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SIX PRIMES AND AN ALMOST PRIME IN FOUR LINEAR EQUATIONS

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ABSTRACT. There are infinitely many triplets of primes p, q, r such that the arithmetic means of any two of them, $\frac{p+q}{2}$, $\frac{p+r}{2}$, $\frac{q+r}{2}$ are also primes. We give an asymptotic formula for the number of such triplets up to a limit. The more involved problem of asking that in addition to the above the arithmetic mean of all three of them, $\frac{p+q+r}{3}$ is also prime seems to be out of reach. We show by combining the Hardy-Littlewood method with the sieve method that there are quite a few triplets for which six of the seven entries are primes and the last is almost prime.

1. **Introduction.** The Hardy-Littlewood method is number theorist's best tool for additive problems, such as a system of homogeneous linear equations in arithmetically interesting sets. A classical example is van der Corput's result on the existence of infinitely many three-term arithmetic progressions in the set of positive primes [vdC 1939]. Generally k equations require at least $2k+1$ primes for a successful attack by the Hardy-Littlewood method. Thus the problem of four-term arithmetic progressions in positive primes is out of reach at present. As an approximation Heath-Brown proved that there are infinitely many non-trivial arithmetic progressions consisting of three primes and an almost-prime of type P_2 [H-B 1981]. Here, as usual, P_r denotes an integer with at most r prime factors counted according to multiplicity.

The author has recently observed that systems of linear equations with a special structure can sometimes break the $2k+1$ barrier. The most exciting example is the "magic triangle" in positive primes, that is the system of equations

$$\begin{cases} a+b=2w \\ a+c=2v \\ b+c=2u \end{cases} \tag{1}$$

has infinitely many non-trivial solutions in positive primes a, b, c, u, v, w. Actually more is true. For any integer $m \geq 2$ let $N_m(X)$ be denote the number of prime m-tuplets $p_1 \leq X$, $p_2 \leq X, \ldots, p_m \leq X$ such that the arithmetic mean $\frac{1}{2}(p_i+p_j)$ of any two of these primes is also a prime. Thus $N_2(X)$ counts the three-term arithmetic progressions while $N_3(X)$ counts the "magic triangles". In [B 1992] it is proved that for any fixed $m \geq 2$ we have

$$N_m(X) \gg \frac{X^m}{(\log X)^{m(m+1)/2}}. \tag{2}$$

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It is worth emphasising that the corresponding system consists of $k = \frac{1}{2}m(m-1)$ equations in $n = \frac{1}{2}m(m+1) \sim k + \sqrt{2k} + 1$ variables.

Note that in [B,B 1995] Balog and Brüdern investigated the same question where the set of primes was changed to the set of integers representable as a sum of three positive cubes.

Having the result for "magic triangles" one can try to add one more equation and variable by asking whether there are infinitely many prime triplets such that not only the arithmetic mean of any two of them but also the arithmetic mean of all three of them is prime, that is adding the equation $a + b + c = 3z$ to (1) getting

$$\begin{cases} a + b = 2w \\ a + c = 2v \\ b + c = 2u \\ a + b + c = 3z. \end{cases} \tag{3}$$

Like the four-term arithmetic progression case, this is too much to ask. Following the foot steps of Heath-Brown we mix the Hardy-Littlewood method with sieve methods to produce solutions of (3) in six primes and an almost-prime. Hardly can one hope a sieve result for (3) without an asymptotic to (1). Indeed, as a byproduct we get

THEOREM 1.

$$N_3(X) \sim \frac{27}{16} \prod_{p>3} \frac{p^3(p^2 - 5p + 7)}{(p-1)^5} \frac{X^3}{(\log X)^6}, \quad \textit{as } X \longrightarrow \infty. \tag{4}$$

According to the location of the almost-prime among the variables a, b, c, u, v, w and z we can state three results. $N(X\,;\, z = P_3)$ denotes the number of solutions of (3) in positive primes $a \le X$, $b \le X$, $c \le X$, $u \le X$, $v \le X$, $w \le X$, and $z = P_3 \le X$ with the obvious extension to other variables. We have

THEOREM 2.

$$N(X\,;\, z = P_3) \gg \frac{X^3}{(\log X)^7}, \tag{5}$$

$$N(X\,;\, v = P_2) \gg \frac{X^3}{(\log X)^7}, \tag{6}$$

$$N(X\,;\, a = P_3) \gg \frac{X^3}{(\log X)^7}, \tag{7}$$

for $X > X_0$.

The difference between the quality of the results (5), (7) and (6) lays in the difference between the level of distribution in the sieve error terms we can reach in the different situations. The proof of (7) is analogous to the proof of (5), but we have to use a completely different argument in (6) when dealing with the sieve error term. We will present the detailed proof only for (5). The proof of (6) has the same structure than the proof of (5) except the "minor arcs" bound, which can be reduced to a lemma of [H-B 1981]. However, this more powerful method is not applicable to the other two situations, because they are too symmetric in a certain sense. In Sections 2–6 we discuss the proof of (5) and as well as the general structure of the application of the circle method to "magic triangles". In Sections 7–8 we comments the alterations to the proof of (6).

Heuristic arguments suggest that $N(X)$, the number of solutions of (3) in seven positive primes each $\leq X$ satisfy

CONJECTURE.

$$N(X) \sim \frac{27}{8} \prod_{p>3} \frac{p^4(p-3)^2}{(p-1)^6} \frac{X^3}{(\log X)^7}, \quad \text{as } X \longrightarrow \infty. \tag{8}$$

In this point it is worth mentioning that the same heuristic arguments give that the number of four-term arithmetic progressions in positive primes each $\leq X$ must be

$$\sim \frac{3}{2} \prod_{p>3} \frac{p^2(p-3)}{(p-1)^3} \frac{X^2}{(\log X)^4}. \tag{9}$$

It is transparent that the factors corresponding to $p > 3$ in (8) are just the squares of the factors in (9), however one could not guess this relation from the structure of the equations only.

We want to stress here, that the application of the Hardy-Littlewood circle method as described in Section 3, is quite general, and capable to give asymptotics for "magic triangles" in other sets than the set of primes or even for other systems of linear equations with "fewer than expected" variables. In a forthcoming paper we plan to use the present method to derive a lower bound for the number of solutions of (3) in integers representable as a sum of two squares.

Throughout the paper we will use the standard notations of analytic number theory. p will always denote a positive prime and a, b, c, u, v, w, z will always be the variables of (3).

2. **Preparation.** Let us concentrate first to the proof of (5). Theorem 1, the asymptotic result, will follow from this discussion, and it will also be clear that (7) can be proved much the same way.

If one of the variables in (1) is fixed then choosing the value of two other variables determines the rest. Thus the number of "magic triangles" with a fixed entry is $O(X^2/(\log X)^2)$. Therefore we can suppose that neither of the variables is divisible by 2 or 3, which sets up the following conditions.

$$2 \nmid a, b, c\,;$$
$$3 \nmid a, b, c, a+b, a+c, b+c\,;$$
$$2 |, 4 \nmid a+b, a+c, b+c.$$

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One can easily check that these conditions are equivalent to

$$a \equiv b \equiv c \equiv \begin{cases} 1 \\ 5 \\ 7 \\ 11 \end{cases} \bmod 12.$$

Similarly we can suppose that neither of the variables in (3) is divisible by 2 or 3, which sets up the following conditions

$$2 \not| a, b, c, a+b+c\,;$$
$$3 \not| a, b, c, a+b, a+c, b+c\,;$$
$$2|, 4 \not| a+b, a+c, b+c\,;$$
$$3|, 9 \not| a+b+c.$$

One can easily check that these conditions are equivalent to

$$\{a,b,c\} \equiv \begin{cases} \{1,1,1\} \\ \{3,3,3\} \end{cases} \bmod 4, \quad \{a,b,c\} \equiv \begin{cases} \{1,1,1\} \\ \{2,2,2\} \\ \{4,4,4\} \\ \{5,5,5\} \\ \{7,7,7\} \\ \{8,8,8\} \\ \{1,1,4\} \\ \{2,2,8\} \\ \{4,4,7\} \\ \{2,5,5\} \\ \{1,7,7\} \\ \{5,8,8\} \\ \{1,4,7\} \\ \{2,5,8\} \end{cases} \bmod 9.$$

By the Chinese Remainder Theorem the triplet (a,b,c) is fixed to be congruent to one of the 72 admissible triplets modulo 36 when dealing with (3). The 4 admissible triplets modulo 12 when dealing with (1) can be written as 108 admissible triplets modulo 36. Let us fix an admissible triplet (a_0, b_0, c_0).

