' \cong (C_p)^2$ we have that there are $\alpha, \beta, \gamma \in \mathbb{F}_p$ such that at least one of α, β, γ is nonzero and $[x, y]^\alpha [x, z]^\beta [y, z]^\gamma = 1$. If $\alpha = \beta = 0$ then $\gamma \neq 0$, and $[y, z] = 1$ follows. If $\alpha = 0$ and $\beta \neq 0$ then $[x^\beta y^\gamma, z] = 1$. Now replacing x by $x^\beta y^\gamma$ we find that in the new generating set $[x, z] = 1$ holds. Similarly, if $\alpha \neq 0$ and $\beta = 0$ then $[y, x^{-\alpha} z^\gamma] = 1$ and replacing x by $x^{-\alpha} z^\gamma$ we obtain that $[x, y] = 1$ holds in the new generating set. Finally if $\alpha\beta \neq 0$, then we replace x by $x^{\beta/\alpha} y^{\gamma/\alpha}$ and y by $yz^{\beta/\alpha}$ to obtain that $[x, y] = 1$. Thus, after applying one of the substitutions above and possibly renaming the generators, $[y, z] = 1$ holds, and the claim is valid.

We continue by verifying the following claim: if H is a subgroup in G with order p^2 and $H \cap Z = 1$ then there are $c, d \in Z$ such that $H = \langle yc, zd\rangle$.

Assume that H is a subgroup or order p^2 that does not intersect Z. Then $HZ/Z \cong H/(H \cap Z) = H$ and so $H \cong C_p \times C_p$. In particular H can be generated by two elements of the form $u = x^{\alpha_1} y^{\beta_1} z^{\gamma_1} c_1$ and $v = x^{\alpha_2} y^{\beta_2} z^{\gamma_2} c_2$ where $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}_p$, $c_i \in Z$ and $\langle uZ, vZ\rangle \cong C_p \times C_p$. Since $[u, v] = 1$ we obtain that

$$1 = [u, v] = [x^{\alpha_1} y^{\beta_1} z^{\gamma_1} c_1, x^{\alpha_2} y^{\beta_2} z^{\gamma_2} c_2] = [x, y]^{\alpha_1\beta_2 - \alpha_2\beta_1} [x, z]^{\alpha_1\gamma_2 - \alpha_2\gamma_1}.$$

Thus $\alpha_1\beta_2 - \alpha_2\beta_1 = \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. Note that these two expressions can be viewed as determinants of suitable 2×2 matrices. If $(\alpha_1, \alpha_2) \neq (0, 0)$ then the vectors (β_1, β_2) and (γ_1, γ_2) are both multiples of (α_1, α_2) and so the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}$$

has row-rank 1. Since the row-rank of a matrix is the same as the column-rank, this also shows that the vector $(\alpha_2, \beta_2, \gamma_2)$ is a multiple of the vector $(\alpha_1, \beta_1, \gamma_1)$ and so $uZ = vZ$, which gives $HZ/Z \cong C_p$, a contradiction. Thus $(\alpha_1, \alpha_2) = (0, 0)$; that is $u = y^{\beta_1} z^{\gamma_1} c_1$ and $v = y^{\beta_2} z^{\gamma_2} c_2$. Since $\langle uZ, vZ\rangle \cong C_p \times C_p$, we must have that $\beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$. Also, if $\beta_1, \beta_2 = 0$ then $HZ/Z \cong C_p$, and so we may assume that $\beta_1 \neq 0$. Change v to $u^{-\beta_2/\beta_1} v$; then $\langle u, v\rangle = H$ and v is of the form $z^\gamma d'$, where $d' \in Z$. Now change u to $uv^{-\gamma_1/\gamma_2}$. Then $\langle u, v\rangle = H$ still holds and now u is of the form $y^\beta c'$, where $c' \in Z$. Now $u^{\beta^{-1}}$ and $v^{\gamma^{-1}}$ are as required.

