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NONEXPANSIVE UNIFORMLY ASYMPTOTICALLY STABLE FLOWS ARE LINEAR

BY

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ABSTRACT. We show that if a flow (R, X, π) on a separable metric space (X, d) satisfies (i) the transition mapping $\pi(t, \cdot): X \to X$ is non-expansive for every $t \geq 0$; (ii) X contains a globally uniformly asymptotically stable compact invariant subset, then the flow (R, X, π) is linear in the sense that it can be topologically and equivariantly embedded into a flow $(R, l_2, \hat{\pi})$ on the Hilbert space l_2 for which all of the transition mappings $\hat{\pi}(t, \cdot)$ are linear operators on l_2.

1. **Introduction.** By a flow (R, X, π) on a topological space X we mean a continuous group action $\pi: R \times X \to X$ of the additive group of reals R on the space X. A compact subset M of X is called stable if for any neighborhood U of M there is a neighborhood V of M such that $\pi(R^+, V) \subset U$. A stable subset M is called globally asymptotically stable if for any neighborhood U of M and for any $x \in X$ there is a $T \in R$ such that $\pi([T, \infty), x) \subset U$. If, in addition, there is a neighborhood U of M such that for any neighborhood $V \subset U$ of M there exists $T \in R$ such that $\pi([T, \infty), U) \subset V$ the set M is called globally uniformly asymptotically stable. If X is locally compact and metric then a globally asymptotically stable compact subset M is necessarily globally uniformly asymptotically stable. This can be easily proved using the fact that in such a case there is a continuous (Liapunov) function $\phi: X \to [0, \infty)$ such that $\phi(x) > 0$ if $x \notin M$, $\phi(x) = 0$ if $x \in M$, and $\phi(\pi(t, x)) = e^{-t}\phi(x)$ for every $x \in X$ and $t \geq 0$, ([2, Theorem 2.7.14]). We say that a flow (R, X_1, π_1) can be embedded into another flow (R, X_2, π_2) if there exists a topological embedding $i: X_1 \to X_2$ which is equivalent, i.e., such that it makes the diagram

$$\begin{array}{ccc} R \times X_1 & \xrightarrow{\pi_1} & X_1 \\ \Big\downarrow{\scriptstyle I \times i} & & \Big\downarrow{\scriptstyle i} \\ R \times X_2 & \xrightarrow{\pi_2} & X_2 \end{array}$$

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commutative, where I is the identity mapping on R. If L is a topological vector space then a flow (R, L, π) on L is said to be linear if all of the transition mappings $\pi(t, \cdot): L \to L$ are linear operators on L. In the sequel we confine ourselves always to the case where the underlying space X of the flow (R, X, π) is separable and metrizable. The aim of this note is to find sufficient conditions under which such a flow is linear in the sense that it can be embedded in a linear flow $(R, l_2, \hat{\pi})$ on the Hilbert space l_2 equipped with its norm topology.

In recent years several researchers, e.g. M. Edelstein, L. Janos, R. McCann, and others (see [3], [5], [8]) found conditions under which a flow (R, X, π) or a semiflow (R^+, X, π) can be embedded into a radial flow (R, l_2, ρ) on l_2 defined by $\rho(t, x) = c^t x$ for $t \in R$, $x \in l_2$ where c is a constant in $(0, 1)$. The common feature of these cases is that the transition mappings $\pi(t, \cdot)$ shrink, in some sense, the space X to a point as $t \to \infty$. Different kinds of conditions had been found earlier by J. de Groot, P. C. Baayen, J. de Vries and others (see [1], [11]) ensuring that a flow (R, M, π) on a compact metric space can be embedded into a linear flow $(R, l_2, \hat{\pi})$ where the transition functions $\hat{\pi}(t, \cdot)$ are orthogonal operators on l_2. The first author's result (see [4]) on factorizing a selfmapping $f: X \to X$ on a compact space X into its more simple components gives us an idea of how to proceed in a more general case, subsuming the above two cases treated previously, namely when the flow (R, X, π) is such that the transition mappings $\pi(t, \cdot)$ shrink the space X in a certain sense to a compact subset M of X as $t \to \infty$. The method rests heavily on the existence of a retraction $a: X \to M$ exhibited already in previous works (see [4], [6], [10]).

Our main result is as follows:

THEOREM 1.1. *Let* (R, X, π) *be a flow on a separable metric space* (X, d) *satisfying the conditions:*

(i) *the mapping* $\pi(t, \cdot): X \to X$ *is non-expansive for* $t \geq 0$, *i.e.,* $d(\pi(t, x), \pi(t, y)) \leq d(x, y)$ *for every* $x, y \in X$ *and* $t \geq 0$.