We will apply the linear sieve to the weighted sequence of positive integers z with weight

$$\omega(z) = \sum\sum\sum \Lambda(a)\Lambda(b)\Lambda(c)\Lambda\Big(\frac{a+b}{2}\Big)\Lambda\Big(\frac{a+c}{2}\Big)\Lambda\Big(\frac{b+c}{2}\Big),$$

where the triple sum is extended to $a \le X$, $b \le X$, $c \le X$, $a+b+c = 3z$ and $(a,b,c) \equiv (a_0, b_0, c_0) \bmod 36$. Here $\Lambda(n)$ is the von Mangoldt's function.

Now let $\{\lambda_d\,;\, d \le D, (d,6) = 1\}$ be a set of sieveing weights, in this stage of the proof they are arbitrary real (or complex) numbers defined on square-free numbers coprime to 6 and satisfying $|\lambda_d| \le 1$. We are going to study the sum

$$\sum_{d \le D} \lambda_d \sum_z \omega(dz).$$

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Actually we will need a more general sum, namely let $\omega^*(z)$ be defined similarly to $\omega(z)$ but $\Lambda(a)$ is changed to an arbitrary function $\Lambda^*(a)$ satisfying only $|\Lambda^*(a)| \leq \log a$, that is

$$\omega^*(z) = \sum\sum\sum \Lambda^*(a)\Lambda(b)\Lambda(c)\Lambda\Big(\frac{a+b}{2}\Big)\Lambda\Big(\frac{a+c}{2}\Big)\Lambda\Big(\frac{b+c}{2}\Big), \tag{10}$$

We will prove the following theorem.

THEOREM 3. *For any $N > 0$ there is an $M = M(N) > 0$ such that*

$$\sum_{d\leq D} \lambda_d \sum_z \omega^*(dz) = \frac{3}{16}\prod_{p>3}\frac{p^3(p^2-5p+7)}{(p-1)^5}X^2 \sum_{d\leq D}\lambda_d \prod_{p|d}\frac{p-2}{p^2-5p+7}$$
$$\times \sum_{\substack{a\leq X\\ a\equiv a_0(36)\\ (a,d)=1}} \Lambda^*(a)\prod_{p|a}\frac{p^2-3p+2}{p^2-5p+7} + O\Big(\frac{X^3}{(\log X)^N}\Big),$$

whenever $D \leq \frac{X^{1/3}}{(\log X)^M}$. Especially we have

$$\sum_{d\leq D}\lambda_d \sum_z \omega(dz) = \frac{1}{64}\prod_{p>3}\frac{p^3(p^2-5p+7)}{(p-1)^5}X^3 \sum_{d\leq D}\lambda_d\prod_{p|d}\frac{p-2}{p^2-5p+7} + O\Big(\frac{X^3}{(\log X)^N}\Big),$$

and with $D = 1$, $\lambda_1 = 1$

$$\sum_z \omega(z) = \frac{1}{64}\prod_{p>3}\frac{p^3(p^2-5p+7)}{(p-1)^5}X^3 + O\Big(\frac{X^3}{(\log X)^N}\Big).$$

Theorem 1 follows from this last result by adding to the 108 admissible triplets of residues modulo 36 and by standard bunds for prime powers. We leave the details to the reader.

To get better almost-primes we have to use more than a straightforward sieving process. One of the important steps is the "reversal of roles" principle, developed by Iwaniec [I 1972] and Chen [C 1973] independently. This will require the study of a different sum where the roles of a and z are switched. To simplify the notation we restrict our functions $\Lambda(n)$ and $\Lambda^*(n)$ to the interval $\{0 < n \leq X\}$. Let $\Omega^*(a)$ be the number of triangles of type (3) with a fixed vertex a and weights $\Lambda(n)$ at the other six locations, that is

$$\Omega^*(a) = \sum\sum\sum \Lambda^*(z)\Lambda(b)\Lambda(c)\Lambda\Big(\frac{3z-b}{2}\Big)\Lambda\Big(\frac{3z-c}{2}\Big)\Lambda\Big(\frac{b+c}{2}\Big),$$

where the triple sum is extended to $3z-b-c = a$ and $(b,c) \equiv (b_0,c_0) \bmod 36$, while $z \equiv z_0 \equiv (a_0+b_0+c_0)/3 \bmod 12$ with the fixed admissible triplet (a_0,b_0,c_0) of residues modulo 36. We also restrict $\Omega^*(a)$ to $0 < a \leq X$. Theorem 3 remains valid with little alterations when $\omega^*(z)$ is changed to $\Omega^*(a)$. However, the main term becomes slightly more clumsy as we have the additional condition that $0 < 3z - b \leq 2X$. We introduce the function

$$H(z) = H(z\,;X) = 3\min\Big(\frac{z}{X}, \frac{1}{3}, 1-\frac{z}{X}\Big).$$

We have

THEOREM 3′. *For any $N > 0$ there is an $M = M(N) > 0$ such that*

$$\sum_{d\le D}\lambda_d\sum_a \Omega^*(da) = \frac{3}{16}\prod_{p>3}\frac{p^3(p^2-5p+7)}{(p-1)^5}X^2\sum_{d\le D}\lambda_d\prod_{p|d}\frac{p-2}{p^2-5p+7}$$
$$\times \sum_{\substack{z\le X\\ z\equiv z_0(12)\\ (z,d)=1}}\Lambda^*(z)\big(H(z)\big)^2\prod_{p|z}\frac{p^2-3p+2}{p^2-5p+7}+O\Big(\frac{X^3}{(\log X)^N}\Big),$$

whenever $D \le \frac{X^{1/3}}{(\log X)^M}$.

The proof of Theorem 3′ follows closely the proof of Theorem 3. We will indicate the slight differences in due course. This strategy seems more illuminating than proving the general statement. In Section 6 we will explaine how to derive (5) and (6) from Theorems 3 and 3′ via the weighted linear sieve but we will be very brief with respect to the sieve. Before that we prove these theorems using the Hardy-Littlewood method. Section 3 is devoted to the "minor arcs", Section 4 to the "major arcs" and Section 5 to the "singular series" although these terms will not appear in their conventional form.

3. **Application of the circle method.** We will detect the equation $b + c = 2u$ in (10) by the circle method, and the equation $a + b + c = 3dz$ by $d|a + 2u$, $3|a + b + c$ is immediate from the choice of (a_0, b_0, c_0). (The equation $3z - b - c = da$ is detected by $d|3z - 2u$ instead.) We have for fixed a, b and c

$$\Lambda\Big(\frac{b+c}{2}\Big) = \int_0^1 \sum_{\substack{u\le X\\ d|a+2u}}\Lambda(u)e\big(\alpha(b+c-2u)\big)\,d\alpha.$$

Summing over a, b and c we arrive at

$$\sum_z \omega^*(dz) = \sum_{\substack{a\le X\\ a\equiv a_0(36)}}\Lambda^*(a)\int_0^1\sum_{\substack{b\le X\\ b\equiv b_0(36)}}\Lambda(b)\Lambda\Big(\frac{a+b}{2}\Big)e(\alpha b)\times$$
$$\times\sum_{\substack{c\le X\\ c\equiv c_0(36)}}\Lambda(c)\Lambda\Big(\frac{a+c}{2}\Big)e(\alpha c)\sum_{\substack{u\le X\\ d|a+2u}}\Lambda(u)e(-2\alpha u)\,d\alpha.$$

We introduce the next functions defined for any fixed positive integers a and d.

$$f(\alpha\,;d,a) = \sum_{\substack{bu\le X\\ d|a+2u}}\Lambda(u)e(\alpha u),\quad f_{a,b_0}(\alpha) = \sum_{\substack{b\le X\\ b\equiv b_0(36)}}\Lambda(b)\Lambda\Big(\frac{a+b}{2}\Big)e(\alpha b).$$

For Theorem 3′ these definitions alter to

$$f(\alpha\,;d,z) = \sum_{\substack{u\le X\\ d|3z-2u}}\Lambda(u)e(\alpha u),\quad f_{z,b_0}(\alpha) = \sum_{\substack{b\le X\\ b\equiv b_0(36)}}\Lambda(b)\Lambda\Big(\frac{3z-b}{2}\Big)e(\alpha b).$$