Let us now prove that G does not admit a Kantor family. We argue by contradiction and assume that $(\mathcal{F}, \mathcal{F}^*)$ is a Kantor family of order (p^2, p) for G. If A, B are distinct elements of $\mathcal{F}$ such that $A \cap Z = B \cap Z = 1$, then the claim above implies that $[A, B] = 1$, and so AB is a subgroup of G with order p^4. Thus, if $C \in \mathcal{F} \setminus \{A, B\}$, then $AB \cap C \neq 1$, which contradicts the triple condition. Thus $\mathcal{F}$ has at most one member that avoids the centre. Let us suppose now that A, B, C are pairwise distinct members of $\mathcal{F}$ such that $A \cap Z$, $B \cap Z$ and $C \cap Z$ are nontrivial. As $A \cap B = A \cap C = B \cap C = 1$, we obtain that $|A \cap Z| = |B \cap Z| = |C \cap Z| = p$ and that $A \cap Z$, $B \cap Z$, $C \cap Z$ are three distinct subgroups of Z. This, however, implies that $Z = (A \cap Z)(B \cap Z)$, and, in turn, that $C \cap Z \leqslant (A \cap Z)(B \cap Z)$, which violates the triple condition.

The argument in the last paragraph implies that at most two members of $\mathcal{F}$ can intersect Z nontrivially, and at most one member of $\mathcal{F}$ can avoid the centre. Thus $|\mathcal{F}| \leqslant 3$, which is a contradiction as p is odd and $|\mathcal{F}| = p + 1$. Therefore, case (II) is impossible.

Case (III). As G is not extraspecial, $|Z| = p^3$, and Lemma 4.4(iv) implies that the members of $\mathcal{F}^*$ are abelian. In this case [Hac96, Theorem 3.2, Lemmas 2.1 and 2.2] show that the subgroups $A \cap Z$ and $A^* \cap Z$ (with $A \in \mathcal{F}$ and $A^* \in \mathcal{F}^*$) form a Kantor family of order p for Z. However, we have a contradiction to Theorem 4.2 since the associated subquadrangle of order p is not a translation generalized quadrangle.

As none of the possibilities listed at the beginning of the section can occur, Theorem 1.2 must hold.

6. Proof of Theorem 1.1

Here we prove Theorem 1.1, but first we show that applying the Knarr construction to a BLT-set of lines arising from a Kantor family $(\mathcal{F}, \mathcal{F}^*)$ of the flock elation group results in an elation generalized quadrangle isomorphic to that directly associated to $(\mathcal{F}, \mathcal{F}^*)$.

THEOREM 6.1. *Let G be the flock elation group of order p^5, p odd, and suppose that G admits a Kantor family $(\mathcal{F}, \mathcal{F}^*)$ giving rise to an elation generalized quadrangle $\mathcal{E}$. Consider the BLT-set of lines $\mathcal{L}$ of $\mathrm{W}(p)$ obtained by taking the image of $\mathcal{F}^*$ under the natural projection map from G onto $G/Z(G)$. Then the flock quadrangle arising from $\mathcal{L}$ via the Knarr construction is equivalent to $\mathcal{E}$.*

PROOF. First note that G is extraspecial of exponent p, and observe that the matrices of the form

$$\begin{pmatrix} 1 & a & b & c & d & e \\ 0 & 1 & 0 & 0 & 0 & d \\ 0 & 0 & 1 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & -b \\ 0 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad a, b, c, d, e \in \mathrm{GF}(p),$$

define a representation of G into the symplectic group $\mathrm{PSp}(6, p)$ with its associated null polarity given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, the centre of G consists only of those upper triangular matrices with zeros everywhere above the diagonal except possibly the top right corner, and G fixes the projective point P represented by $(1, 0, 0, 0, 0, 0)$. Hence G induces an action on the quotient $P^{\perp}/P \equiv \mathrm{W}(p)$. It is not difficult to show that the right coset action of G on $G/Z(G)$ is permutationally isomorphic to the action of G on $P^{\perp}/P$ (as a projective right-module). To be more specific, the representatives of $G/Z(G)$ are in a bijection with matrices of the form

$$\begin{pmatrix} 1 & a & b & c & d & 0 \\ 0 & 1 & 0 & 0 & 0 & d \\ 0 & 0 & 1 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & -b \\ 0 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad a, b, c, d \in \mathrm{GF}(p),$$

and $P^{\perp}/P$ can naturally be identified with vectors of the form $(0, a, b, c, d, 1)$. Thus we have a bijection from $G/Z(G)$ onto $P^{\perp}/P$ given by

$$\begin{pmatrix} 1 & a & b & c & d & 0 \\ 0 & 1 & 0 & 0 & 0 & d \\ 0 & 0 & 1 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & -b \\ 0 & 0 & 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mapsto P + (0, a, b, c, d, 1)$$

such that the right coset action of G is equivalent to the right-module action of G on $P^{\perp}/P$.