(ii) *there is a compact invariant subset* M *of* X *which is globally uniformly asymptotically stable.*

Then the flow (R, X, π) *can be embedded into a linear flow on the Hilbert space* l_2.

2. **Auxiliary results.** For the sake of the reader's convenience we formulate here three known results which will enable us to carry out the plan outlined in the introduction.

THEOREM A (R. C. McCann, [9]). *Let* (R, X, π) *be a flow on a separable metric space* X *which has a globally uniformly asymptotically stable critical point. Then* (R, X, π) *can be embedded into a radial flow on* l_2

THEOREM B. *Let* (R, M, π) *be a flow on a compact metric space* (M, d) *such that all of the transition mappings* $\pi(t, \cdot)$ *are isometries.*

Then (R, M, π) can be embedded into a linear flow $(R, l_2, \hat{\pi})$ on l_2 for which the mappings $\hat{\pi}(t, \cdot)$ are orthogonal operators on l_2.

Proof. We first assume that the action π is effective, i.e., $\pi(t, \cdot)$ is not the identity on M unless $t=0$. Since the family $\{\pi(t, \cdot): t \in R\}$ is evidently equicontinuous, the Arzelà–Ascoli theorem implies that the closure G of this family in the topological semigroup M^M (the set of all continuous selfmappings of M into itself equipped with the uniform convergence topology) is a compact group acting continuously on M. Thus, we consider the dynamical system (G, M, π_1) on M where G is a compact group containing R as a dense subgroup (due to effectiveness of π) and π_1 is the unique extension from $R \times M$ to $G \times M$. Now Theorem 7.2 of [1] assures that there exists a topological embedding $i: M \to l_2$ and a dynamical system $(G, l_2, \hat{\pi}_1)$ on l_2 with $\hat{\pi}_1(g, \cdot)$ an orthogonal transformation for each $g \in G$ such that i is an equivariant embedding of (G, M, π) into $(G, l_2, \hat{\pi}_1)$. The required continuity of $\hat{\pi}_1$ follows from the observation made in [11] where it is shown that the linearization process described in [1] is continuous with respect to the strong operator topology. Restricting $\hat{\pi}_1$ to $R \times l_2$ we obtain our statement. In case π is not effective the set of those $t \in R$ for which $\pi(t, \cdot)$ is the identity is a discrete subgroup, say K of R and the quotient group $C = R/K$ is the circle group. Thus our original flow (R, M, π) gives rise to the group action $\pi^*: C \times M \to M$ defined by $\pi^*(c, x) = \pi(t, x)$ where $t \in R$ is such that $\alpha(t) = c$ with α being the natural map $\alpha: R \to C$. Applying the above reasoning to the dynamical system (C, M, π^*) we are done because of the continuity of $\alpha: R \to C$.

REMARK. The quoted papers [1] and [11] deal with much more general case where G is only locally compact and M an arbitrary metric space. Applying the construction of [1] as outlined on page 364 to our case we may obtain the desired embedding $i: M \to l_2$ explicitly as follows:

First we identify M with a subset of l_2, and consider the space K of all square-summable functions from G to l_2 with respect to the Haar measure in G. Because of the compactness of G, the orbits $g\xi$ of every point $\xi \in M$ are square-summable functions which offers a natural embedding of M into K defined by sending ξ into its G-orbit. The space K is again a separable Hilbert space with the norm defined by $[\int_G \|f(g)\|^2 \, dg]^{1/2}$ for $f \in K$, where $\| \ \|$ stands for the norm in l_2. The group G acts on K by sending the function $f(g)$ into $f(gg_0)$ for $g_0 \in G$. The invariance of the Haar measure yields readily that this action preserves the norm in K, thus G acts on K as a group of orthogonal transformations. Finally K can be identified with l_2 and restricting the action of G to R we are done.

THEOREM C. *Let a flow (R, X, π) on a separable metric space (X, d) satisfy the conditions* (i) *and* (ii) *of Theorem* 1.1. *Then there exists a retraction $a: X \to M$ commuting with $\pi(t, \cdot)$ for every $t \in R$.*

Proof. From (ii) it follows that for every $x \in X$ the positive semi-orbit $\{\pi(t, x) : t \geq 0\}$ is precompact. This shows that the hypotheses of Theorem 4 of [10] are met in our case. Observing that no linear structure is used in the proof of this theorem (formulated for non-expansive semiflows on the unit ball of a Banach space) we are done. The desired value $a(x) \in M$ for $x \in X$ of the retraction $a : X \to M$ is simply the uniquely determined point in M which is asymptotic to x, i.e., such that $d(\pi(t, x), \pi(t, a(x))) \to 0$ as $t \to +\infty$.