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The first one is the well-known generating function of primes in an arithmetic progression, and well studied in the literature, while the second one is the anologue for a kind of prime twins, and almost nothing is known about, except on average over a. We have

$$\sum_z \omega^*(dz) = \sum_{\substack{a \le X \\ a \equiv a_0(36)}} \Lambda^*(a) \int_0^1 f_{a,b_0}(\alpha) f_{a,c_0}(\alpha) f(-2\alpha\,;d,a)\,d\alpha \tag{11}$$

and for Theorem 3′ we have similarly

$$\sum_a \Omega^*(da) = \sum_{\substack{z \le X \\ z \equiv z_0(12)}} \Lambda^*(z) \int_0^1 f_{z,b_0}(\alpha) f_{z,c_0}(\alpha) f(-2\alpha\,;d,z)\,d\alpha. \tag{11′}$$

Although not necessary, it will be convenient to use the same weight function in the integral as is used in [B 1990]. Let $Q \ge 1$ and $L \ge 1$ be arbitrary real numbers and

$$W(\alpha) = 1 - \sum_{q \le Q} \frac{1}{qL} \sum_{(f,q)=1} \sum_{0 \le \ell < qL} e\Big(\Big(\alpha - \frac{f}{q}\Big)\ell\Big). \tag{12}$$

The important properties of this function are summarized in the following lemma (see Lemma 1 in [B 1990]).

LEMMA 1. *We have uniformly in Q, L and α*

$$W(\alpha) \ll 1 + \frac{Q^2}{L},$$

$$W(\alpha) \ll \min_{q \le Q} L\|\alpha q\| + \frac{Q^2}{L}.$$

Actually this lemma is proved with Q^{15} in place of Q^2 in the upper bounds which is perfectly enough for our purposes as Q will be a power of $\log X$ while L will be X over another power of $\log X$. However, the lemma is true in the above stronger form and the proof is straightforward, we leave it to the reader. We will also need Lemma 2 of [B 1990]

LEMMA 2. *For any $N > 0$ there is an $M = M(N) > 0$ such that if*

$$D \le \frac{x^{1/3}}{(\log X)^M},\ (\log X)^M \le s \le \frac{x}{(\log X)^M},\ \Big\|\alpha - \frac{r}{s}\Big\| < \frac{1}{s^2},\quad (r,s) = 1 \tag{13}$$

then

$$\sum_{d \le D} \max_{(f,d)=1} \Big| \sum_{\substack{u \le X \\ u \equiv f(d)}} \Lambda(u) e(\alpha u) \Big| \ll \frac{x}{(\log X)^N}.$$

Especially we have

$$\Big| \sum_{d \le D} \lambda_d f(\alpha\,;d,a) \Big| \ll \frac{x}{(\log X)^N}, \tag{14}$$

under the conditions (13) on α and D, and uniformly in a (or in z).

Let us fix $N > 0$, the exponent in the error term in Theorem 3, and choose

$$Q = (\log X)^M, \quad L = \frac{X}{(\log X)^{M+N+4}}, \tag{15}$$

where $M = M(N+4) > N+4$ is given by Lemma 2. By the Dirichlet's Approximation Theorem we can find for any α a rational number r/s such that

$$\left|\alpha - \frac{r}{s}\right| < \frac{(\log X)^M}{sX}, \quad (r,s) = 1,\ s \leq \frac{X}{(\log X)^M}. \tag{16}$$

If $s > Q = (\log X)^M$ then Lemma 2 and the first bound of Lemma 1 provide

$$\left|\sum_{d\leq D} \lambda_d f(\alpha\,;d,a)W(\alpha)\right| \ll \frac{X}{(\log X)^{N+4}}, \tag{17}$$

while if $s \leq Q$ then $\min_{q\leq Q} L\|\alpha q\| \leq (\log X)^{-N-4}$ and (17) follows from the second bound of Lemma 1 and from the trivial bound $f(\alpha\,;d,a) \ll X/d$. By Cauchy-Schwarz inequality, Parseval identity, (17) and trivial upper bounds for the number of primes we get

$$\begin{aligned} I &= \sum_{\substack{a\leq X\\ a\equiv a_0(36)}} \Lambda^*(a) \int_0^1 f_{a,b_0}(\alpha) f_{a,c_0}(\alpha) \sum_{d\leq D} \lambda_d f(-2\alpha\,;d,a) W(-2\alpha)\,d\alpha \\ &\leq \sum_{a\leq x} |\Lambda^*(a)| \sup_{\alpha} \left|\sum_{d\leq D} \lambda_d f(\alpha\,;d,a)W(\alpha)\right| \sum_{b\leq X}\left(\Lambda(b)\Lambda\left(\frac{a+b}{2}\right)\right)^2 \\ &\ll \sup_{\alpha,a}\left|\sum_{d\leq D} \lambda_d f(\alpha\,;d,a)W(\alpha)\right| X^2(\log X)^4 \ll \frac{X^3}{(\log X)^N}. \end{aligned} \tag{18}$$

4. **The "major arcs".** In this section we evaluate I. The structure of $W(\alpha)$ implies that I is the sum we are interested in minus a complicated looking average of it, more precisely we have

$$\begin{aligned} I = \sum_{d\leq D} \lambda_d \sum_z \omega^*(dz) - \sum_{\substack{a\leq X\\ a\equiv a_0(36)}} \Lambda^*(a) \sum_{q\leq Q} \sum_{(f,q)=1} \sum_{\substack{b\leq X\\ b\equiv b_0(36)}} \sum_{\substack{c\leq X\\ c\equiv c_0(36)}} \\ \times \Lambda(b)\Lambda(c)\Lambda\left(\frac{a+b}{2}\right)\Lambda\left(\frac{a+c}{2}\right)\frac{1}{qL}\sum_{d\leq D}\lambda_d \sum_{\substack{u\leq X\\ d\mid a+2u\\ 0\leq \ell<qL\\ b+c=2(u+\ell)}} \Lambda(u) e\left(-\frac{f\ell}{q}\right). \end{aligned} \tag{19}$$

After splitting the summation over u into residue classes g modulo q we can evaluate the

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inner sum by the Bombieri-Vinogradov Theorem.

$$\frac{1}{qL}\sum_{d\le D}\lambda_d \sum_{\substack{u\le X\\ d|a+2u\\ 0\le \ell<qL\\ b+c=2(u+\ell)}} \Lambda(u)e\Big(-\frac{f\ell}{q}\Big)$$

$$= \sum_{(g,q)=1} e\Big(-\frac{f}{q}\Big(\frac{b+c}{2}-g\Big)\Big)\frac{1}{qL}\sum_{d\le D}\lambda_d \sum_{\substack{\frac{b+c}{2}-qL<u\le\frac{b+c}{2}\\ u\equiv g \bmod q\\ 2u\equiv -a \bmod d}} \Lambda(u)$$

$$= \frac{1}{\phi[q,d]}\sum_{\substack{(g,q)=1\\ 2g\equiv -a(q,d)}} e\Big(-\frac{f}{q}\Big(\frac{b+c}{2}-g\Big)\Big)\sum_{\substack{d\le D\\ (a,d)=1}}\lambda_d + O\Big(\frac{1}{(\log X)^{2M+N+3}}\Big).$$

Here and later we abbreviate $\phi([q,d])$ by $\phi[q,d]$, where $[q,d]$ is the least common multiple. A similar convention applies to the greatest common divisor, already in the above formula. (For Theorem 3′ we have the condition $2g\equiv 3z \bmod (q,d)$ rather than $2g\equiv -a \bmod (q,d)$.) Writing this into (19) and using the bound (18) to I we get

$$\sum_{d\le D}\lambda_d\sum_z \omega^*(dz)$$

$$= \sum_{d\le D}\lambda_d \sum_{\substack{a\le X\\ (a,d)=1\\ a\equiv a_0(36)}} \Lambda^*(a)\sum_{q\le Q}\frac{1}{\phi[q,d]}\sum_{(f,q)=1}\sum_{\substack{(g,q)=1\\ 2g\equiv -a(q,d)}} e\Big(\frac{fg}{q}\Big)$$

$$\times \sum_{\substack{b\le X\\ b\equiv b_0(36)}}\sum_{\substack{c\le X\\ c\equiv c_0(36)}} \Lambda(b)\Lambda(c)\Lambda\Big(\frac{a+b}{2}\Big)\Lambda\Big(\frac{a+c}{2}\Big)e\Big(-\frac{f(b+c)}{2q}\Big) + O\Big(\frac{X^3}{(\log X)^N}\Big),$$