Let $(\mathcal{F}, \mathcal{F}^*)$ be a Kantor family for G and let $\mathcal{E}$ be the associated elation generalized quadrangle with points:

(i) elements of g;
(ii) right cosets $A_i^* g$ of elements of $\mathcal{F}^*$;
(iii) ∞;

and lines:

(a) right cosets $A_i g$ of elements of $\mathcal{F}$;
(b) symbols $[A_i]$ where $A_i \in \mathcal{F}$.

Let $Q = (0, 0, 0, 0, 0, 1)$ and note that Q is opposite to P. Let $\mathcal{K}$ be the flock quadrangle associated to $\mathcal{L}$ constructed from the point P, and define a map from $\mathcal{E}$ to $\mathcal{K}$ as follows:

$$\infty \mapsto P, \quad [A_i] \mapsto \pi_i, \quad A_i^* g \mapsto z_i^g, \quad A_i g \mapsto M_i^g, \quad g \mapsto Q^g.$$

We will show that this map defines an isomorphism of generalized quadrangles. Since the action of G on $P^\perp/P$ is permutationally isomorphic to the right coset action of G on $G/Z(G)$, we have that the stabilizer of the subspace corresponding to a subgroup H containing $Z(G)$ is just H itself. Therefore A_i fixes z_i and A_i^* fixes M_i (for all i), and so the map above is well defined. Now we verify that the four types of incidences are compatible.

Incidence of ∞ and $[A_i]$. It is clear that $P \sim \pi_i$ for all i.

Incidence of $A_i^* g$ and $[A_i]$. We want to show that $\pi_i \sim z_i^g$ given that we know that $\pi_i \sim z_i$. Now G fixes every subspace of $P^\perp$ on P, and hence G fixes π_i. Therefore $z_i^g \sim \pi_i^g = \pi$ (note that g is a collineation).

Incidence of $A_i^* g$ and $A_i h$. So $A_i h \subset A_i^* g$. We want to show that $M_i^h \sim z_i^g$. By definition, z_i is the unique line of π_i (not on P) that is on a plane M_i on Q. We know that $M_i \sim z_i$. Since $A_i h \subset A_i Z g$, then there exists an element e of $Z(G)$ such that $hg^{-1}e \in A_i$. It suffices to show that $M_i^{hg^{-1}} \sim z_i$. Now A_i fixes M_i and so $M_i^{hg^{-1}} = M_i^{e^{-1}}$. Now e^{-1} fixes z_i and so $M_i^{hg^{-1}} \sim z_i$.

Incidence of g and $A_i g$. It is clear that G acts regularly on the points opposite P. Since for all i we have $Q \sim M_i$, it follows that $Q^g \sim M_i^g$.

Therefore, the flock quadrangle arising from $\mathcal{L}$ via the Knarr construction is equivalent to $\mathcal{E}$. □

6.1. Theorem 1.1 and its proof In general, we do not know that a Kantor family of the flock elation group must arise from a q-clan (possibly after applying an automorphism of the flock elation group), but we can establish this for q prime, which is the essence of Theorem 1.1, restated slightly differently below.

An elation generalized quadrangle of order (s, p)*, with* p *prime, is a flock quadrangle, isomorphic to* $\mathrm{Q}(4, p)$ *or isomorphic to* $\mathrm{W}(p)$*.*

PROOF OF THEOREM 1.1 Let $\mathcal{S}$ be an elation generalized quadrangle of order (s, p), where p is prime, and suppose that $(G, \mathcal{F}, \mathcal{F}^*)$ is the corresponding Kantor family. By [BTVM96], we may assume that $s = p^2$, and so G has order p^5. By Theorem 1.2, G must be extraspecial. Now the Frattini subgroup of G has order p and so has nontrivial intersection with every subgroup of G that has order at least p^3. Hence $Z(G)$ is contained in every element of $\mathcal{F}^*$. Therefore, by Lemma 4.6 and Theorem 6.1, our generalized quadrangle $\mathcal{S}$ is a flock quadrangle. □

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