REMARK. Another application of this retraction device has been used by the first and the third author (see [6]) to obtain a fixed point for self-mappings on a convex subset of a Banach space and which possess an attractor for compact subsets.

3. **Proof of the theorem.** Assume now that (R, X, π) is a flow on a separable metric space (X, d) satisfying conditions (i) and (ii). Identifying M to a point we form the quotient space X/M which is obviously again separable and metrizable. There exists a unique flow π^* on X/M which makes the diagram

$$\begin{array}{ccc} R \times M & \xrightarrow{\pi} & X \\ {\scriptstyle I \times \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ R \times X/M & \xrightarrow{\pi^*} & X/M \end{array}$$

commutative, where $\alpha : X \to X/M$ is the natural projection and I the identity mapping on R. By (R, M, π^{**}) we denote the flow induced on M by π.

Applying Theorem C we construct the embedding i of X into the cartesian product $X/M \times M$ be setting $i(x) = (\alpha(x), a(x))$ for $x \in X$, where α is the natural projection mentioned above and a the retraction of X onto M ensured by Theorem C. Due to the commutativity of a with $\pi(t, \cdot)$ for every $t \in R$ we conclude readily that i defines an embedding of the flow (R, X, π) into the cartesian product $(R, X/M, \pi^*) \times (R, M, \pi^{**})$. We now prove

LEMMA 3.1. *The first factor* $(R, X/M, \pi^*)$ *satisfies the hypotheses of Theorem A.*

Proof. Since M is globally uniformly asymptotically stable with respect to π, it follows directly that $\alpha(M)$ is globally uniformly asymptotically stable critical point with respect to π^*. Also, as noted at the beginning of this section, X/M is separable and metrizable.

We now turn to the second factor (R, M, π^{**}). It follows easily, e.g., again from [10] that all the translation functions $\pi^{**}(t, \cdot)$ are isometries on (M, d), which implies that this factor satisfies the hypotheses of Theorem B.

Thus, piecing these facts together we conclude that (R, X, π) can be embedded into the product $(R, l_2, \rho)\times(R, l_2, \hat{\pi})$ of two linear flows on l_2 and since $l_2\times l_2$ can be identified again with l_2 our assertion follows.

4. **A topological reformulation of the theorem.** If (R, X, π) is a flow on a separable metrizable space X without any metric given on X the question arises how to reformulate the metric dependent condition (i) in order that the conclusion of our theorem remains valid. We shall show that the concept of even continuity, as treated, e.g. in [7] pp. 235 is the remedy. We need the following result. (For the proof see [7], pp. 241.)

LEMMA 4.1. *Let (R, X, π) be a flow on a metrizable space X (not necessarily separable) satisfying the condition*

(i)* *the family $\{\pi(t,\cdot): t\geq 0\}$ is evenly continuous.*

Then for each sequence $\{t_n\}$ of numbers $t_n\geq 0$ and each convergent sequence $\{x_n\}\to x$ in X the following implication is true:

If the sequence $\{\pi(t_n, x)\}$ converges to some point $y\in X$ then the sequence $\{\pi(t_n, x_n)\}$ converges to the same point y.

THEOREM 4.2. *Assume that (R, X, π) is a flow on a separable metrizable space X satisfying the condition* (ii) *of Theorem* 1.1. *and the condition* (i)* *of the above lemma.*

Then the conclusion of Theorem 1.1. *is valid for the flow (R, X, π).*

Proof. Choosing arbitrarily a metric d on X, compatible with the topology of X we show that the metric d^* defined by

$$d^*(x, y)=\sup\{d(\pi(t, x), \pi(t, y)): t\geq 0\}$$

is topologically equivalent to d and such that $\pi(t,\cdot)$ is non-expansive relative to d^* for every $t\geq 0$. Thus the flow (R, X, π) satisfies the condition (i) of Theorem 1.1 relative to d^* and the conclusion of this theorem is therefore valid.