To get further we have to split the sum over b and c into residue classes h and k modulo $2q$. We arrive at

(20)

$$\sum_{d\le D}\lambda_d\sum_z\omega^*(dz) = \sum_{d\le D}\lambda_d\sum_{q\le Q}\frac{1}{\phi[q,d]}\sum_{\substack{a\le X\\ a\equiv a_0(36)}}\Lambda^*(a)$$

$$\times\sum_f\sum_g\sum_h\sum_k e\Big(\frac{fg}{q}\Big)e\Big(\frac{-fh}{2q}\Big)e\Big(\frac{-fk}{2q}\Big)$$

$$\times \sum_{\substack{b\le X\\ b\equiv h(2q)\\ b\equiv b_0(36)}}\sum_{\substack{c\le X\\ c\equiv k(2q)\\ c\equiv c_0(36)}} \Lambda(b)\Lambda(c)\Lambda\Big(\frac{a+b}{2}\Big)\Lambda\Big(\frac{a+c}{2}\Big) + O\Big(\frac{X^3}{(\log X)^N}\Big),$$

where $\sum_f$ and $\sum_g$ runs over sets of reduced residues modulo q while $\sum_h$ and $\sum_k$ runs over sets of reduced residues modulo $2q$ satisfying

(21)
$$(f,q)=1,$$
$$(g,q)=1,\quad 2g+a\equiv 0 \bmod (q,d),$$

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$$(h,2q)=1,\quad \Big(\frac{a+h}{2},q\Big)=1,\quad h\equiv b_0 \bmod (36,2q),$$
$$(k,2q)=1,\quad \Big(\frac{a+k}{2},q\Big)=1,\quad k\equiv c_0 \bmod (36,2q).$$

For Theorem 3′ we have to alter the conditions slightly

$$(21') \qquad \begin{gathered} (f,q)=1,\\ (g,q)=1,\quad 2g\equiv 3z \bmod (q,d),\\ (h,2q)=1,\quad \Big(\frac{3z-h}{2},q\Big)=1,\quad h\equiv b_0 \bmod (36,2q),\\ (k,2q)=1,\quad \Big(\frac{3z-k}{2},q\Big)=1,\quad k\equiv c_0 \bmod (36,2q). \end{gathered}$$

The main result of [B 1990] is that the prime k-tuplets conjecture is true on average. To evaluate the summation over b and over c in (20) we will need a special case of this. Note that what we really need, asymptotics for prime-twins on average, was available in the 1960's by the works of Lavrik [L 1961].

LEMMA 3. *Let A, B, C, f, q be integers, $ACfq\neq 0$, $(Af+B,q)=1$ and $N>0$ be real. We have*

$$\sum_{(Cf+D,q)=1}\Big|\sum_n \Lambda(An+B)\Lambda(Cn+D)-\sigma(A,B,C,D\,;f,q)\sum_n 1\Big| \ll \frac{X^2}{(\log X)^N},$$

where $\sum_n$ runs over all integers n satisfying $0<An+B\le X$, $0<Cn+D\le X$ and $n\equiv f \bmod q$, moreover

$$\sigma(A,B,C,D\,;f,q)=\prod_{p|q}\frac{p}{p-\rho(p)}\prod_p\Big(1-\frac{1}{p}\Big)^{-2}\Big(1-\frac{\rho(p)}{p}\Big)$$

and $\rho(p)$ is the number of solutions of $(An+B)(Cn+D)\equiv 0 \bmod p$. The implied constant depends on A, C and N only.

The summation over b and similarly over c can be evaluated by Lemma 3. Writing $b=36n+b_0$ the summation over b becomes $\sum_n \Lambda(36n+b_0)\Lambda(18n+\frac{a+b_0}{2})$ over n satisfying $0<36n+b_0\le X$ and $36n+b_0\equiv h \bmod 2q$. Let $\rho(p)$ be the number of solutions of the congruence $(36n+b_0)(18n+\frac{a+b_0}{2})\equiv 0 \bmod p$. As $(a,6)=(b_0,6)=(\frac{a+b_0}{2},6)=1$ for all a we are interested in, we have that

$$\rho(p)=\begin{cases}0 & \text{if } p=2,3,\\ 1 & \text{if } p|a,\\ 2 & \text{if } p\nmid a,\ p>3.\end{cases}$$

Lemma 3 implies that

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$$\sum_{\substack{a\le X\\(a,d)=1\\a\equiv a_0(36)b}}\left|\sum_{\substack{b\le X\\b\equiv h(2q)\\b\equiv b_0(36)}}\Lambda(b)\Lambda\Big(\frac{a+b}{2}\Big)-\frac{(18,q)}{4q}\prod_{p>3}\frac{p(p-2)}{(p-1)^2}\prod_{\substack{p\mid q\\p>3}}\frac{p}{p-2}\prod_{\substack{p\mid a\\p\nmid q}}\frac{p-1}{p-2}X\right| \ll \frac{X^2}{(\log X)^{6M+N+2}}.$$

For the other situation Lemma 3 implies

$$\sum_{\substack{z\le X\\(z,d)=1\\z\equiv z_0(12)}}\left|\sum_{\substack{b\le X\\b\equiv h(2q)\\b\equiv b_0(36)}}\Lambda(b)\Lambda\Big(\frac{3z-b}{2}\Big)-\frac{(36,q)}{4q}\prod_{p>3}\frac{p(p-2)}{(p-1)^2}\prod_{\substack{p\mid q\\p>3}}\frac{p}{p-2}\prod_{\substack{p\mid z\\p\nmid q}}\frac{p-1}{p-2}XH(z)\right| \ll \frac{X^2}{(\log X)^{6M+N+2}}.$$

Writing these into (20) first we change the sum over b than the sum over c into their expected main term. We arrive at

(22)

$$\sum_{d\le D}\lambda_d\sum_z\omega^*(dz)=\frac{1}{16}\prod_{p>3}\frac{p^2(p-2)^2}{(p-1)^4}X^2\sum_{d\le D}\lambda_d\sum_{\substack{a\le X\\(a,d)=1\\a\equiv a_0(36)}}\Lambda^*(a)$$
$$\times\sum_{q\le Q}\frac{(18,q)^2\kappa(q)}{q^2\phi[q,d]}\prod_{\substack{p\mid q\\p>3}}\Big(\frac{p}{p-2}\Big)^2\prod_{\substack{p\mid a\\p\nmid q}}\Big(\frac{p-1}{p-2}\Big)^2+O\Big(\frac{X^3}{(\log X)^N}\Big),$$

where

$$\kappa(q)=\kappa_{d,a}(q)=\sum_f\sum_g\sum_h\sum_k e\Big(\frac{fg}{q}\Big)e\Big(\frac{-fh}{2q}\Big)e\Big(\frac{-fk}{2q}\Big),$$

and $\sum_f$ and $\sum_g$ runs over sets of reduced residues modulo q while $\sum_h$ and $\sum_k$ runs over sets of reduced residues modulo $2q$ satisfying (21) or (21').

In the next section we will prove the following explicite formula for $\kappa(q)$.

