A NOTE ON THE FINITE FOURIER TRANSFORM

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1. One of the best known theorems on the finite Fourier transform is:— The integral function $F(z)$ is of the exponential type C and belongs to L^2 on the real axis, if and only if, there exists an $f(x)$ belonging to $L^2(-C, C)$ such that

$$F(z) = (2\pi)^{-\frac{1}{2}} \int_{-C}^{C} e^{izx} f(x) dx \tag{1.1}$$

(Boas [1], 6.8). Additionally, if $f(x)$ vanishes almost everywhere in a neighbourhood of C and also in a neighbourhood of $-C$, then $F(z)$ is of an exponential type lower than C.

Thus we may write

$$F(z) = (2\pi)^{-\frac{1}{2}} \int_{B}^{A} e^{izx} f(x) dx \tag{1.2}$$

where the interval (B, A) is enclosed in the interval $(-C, C)$ and if $B \neq -C$, then $A = C$. Also $f(x)$ is not zero almost everywhere in any neighbourhood of A or of B.

We will assume that

$$f(x) \sim K(A - x)^p, \quad \operatorname{Re} p > -\tfrac{1}{2} \tag{1.3}$$

as $x \to A-$ (K, constant $\neq 0$), and that

$$f(x) \sim M(x - B)^q, \quad \operatorname{Re} q > -\tfrac{1}{2} \tag{1.4}$$

as $x \to B+$ (M, constant $\neq 0$).

The purpose of this note is to indicate the connection between equations (1.3) and (1.4) and the asymptotic behaviour of $F(iv)$ for large $|v|$.

2. We assume that $v > 0$, then

$$(2\pi)^{\frac{1}{2}} F(-iv) = \int_{B}^{A} e^{vx} f(x) dx;$$

by a change in variable we obtain

$$(2\pi)^{\frac{1}{2}} e^{-vA} F(-iv) = \int_{0}^{A-B} e^{-vy} f(A - y) dy. \tag{2.1}$$

Writing $U(x)$ as the unit function ($= 1$, $x > 0$ and $= 0$ otherwise) we immediately observe that equation (2.1) shows that $(2\pi)^{1/2}e^{-vA}F(-iv)$ is the Laplace transform of $f(A-y)U(A-B-y)$.

If we note also that equation (1.3) may be written in the form

$$f(A-y)\sim Ky^{p},\quad \operatorname{Re} p > -\tfrac{1}{2}$$

as $y\to 0+$, we may apply Theorem 1 p. 473 of Doetsch [2] to obtain

$$(2.2)\qquad (2\pi)^{1/2}e^{-vA}F(-iv)\sim K\Gamma(p+1)v^{-p-1}$$

as $v\to+\infty$.

Similarly, we have

$$(2\pi)^{1/2}e^{vB}F(iv)=\int_0^{A-B}e^{-vy}f(B+y)dy$$

which will show that

$$(2\pi)^{1/2}e^{vB}F(iv)\sim M\Gamma(q+1)v^{-q-1}$$

as $v\to+\infty$.

These results are summarized by

THEOREM 2. If $f(x)$ belongs to $L^2(B, A)$ and

$$(2.3)\quad \text{(i)}\qquad F(z)=(2\pi)^{-1/2}\int_B^A e^{izx}f(x)dx,$$

$$(2.4)\quad \text{(ii)}\qquad f(x)\sim K(A-x)^{p},\quad \operatorname{Re} p>-\tfrac{1}{2},$$

as $x\to A-$, where K is a constant,

$$(2.5)\quad \text{(iii)}\qquad f(x)\sim M(x-B)^{q},\quad \operatorname{Re} q>-\tfrac{1}{2},$$

as $x\to B+$, where M is a constant, then

$$(2.6)\qquad F(iv)\sim(2\pi)^{-1/2}M\Gamma(q+1)v^{-q-1}e^{-vB}$$

as $v\to+\infty$, and

$$(2.7)\qquad F(iv)\sim(2\pi)^{-1/2}K\Gamma(p+1)(-v)^{-p-1}e^{-vA}$$

as $v\to-\infty$.