The fact that $d^*(x, y)$ is finite for every $x, y\in X$ follows from the fact that the semiorbits $\{\pi(t, x): t\geq 0\}$ are precompact. The verification that d^* is a metric on X and that $\pi(t,\cdot)$ is non-expansive relative to d^* for $t\geq 0$ follows directly from the definition of d^*. It remains to show that d^* is topologically equivalent to d. Since evidently $d^*\geq d$ all we need to show is that if a sequence $\{x_n\}$ in X converges to a point $x\in X$ then the numerical sequence $d^*(x_n, x)$ converges to zero. Assume this is not so. Then, passing to a subsequence, if necessary, we may assume that for some $\varepsilon>0$ we have $d^*(x_n, x)\geq\varepsilon$ for every $n=1, 2,\ldots$. From the definition of d^* follows that for each $n=1, 2,\ldots$ we can find $t_n\geq 0$ such that

$$d(\pi(t_n, x_n), \pi(t_n, x))\geq\varepsilon/2.$$

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Passing again to a subsequence, if necessary, we can achieve that the sequence $\{\pi(t_n, x)\}$ converges to a point $y \in X$. But then Lemma 4.1 implies that the sequence $\{\pi(t_n, x_n)\}$ converges to the same point which is evidently impossible and which completes the proof of our statement.

5. **An example and conclusion.** Let X be the subset of the plane R^2 with the open unit disc removed, i.e., in polar coordinates $X = \{(r, \phi) : r \geq 1\}$, and let (R, X, π) be the flow on X defined by the autonomous system

$$\frac{dr}{dt} = r(1-r), \qquad \frac{d\phi}{dt} = 1.$$

The trajectories spiral around the unit circle C and approach it as $t \to \infty$, thus C is evidently a globally uniformly asymptotically stable invariant subset. It is also evident that the action of π is non-expansive relative to the Euclidean metric on X so that both conditions (i) and (ii) are satisfied and (R, X, π) can be embedded into a linear flow on l_2 in accordance to Theorem 1.1.

A natural question arises as to the dimension of the linear space in which the flow (R, X, π) can be linearized. Does there exist a linear flow $(R, R^n, \hat{\pi})$ on a finite dimensional Euclidean space R^n in which (R, X, π) can be embedded? From the proof of Theorem 1.1. we know that (R, X, π) can be embedded in the product, $(R, X/C, \pi^*) \times (R, C, \pi^{**})$. We observe that X/C is homeomorphic to R^2, hence the first factor satisfies hypotheses of Corollary 9 [8] yielding that this factor is isomorphic to a radial flow on R^2. Since the second factor represents a simple rotation along the circle C we conclude that our flow can be linearized on the space $R^4 = R^2 \times R^2$.

This question, namely to find an easily verifiable criterion as to whether or not a flow on a finite dimensional separable metric space can be linearized in an Euclidean space will be dealt with in our forthcoming paper.

References

1. P. C. Baayen and J. de Groot, *Linearization of Locally Compact Transformation Groups in Hilbert Space*. Math. Systems Theory, Vol. **2** No. 4, 363–379.

2. N. P. Bhatia and G. P. Szegö, *Dynamical systems: Stability Theory and Applications, Lecture Notes in Math.* **35,** Springer Verlag, Berlin, 1967.

3. Michael Edelstein, *On the Homomorphic and Isomorphic Embeddings of a Semiflow into a Radial Flow.* Pacific J. Math., 91 (1980) 281–291.

4. Ludvik Janos, *On Representation of Self-mappings.* Proc. Amer. Math. Soc. **26** (1970), 529–533.

5. Ludvik, Janos, *A linerization of Semiflows in the Hilbert Space* l_2. Topology proceedings, volume **2** (1977), 219–232.

6. Ludvik Janos and J. L. Solomon, *A Fixed Point Theorem and Attractors.* Proc. Amer. Math. Soc. **71,** (2), (1978), 257–262.

7. J. L. Kelley, *General Topology*. Van Nostrand, New York, 1955.

8. Roger C. McCann, *Asymptotically Stable Dynamical Systems are Linear,* Pacific J. Math, **81** (1979), 475–479.

9. Roger C. McCann, *Embedding Asymptotically Stable Dynamical Systems into Radial Flows in* l_2. Pacific J. Math. 90 (1980) 425–429.

10. Robert Sine, *Structure and Asymptotic Behavior of Abstract Dynamical Systems.* Non-linear Analysis, Theory, Methods and Applications. Vol. **2,** No. 1 (1978), 119–127.

11. J. de Vries, *A Note on Topological Linearization of Locally Compact Transformation Groups in Hilbert Space.* Math. System Theory, Vol. **6,** No. 1, 49–59.

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