LEMMA 4. *For any admissible triplet* (a_0,b_0,c_0) *and integers* $(a,d)=(d,6)=1$ *with* $a\equiv a_0 \bmod 36$ *and* d *is square-free we have that* $\kappa(q)$ *is multiplicative,* $\kappa(p^r)=0$ *if* $r\ge 2$ *and*

$$\kappa(p)=\begin{cases}1, & \text{if } p=2,3,\\ 1-p, & \text{if } p\mid a,\ p\nmid d,\\ 4-p, & \text{if } p\nmid a,\ p\nmid d,\\ 2p-4, & \text{if } p\nmid a,\ p\mid d.\end{cases}$$

(The same is true for the altered $\kappa(q)$ just every a must be changed to z.) Having this lemma we can evaluate the summation over q. It turns out that the sum converges at a rate $\tau(q)q^{-2}$, where $\tau(q)$ is the divisor function. We get

$$\sum_{q\le Q}\frac{(18,q)^2\kappa(q)}{q^2\phi[q,d]}\prod_{\substack{p|q\\ p>3}}\Big(\frac{p}{p-2}\Big)^2\prod_{\substack{p|a\\ p\nmid q}}\Big(\frac{p-1}{p-2}\Big)^2$$
$$=\prod_{p|d}\frac{1}{p-1}\prod_{p|a}\Big(\frac{p-1}{p-2}\Big)^2\sum_{q\le Q}\mu^2(q)\prod_{p|q}\frac{(6,p)^2}{p^2(p-1)}\prod_{\substack{p|q\\ p\nmid 6}}\Big(\frac{p}{p-2}\Big)^2$$
$$\times\prod_{\substack{p|(a,q)\\ p\nmid 6d}}\frac{(p-2)^2}{1-p}\prod_{\substack{p|q\\ p\nmid 6ad}}(4-p)\prod_{\substack{p|(q,d)\\ p\nmid 6}}(2p-4)(p-1)$$
$$=\prod_{p|d}\frac{1}{p-1}\prod_{p|a}\Big(\frac{p-1}{p-2}\Big)^2(1+1)(1+\frac{1}{2})\prod_{p\nmid 6ad}\Big(1+\frac{4-p}{(p-2)^2(p-1)}\Big)$$
$$\times\prod_{\substack{p|a\\ p\nmid 6d}}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p|d\\ p\nmid 6a}}\Big(1+\frac{2p-4}{(p-2)^2}\Big)+O\Big(\frac{\tau(d)(\log X)^3}{dQ}\Big).$$

Note that the "strange" factors $(1+1)(1+\frac{1}{2})$ correspond to the primes 2 and 3 when the infinite sum over q is written in product form. After writing this into (22) Theorem 3 and Theorem 3′ follow by trivial transformations.

5. **The singular series.** In this section we prove Lemma 4. First we write $q=2^rq'$, where $2\nmid q'$ and $r\ge 0$. We fix two integers $\bar{2}$ and $\bar{q}$ satisfying

$$2\bar{2}\equiv 1 \bmod q',\quad q'\bar{q}\equiv 1\bmod 2^{r+1}.$$

We write

$$f=f_1 2^r+f_2 q',$$
$$g=g_1 2^r\bar{2}^r+g_2 q'\bar{q},$$
$$h=h_1 2^{r+1}\bar{2}^{r+1}+h_2 q'\bar{q},$$
$$k=k_1 2^{r+1}\bar{2}^{r+1}+k_2 q'\bar{q},$$

and one can easily check that the conditions (21) are equivalent to the following two sets of conditions.

$$(f_1,q')=1,$$
$$(g_1,q')=1,\quad 2g_1+a\equiv 0\bmod (q',d),$$
$$(h_1,q')=1,\quad (a+h_1,q')=1,\quad h_1\equiv b_0\bmod (9,q'),$$
$$(k_1,q')=1,\quad (a+k_1,q')=1,\quad k_1\equiv c_0\bmod (9,q')$$

and

$$\begin{aligned} (f_2, 2^r) &= 1, \\ (g_2, 2^r) &= 1, \\ (h_2, 2^{r+1}) &= 1, \quad h_2 \equiv b_0 \bmod (4, 2^{r+1}), \\ (k_2, 2^{r+1}) &= 1, \quad k_2 \equiv c_0 \bmod (4, 2^{r+1}). \end{aligned}$$

For the altered function $\kappa(q)$ the first set of conditions is replaced by

$$\begin{aligned} (f_1, q') &= 1, \\ (g_1, q') &= 1, \quad 2g_1 \equiv 3z \bmod (q', d), \\ (h_1, q') &= 1, \quad (3z - h_1, q') = 1, \quad h_1 \equiv b_0 \bmod (9, q'), \\ (k_1, q') &= 1, \quad (3z - k_1, q') = 1, \quad k_1 \equiv c_0 \bmod (9, q'). \end{aligned}$$

Writing these into the definition of $\kappa(q)$ we arrive at

$$\begin{aligned} \kappa(q) &= \sum_f \sum_g \sum_h \sum_k e\Big(\frac{fg}{q}\Big) e\Big(\frac{-fh}{2q}\Big) e\Big(\frac{-fk}{2q}\Big) \\ &= \sum_{f_1} \sum_{g_1} \sum_{h_1} \sum_{k_1} \sum_{f_2} \sum_{g_2} \sum_{h_2} \sum_{k_2} e\Big(\frac{(f_1 2^r + f_2 q')(g_1 2^r \bar{2}^r + g_2 q' \bar{q})}{2^r q'}\Big) \\ &\qquad \times e\Big(\frac{-(f_1 2^r + f_2 q')(h_1 2^{r+1} \bar{2}^{r+1} + h_2 q' \bar{q})}{2^{r+1} q'}\Big) \\ &\qquad e\Big(\frac{-(f_1 2^r + f_2 q')(k_1 2^{r+1} \bar{2}^{r+1} + k_2 q' \bar{q})}{2^{r+1} q'}\Big) \\ &= \sum_{f_1} \sum_{g_1} \sum_{h_1} \sum_{k_1} e\Big(\frac{f_1 g_1}{q'}\Big) e\Big(\frac{-f_1 h_1 \bar{2}}{q'}\Big) e\Big(\frac{-f_1 k_1 \bar{2}}{q'}\Big) \\ &\qquad \times \sum_{f_2} \sum_{g_2} \sum_{h_2} \sum_{k_2} e\Big(\frac{f_2 g_2}{2^r}\Big) e\Big(\frac{-f_2 h_2}{2^{r+1}}\Big) e\Big(\frac{-f_2 k_2}{2^{r+1}}\Big) \\ &= \kappa_1(q') \kappa_2(2^r) \end{aligned}$$

In the calculations below we will frequently use, actually we will only use the following sums

$$\sum_{g=1}^{q} e\Big(\frac{fg}{q}\Big) = 0, \qquad \sum_{(g,q)=1} e\Big(\frac{fg}{q}\Big) = \mu(q)$$

whenever $(f, q) = 1$ and $q > 1$. Here $\mu(q)$ is the Möbius function, that is $(-1)^s$ if q is a product of s different primes and 0 otherwise. We can evaluate the summation over g_2 easily getting

$$\kappa_2(2^r) = \begin{cases} 1 & \text{if } r = 0, 1, \\ 0 & \text{if } r > 1. \end{cases}$$

Making the change of variable $f_1 \bar{2} = f$ and $2g_1 = g$ in the definition of $\kappa_1(q')$, and also writing q, h, k for q', h_1, k_1 we get that (for $2 \nmid q$)

$$\kappa_1(q) = \sum_f \sum_g \sum_h \sum_k e\Big(\frac{fg}{q}\Big) e\Big(\frac{-fh}{q}\Big) e\Big(\frac{-fk}{q}\Big),$$

with the new set of conditions

$$(21'') \qquad \begin{gathered}(f,q)=1,\\ (g,q)=1, \quad g+a\equiv 0 \bmod (q,d),\\ (h,q)=1, \quad (a+h,q)=1, \quad h\equiv b_0 \bmod (9,q),\\ (k,q)=1, \quad (a+k,q)=1, \quad k\equiv c_0 \bmod (9,q).\end{gathered}$$

We can prove that $\kappa_1(q)$ is multiplicative by much the same argument as the one we used in order to separate $\kappa(2^r q')=\kappa_1(q')\kappa_2(2^r)$, we omit the details. We only have to calculate $\kappa_1(p^r)$ for all odd primes p. We start with $p=3$. As $(d,6)=1$ the congruence condition on g is abundant and we can evaluate the summation over g to get -1 or 0 according to $r=1$ or $r>1$. For $r=1$ the summation over h and over k contains the single terms $h=b_0$ and $k=c_0$. We arrive at

$$\kappa_1(3)=-\Big(e\Big(-\frac{1}{3}(b_0+c_0)\Big)+e\Big(-\frac{2}{3}(b_0+c_0)\Big)\Big)=1$$

and we have

$$\kappa_1(3^r)=\begin{cases}1 & \text{if } r=0,1,\\ 0 & \text{if } r>1.\end{cases}$$