Note:— A modification of the proof of Theorem 2 will allow equations (2.6) and (2.7) to be replaced by

$$(2.8)\qquad F(z)\sim(2\pi)^{1/2}M\Gamma(q+1)(-iz)^{-p-1}e^{izB}$$

as $|z|\to\infty$ with $0<\arg z<\pi$, and

$$(2.9)\qquad F(z)\sim(2\pi)^{1/2}K\Gamma(p+1)(iz)^{-q-1}e^{izA}$$

as $|z|\to\infty$ with $-\pi<\arg z<0$.

3. We now assume that $F(z)$ is an integral function of type C which belongs to $L^2(-\infty,\infty)$ on the real axis and that equations (2.6) and (2.7) hold. Then the theorems of Boas [1] quoted at the beginning of this note show that there is a function $f(x)$ belonging to $L^2(-C, C)$ such that

$$(2\pi)^{1/2}F(z) = \int_P^Q e^{izx}f(x)dx \tag{3.1}$$

where the interval (P, Q) is included in the interval $(-C, C)$.

With a change of variable, we obtain

$$(2\pi)^{1/2}e^{vP}F(iv) = \int_0^{Q-P} e^{-vy}f(P+y)dy$$

which may be modified to

$$\begin{aligned}(2\pi)^{1/2}e^{vB}F(iv)v^{q+1} &= e^{v(B-P)}v^{q+1}\int_0^{\eta} e^{-vy}f(P+y)dy \\ &\quad + e^{v(B-P-\eta)}v^{q+1}\int_0^{Q-P-\eta} e^{-vy}f(P+\eta+y)dy\end{aligned} \tag{3.2}$$

where $0 < \eta < Q - P$.

Since $f(x)$ belongs to $L^1(P, Q)$, the integrals on the right side of equation (3.2) are bounded uniformly in v. Using equation (2.6) we see that $P = B$ and that $f(P+y)$ is not zero almost everywhere in $0 < y < \eta$ and also that

$$\begin{aligned}\lim_{v\to+\infty} v^{q+1}\int_0^{\eta} e^{-vy}f(P+y)dy &= \lim_{v\to+\infty}(2\pi)^{1/2}e^{vB}F(iv)v^{q+1} \\ &= M\Gamma(q+1).\end{aligned} \tag{3.3}$$

That is

$$\int_0^{\eta} e^{-vy}f(P+y)dy \sim M\Gamma(q+1)v^{-q-1} \tag{3.4}$$

as $v \to +\infty$.

Now chapter 16 of Doetsch [2] is devoted almost entirely to the discussion of the implications of equations of the type (3.4). Possibly the most general result is obtained from his Theorem 4 of page 512. Here q is restricted to have real values and it must be known that $f(P+y) > -Sy^q$ for some S. We may then infer that

$$\int_0^y f(P+y)dy \sim M\frac{y^{q+1}}{q+1}$$

as $y \to 0+$.

We may now return to equation (3.1) and examine the upper limit Q. We can show that $A = Q$ and derive similar conclusions to those just made. We summarize this work in

THEOREM 3. If $F(z)$ is an integral function of the exponential type such that

(i) $F(x)$ belongs to $L^2(-\infty, \infty)$ on the real axis, and

(ii) equations (2.6) and (2.7) hold, then there exists a function $f(x)$ which belongs to $L^2(B, A)$ such that

(α) equation (2.3) holds, and

(β) $f(x)$ does not vanish almost everywhere in any neighbourhood of B, or in any neighbourhood of A.

Additionally, if M and q are real, and if $f(x) > -S_1(x-B)^q$ for some constant upper neighbourhood of B, then

$$\int_B^x f(t)dt \sim \frac{M(x-B)^{q+1}}{p+1} \tag{3.5}$$

as $x \to B+$; and if K and p are real, and if $f(x) > -S_2(A-x)^p$ for some constant S_2 in a lower neighbourhood of A, then

$$\int_x^A f(t)dt \sim \frac{K(A-x)^{p+1}}{q+1} \tag{3.6}$$

as $x \to A-$.

The estimates (3.5) and (3.6) still hold if one (or both) of the inequality signs in the last paragraph is changed from $>$ to $<$.

References

[1] Boas Jnr, R. P., *Entire Functions*, Academic Press, New York (1954).
[2] Doetsch, G., *Handbuch der Laplace-Transformation*, Bd 1, Birkhäuser, Basel (1950).

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