Next we study primes $p|a$ and therefore $p \nmid d$. (If we do not assume that $(a,d)=1$ then here we realize that no g satisfies (21') and so $\kappa_1(p^r)=0$.) We have no extra condition on g and we can evaluate the summation over g again to get -1 or 0 according to $r=1$ or $r>1$. We also have no extra condition on h and k because $p|h$ iff $p|a+h$. We arrive at

$$\kappa_1(p)=-\sum_{f=1}^{p-1}\sum_{h=1}^{p-1}\sum_{k=1}^{p-1}e\Big(\frac{-fh}{p}\Big)e\Big(\frac{-fk}{p}\Big)=1-p.$$

If $p \nmid a$ and $p \nmid d$ then again no extra condition on g and we get

$$\begin{aligned}\kappa_1(p)&=-\sum_{f=1}^{p-1}\Big(\sum_{\substack{h=1\\ p\nmid a+h}}^{p-1}e\Big(\frac{-fh}{p}\Big)\Big)^2=-\sum_{f=1}^{p-1}\Big(-1-e\Big(\frac{af}{p}\Big)\Big)^2\\ &=-\sum_{f=1}^{p-1}\Big(1+2e\Big(\frac{af}{p}\Big)+e\Big(\frac{2af}{p}\Big)\Big)=4-p.\end{aligned}$$

Only the case of $p \nmid a$ and $p|d$ left. Remember that d is square-free and therefore $(p^r,d)=p$. Thus we need $g+a\equiv 0 \bmod p$ and we can list these integers by $g=-a+mp$, $m=1,\ldots,p^{r-1}$ and we get

$$\sum_g e\Big(\frac{fg}{p^r}\Big)=e\Big(-\frac{af}{p^r}\Big)\sum_{m=1}^{p^{r-1}}e\Big(\frac{fm}{p^{r-1}}\Big)=\begin{cases}e(-\frac{af}{p^r}) & \text{if } r=1,\\ 0 & \text{if } r>1.\end{cases}$$

In the evaluation of the summation over h we can suppose that $r=1$.

$$\sum_h e\Big(-\frac{fh}{p}\Big)=\sum_{h=1}^{p}e\Big(-\frac{fh}{p}\Big)-1-e\Big(\frac{af}{p}\Big)=-1-e\Big(\frac{af}{p}\Big).$$

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Finally we get

$$\kappa_1(p)=\sum_{f=1}^{p-1} e\Big(-\frac{af}{p}\Big)\Big(-1-e\Big(\frac{af}{p}\Big)\Big)^2=\sum_{f=1}^{p-1}\Big(e\Big(-\frac{af}{p}\Big)+2+e\Big(\frac{af}{p}\Big)\Big)=2p-4.$$

The proof of Lemma 4 is complete. (The proof with the altered definition of $\kappa(q)$ proceeds exactly the same way.)

6. **A weighted sieve.** In this section we apply a weighted linear seive. We use constant weights of Kuhn's type, because they are relatively simple and serve our purposes very well. Let us define

$$\Pi=\prod_{3<p<X^{1/12}} p.$$

First we show that

$$\begin{aligned}\sum_{z=P_3}\omega(z)\geq\sum_{(z,\Pi)=1}\omega(z)\Big(1-\sum_{\substack{p\mid z\\ X^{1/12}\leq p<X^{1/4}}}\frac{1}{3}\Big)\\ -\frac{2}{3}\sum_{\substack{z=p_1p_2p_3p_4\\ X^{1/12}\leq p_1<X^{1/4}\leq p_2<p_3<p_4}}\omega(z)-\frac{1}{3}\sum_{\substack{z=p_1p_2p_3p_4\\ X^{1/12}\leq p_1<p_2<X^{1/4}\leq p_3<p_4}}\omega(z)\\ -\frac{1}{3}\sum_{\substack{z=p_1p_2p_3p_4p_5\\ X^{1/12}\leq p_1<p_2<X^{1/4}\leq p_3<p_4<p_5}}\omega(z)-\sum_{X^{1/12}\leq p}\sum_{z}\omega(p^2z)\\ =\sum_1-\frac{2}{3}\sum_2-\frac{1}{3}\sum_3-\frac{1}{3}\sum_4-\sum_5.\end{aligned}\tag{23}$$

To verify this inequality we have to check what z have positive weight on the right hand side. We clearly have $\omega(z)=0$ for $z>X$, and the weights attached to z have the structure $\omega(z)\times$ a coefficient. This coefficient is always ≤ 1. If z is not square-free and counted in the first sum, then also in the last sum and the attached coefficient is ≤ 0. We can restrict ourselves to square-free z. z has a chance for positive coefficients if $(z,\Pi)=1$ and z has at most two prime factors in $I=\{X^{1/12}\leq p<X^{1/4}\}$. However, z can have at most three prime factors in $J=\{X^{1/4}\leq p\}$. If it has no prime factor in I then $z=P_3$. If z has exactly one prime factor in I and exactly three in J then the attached coefficient is 0 by the second sum. If z has exactly two prime factors in I and two or three prime factors in J then the attached coefficient is 0 again by the third or fourth sum. In the remaining cases $z=P_3$.

The last sum can be handled trivially. For any z, after choosing a and b there is at most one choice of c, thus

$$\begin{aligned}\sum_5=\sum_{X^{1/12}\leq p}\sum_{z}\omega(p^2z)\leq\sum_{X^{1/12}\leq p}\sum_{z\leq X/p^2}X^2(\log X)^6\\ \leq\sum_{X^{1/12}\leq p}\frac{X^3(\log X)^6}{p^2}\ll X^{3-1/12}(\log X)^6.\end{aligned}$$

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Σ_1 can be estimated by the linear sieve, see Theorem 8.3 of [H,R 1974]. We are not going to introduce the general sieve notations, neither the precise definition of the functions involved in the lower and upper bound sieves. All the details together with a very similar application can be found in Chapter 11 of [H,R 1974].

There are some sieveing weights $\{\lambda_d\}$, $d \le D = X^{1/3}/(\log X)^M$, $(d,6)=1$ such that

$$\sum_1 = \sum_{(z,\Pi)=1} \omega(z)\Bigl(1 - \sum_{\substack{p|z \\ X^{1/12}\le p<X^{1/4}}} \frac{1}{3}\Bigr) \tag{24}$$
$$\ge \sum_{d\le D} \lambda_d \sum_z \omega(dz) - \frac{1}{3} \sum_{X^{1/12}\le p<X^{1/4}} \sum_{d\le D/p} \lambda_{pd} \sum_z \omega(pdz)$$

and

$$\sum_{d\le D} \lambda_d \prod_{q|d} \frac{q-2}{q^2-5q+7} - \frac{1}{3} \sum_{X^{1/12}\le p<X^{1/4}} \sum_{d\le D/p} \lambda_{pd} \prod_{q|pd} \frac{q-2}{q^2-5q+7} \tag{25}$$
$$= \bigl(1+o(1)\bigr) \prod_{3<q<X^{1/12}} \Bigl(1 - \frac{q-2}{q^2-5q+7}\Bigr)\Bigl(f(4) - \frac{1}{3}\int_4^{12} F\Bigl(4-\frac{12}{t}\Bigr)\frac{dt}{t}\Bigr).$$

The functions $f(s)$ and $F(s)$ satisfy

$$f(4) - \frac{1}{3}\int_4^{12} F\Bigl(4-\frac{12}{t}\Bigr)\frac{dt}{t} = \frac{e^{\gamma}\log 3}{6}, \tag{26}$$

where γ is the Euler's constant. Theorem 3 together with (25) and (26) implies that

$$\begin{aligned}
\sum_1 &\ge \bigl(1+o(1)\bigr)\frac{1}{64}\prod_{p>3}\frac{p^3(p^2-5p+7)}{(p-1)^5}X^3 \\
&\quad\times \prod_{3<q<X^{1/12}}\Bigl(1-\frac{q-2}{q^2-5q+7}\Bigr)\Bigl(f(4)-\frac{1}{3}\int_4^{12}F\Bigl(4-\frac{12}{t}\Bigr)\frac{dt}{t}\Bigr) \\
&= \bigl(1+o(1)\bigr)\frac{1}{64}\prod_{p>3}\frac{p^3(p^2-5p+7)}{(p-1)^5}\Bigl(1-\frac{p-2}{p^2-5p+7}\Bigr)\Bigl(1-\frac{1}{p}\Bigr)^{-1}X^3 \\
&\quad\times \prod_{3<q<X^{1/12}}\Bigl(1-\frac{1}{q}\Bigr)\Bigl(f(4)-\frac{1}{3}\int_4^{12}F\Bigl(4-\frac{12}{t}\Bigr)\frac{dt}{t}\Bigr) \\
&= \bigl(1+o(1)\bigr)\frac{3\log 3}{32}\prod_{p>3}\frac{p^4(p-3)^2}{(p-1)^6}\frac{X^3}{\log X}.
\end{aligned}$$

In the last step we used Mertens' prime number theorem.

In Σ_2, Σ_3 and Σ_4 we change the roles of z and a and we use an upper bound sieve to the prime a. This "reversal of roles" idea is applied in Chen's fundamental work on the Goldbach's problem much the same way as we apply here. The treatment of the three

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sums are analogous, we give the details for $\sum_4$. We have

$$\begin{aligned}\sum_4 &= \sum_{\substack{z=p_1p_2p_3p_4p_5\\ X^{1/12}\le p_1<p_2<X^{1/4}\le p_3<p_4<p_5}} \omega(z) = \sum_a \Lambda(a)\Omega^*(a)\\ &\le \log X \sum_p \Omega^*(p) + O\big(X^{5/2}(\log X)^6\big)\\ &\le \log X \sum_{d\le D} \lambda_d \sum_a \Omega^*(da) + O\big(X^{5/2}(\log X)^6\big),\end{aligned}$$

where $\{\lambda_d\}$ is a set of upper bound sieving weights and $\Lambda^*(z)$ is the characteristic function of the set $z = p_1p_2p_3p_4p_5 \le X$ where $X^{1/12} \le p_1 < p_2 < X^{1/4} \le p_3 < p_4 < p_5$ and z is restricted to the residue class $z \equiv z_0 \bmod 12$.

Similarly to (25) we have from the upper bound sieve that these sieving weights provide

$$\sum_{d\le D} \lambda_d \prod_{q\mid d} \frac{q-2}{q^2-5q+7} = \big(F(2)+o(1)\big) \prod_{3<q<X^{1/6}} \Big(1-\frac{q-2}{q^2-5q+7}\Big). \tag{25'}$$

Thus Theorem 3′ implies (similarly as above) that

$$\sum_4 \le \big(1+o(1)\big)\frac{27}{8}\prod_{p>3}\frac{p^4(p-3)^2}{(p-1)^6}X^2\sum_z \Lambda^*(z)\big(H(z)\big)^2.$$

Here we used that $F(2) = e^\gamma$. Next we have to evaluate the sum

$$\sum_z \Lambda^*(z)\big(H(z)\big)^2 = \sum_{\substack{p_1p_2p_3p_4p_5\le X\\ p_1p_2p_3p_4p_5\equiv z_0(12)\\ X^{1/12}\le p_1<p_2<X^{1/4}\le p_3<p_4<p_5}} \big(H(p_1p_2p_3p_4p_5)\big)^2.$$

This can be derived from the Prime Number Theorem in a standard way. Similar calculations are made in plenty of places, for example in Chapter 11 of [H,R 1974], we do not work out all details. If we write $p_i = X^{t_i}$ then the size conditions are

$$\frac{1}{12} \le t_1 < t_2 < \frac{1}{4} \le t_3 < t_4 < t_5, \quad t_1+t_2+t_3+t_4+t_5 \le 1,$$

which also imply that

$$t_1 < \frac{1}{8}, \quad t_2 < \frac{1}{4} - t_1, \quad t_3 < \frac{1-t_1-t_2}{3}, \quad t_4 < \frac{1-t_1-t_2-t_3}{2}.$$

We get that

$$\sum_z \Lambda^*(z)\big(H(z)\big)^2 = \frac{\big(1+o(1)\big)X}{12\log X}\int_{\frac{1}{12}}^{\frac{1}{8}}\int_{t_1}^{\frac{1}{4}-t_1}\int_{\frac{1}{4}}^{\frac{1-t_1-t_2}{3}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}} \frac{dt_4\,dt_3\,dt_2\,dt_1}{t_1t_2t_3t_4(1-t_1-t_2-t_3-t_4)}$$

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which leads to

$$\sum_4 \leq \left(1+o(1)\right)\frac{9}{32}\prod_{p>3}\frac{p^4(p-3)^2}{(p-1)^6}\frac{X^3}{\log X} \tag{27}$$
$$\times \int_{\frac{1}{12}}^{\frac{1}{8}}\int_{t_1}^{\frac{1}{4}-t_1}\int_{\frac{1}{4}}^{\frac{1-t_1-t_2}{3}}\int_{t_3}^{\frac{1-t_1-t_2-t_3}{2}}\frac{dt_4dt_3dt_2dt_1}{t_1t_2t_3t_4(1-t_1-t_2-t_3-t_4)}.$$

Much the same way we get the inequalities

$$\sum_3 \leq \left(1+o(1)\right)\frac{9}{32}\prod_{p>3}\frac{p^4(p-3)^2}{(p-1)^6}\frac{X^3}{\log X}\times\int_{\frac{1}{12}}^{\frac{1}{4}}\int_{t_1}^{\frac{1}{4}}\int_{\frac{1}{4}}^{\frac{1-t_1-t_2}{2}}\frac{dt_3dt_2dt_1}{t_1t_2t_3(1-t_1-t_2-t_3)}$$

and

$$\sum_2 \leq \left(1+o(1)\right)\frac{9}{32}\prod_{p>3}\frac{p^4(p-3)^2}{(p-1)^6}\frac{X^3}{\log X}\times\int_{\frac{1}{12}}^{\frac{1}{4}}\int_{\frac{1}{4}}^{\frac{1-t_1}{3}}\int_{t_2}^{\frac{1-t_1-t_2}{2}}\frac{dt_3dt_2dt_1}{t_1t_2t_3(1-t_1-t_2-t_3)}.$$

Using the fact that $\log(3-4u)/(1-u)$ is decreasing in $0\leq u\leq 1/2$ we can easily see that

(28)

$$\begin{aligned}\int_{\frac{1}{12}}^{\frac{1}{4}}\int_{t_1}^{\frac{1}{4}}\int_{\frac{1}{4}}^{\frac{1-t_1-t_2}{2}}\frac{dt_3dt_2dt_1}{t_1t_2t_3(1-t_1-t_2-t_3)} &= \int_{\frac{1}{12}}^{\frac{1}{4}}\int_{t_1}^{\frac{1}{4}}\log(3-4t_1-4t_2)\frac{dt_2dt_1}{t_1t_2(1-t_1-t_2)}\\ &\leq \frac{6}{5}\log\frac{7}{3}\int_{\frac{1}{12}}^{\frac{1}{4}}\int_{t_1}^{\frac{1}{4}}\frac{dt_2dt_1}{t_1t_2}\\ &= \frac{3}{5}\left(\log\frac{7}{3}\right)(\log 3)^2 < 0.61359\end{aligned}$$

We can bound the other two integrals in a similar manner. Collecting all these bounds we get that

$$N(X\,;z=P_3)\geq 0.23562\times\frac{27}{8}\prod_{p>3}\frac{p^4(p-3)^2}{(p-1)^6}\frac{X^3}{(\log X)^7} \tag{29}$$

for sufficienly large X. The proof of (7) is analogous. Note that by numerical integration one can get 1.05357 in place of 0.23562 in (29).

7. **The proof of (6).** As we have indicated in the introduction, we are not going to present the detailed proof for two reasons. First it is a combination of the above ideas and Heath-Brown's large sieve technique on the minor arcs, no further idea is necessary. Second the technical steps would make the paper too lengthy and unreadable. What we do present is how to reach $D=X^{1/2}/(\log X)^M$ in a Theorem 3 like statement. Even in this we will only deal with the "minor arcs" bound as the calculation of the main term is identical to that of in Section 4. The application to the weighted sieve than follows the pattern given in Section 6 and actually well discussed in some places such as [H,R 1974] and [H-B 1981].

This time we list our triangles in a third different way, namely according to w, b, c such that $2w-b$, $\frac{b+c}{2}$, $\frac{2w+c}{3}$ are primes and we sieve the sequence of $v=\frac{2w-b+c}{2}$. If

(a_0, b_0, c_0) is a fixed admissible triplet modulo (36), and we study triangles with corners congruent to this triplet, then in our list we need $w \equiv w_0 \equiv \frac{a_0+b_0}{2} \bmod 18$. We define

$$\Omega(v) = \sum\sum\sum \Lambda(w)\Lambda(b)\Lambda(c)\Lambda(2w-b)\Lambda\Big(\frac{2w+c}{3}\Big)\Lambda\Big(\frac{b+c}{2}\Big),$$

where the triple sum is extended to $w \le X$, $b \le X$, $c \le X$, such that $0 < 2w - b \le X$, $2w - b + c = 2v$ and $(b, c) \equiv (b_0, c_0) \bmod 36$, $w \equiv w_0 \bmod 18$. We simplify notation by restricting our functions to the interval $0 < n \le X$. Actually we have to change $\Lambda(w)$ to a more general function $\Lambda^*(w)$ again, because after reversing the roles of w and v in the application of the weighted sieve, the set of v becomes the set of unwanted products of three primes surviving the sieving process. As we have already seen this does not make any difference, and for simplicity we stay with $\Lambda(w)$.

Now let $\{\lambda_d\,; d \le D, (d, 6) = 1\}$ be a set of sieveing weights. We are going to study the sum

$$\sum_{d \le D} \lambda_d \sum_v \Omega(dv).$$

We will detect the equation $b + c = 2u$ by the circle method, and the divisibility $d|2w - b + c$ by the discrete circle method, namely by

$$\frac{1}{d}\sum_{r=1}^{d} e\Big(\frac{rn}{d}\Big) = \begin{cases} 1 & \text{if } d|n, \\ 0 & \text{if } d \nmid n. \end{cases}$$

We arrive at

$$\begin{aligned}\sum_{d\le D} \lambda_d \sum_v \Omega(dv) &= \sum_{d \le D} \frac{\lambda_d}{d} \sum_{r=1}^{d} \sum_w \Lambda(w) e\Big(\frac{2rw}{d}\Big) \int_0^1 \sum_b \Lambda(b)\Lambda(2w-b) e\Big(\Big(\alpha - \frac{r}{d}\Big)b\Big) \\ &\quad \times \sum_c \Lambda(c)\Lambda\Big(\frac{2w+c}{3}\Big) e\Big(\Big(\alpha + \frac{r}{d}\Big)c\Big) \sum_u \Lambda(u) e(-2\alpha u)\, d\alpha \\ &= \sum_{d \le D} \frac{\lambda_d}{d} \sum_{r=1}^{d} \sum_w \Lambda(w) e\Big(\frac{2rw}{d}\Big) \int_0^1 g_w\Big(\alpha - \frac{r}{d}\Big) h_w\Big(\alpha + \frac{r}{d}\Big) f(-2\alpha)\, d\alpha.\end{aligned}$$

The definition of the functions $g_w(\alpha)$, $h_w(\alpha)$ and $f(\alpha)$ is transparent from the above line. $f(\alpha)$ is the classical trigonometric function over the primes.

We introduce our weight function $W(\alpha)$ of (12) with a similar choice of the parameters as in (15), we specify later, but with a shifted argument. Thus we are interested in estimating

(30)

$$I = \sum_{d \le D} \frac{\lambda_d}{d} \sum_{r=1}^{d} \sum_w \Lambda(w) e\Big(\frac{2rw}{d}\Big) \int_0^1 g_w\Big(\alpha - \frac{r}{d}\Big) h_w\Big(\alpha + \frac{r}{d}\Big) f(-2\alpha) W\Big(-2\alpha - \frac{2r}{d}\Big)\, d\alpha.$$

The big difference is in the upper bound for I, the calculation of the main term is almost identical to the previous calculation in Section 4. Indeed, we have

$$I=\sum_{d\le D}\lambda_d\sum_{v}\Omega(dv)-\sum_{\substack{w\le X\\ w\equiv w_0(18)}}\Lambda(w)\sum_{q\le Q}\sum_{(f,q)=1}\sum_{\substack{b\le X\\ b\equiv b_0(36)}}\sum_{\substack{c\le X\\ c\equiv c_0(36)}}$$
$$\times\Lambda(b)\Lambda(c)\Lambda(2w-b)\Lambda\Big(\frac{2w+c}{3}\Big)\frac{1}{qL}\sum_{d\le D}\lambda_d\sum_{\substack{u\le X\\ d|2w-b+c-2\ell\\ 0\le\ell<qL\\ b+c=2(u+\ell)}}\Lambda(u)e\Big(-\frac{f\ell}{q}\Big),$$

which is pretty much the same as (19). Note that $d|2w-b+c-2\ell$ is the same as $d|w-b+u$ in view of $b+c=2u+2\ell$ and $2\nmid d$, explaining the shift in the argument of the weight function. This makes possible that the divisibility by d is passed to the summation over u and can be handled by the Bombieri-Vinogradov Theorem, as far as $D\le\frac{X^{1/2}}{(\log X)^M}$, while the summations over b and over c are special cases of Lemma 3, exactly like in Section 4. However the conditions $u\equiv b-w \bmod d$, $u\equiv g \bmod q$ and $b\equiv h \bmod q$ introduce the condition $g\equiv h-w \bmod (q,d)$. This establishes a connection between g and h, which was not present in (21) or (21'). In spite of this extra connection the evaluation of the "singular series" is similar to that of in Section 5, we can safely skip the details.

The treatment of the "minor arcs", however, is different. First we list the fractions $\frac{r}{d}$ in their lowest terms $\frac{h}{g}$, $(h,g)=1$ in (30). A given $\frac{h}{g}$ occures for $g|d$ and the attached coefficient is

$$\sum_{\substack{d\le D\\ g|d}}\frac{\lambda_d}{d}\ll\frac{1}{g}\log X.$$

We get

(31)
$$I\ll\log X\sum_{g\le D}\frac{1}{g}\sum_{(h,g)=1}\sum_{w}\Lambda(w)\int_0^1\Big|g_w\Big(\alpha-\frac{h}{g}\Big)h_w\Big(\alpha+\frac{h}{g}\Big)f(-2\alpha)W\Big(-2\alpha-\frac{2h}{g}\Big)\Big|\,d\alpha.$$

We choose $M=M(15N+5)$ given by Lemma 2. For any fixed $\frac{h}{g}$ with $g\le G_0=(\log X)^{14N}$ the classical case ($D=1$) of Lemma 2 and Lemma 1 imply that

$$f(-2\alpha)W\Big(-2\alpha-\frac{2h}{g}\Big)\ll\frac{X}{(\log X)^{15N+5}}+LG_0(\log X)^M+\frac{XQ^2}{L}\ll\frac{X}{(\log X)^{15N+5}},$$

provided that $Q=(\log X)^{M+14N}$ and $L=X(\log X)^{-M-29N-5}$. Parseval identity gives that

$$\sum_{g\le G_0}\frac{1}{g}\sum_{(h,g)=1}\sum_{w}\Lambda(w)\int_0^1\Big|g_w\Big(\alpha-\frac{h}{g}\Big)h_w\Big(\alpha+\frac{h}{g}\Big)f(-2\alpha)W\Big(-2\alpha-\frac{2g}{h}\Big)\Big|\,d\alpha\ll\frac{X^3}{(\log X)^{N+1}}.$$

We split the remaining range of g into diadic intervals of type $G<g\le 2G$ where $G_0\le G\le D$. As $|W(\alpha)|\ll 1$ it is enough to show that uniformly in $w\le X$ we have

$$\sum_{G<g\le 2G}\sum_{(h,g)=1}\int_0^1\Big|g_w\Big(\alpha-\frac{h}{g}\Big)h_w\Big(\alpha+\frac{h}{g}\Big)f(-2\alpha)\Big|\,d\alpha\ll\frac{GX^2}{(\log X)^{N+1}}.\tag{32}$$

This is essentially Lemma 2 of [H-B 1981] and requires only $D \leq X^{1/2}(\log X)^{13N+22}$. Note that he had functions with rather special coefficients in place of f, g_w and h_w, however nothing is used about the coefficients during the proof. Actually Sections 6, 7 and 8 of [H-B 1981] give (32) for any three trigonometric polynomials of length X with bounded coefficients. The main tools of the proof are the Large Sieve and Lemma 3 of [H-B 1981] which reflects an elementary property of the rational numbers.

8. **Concluding remarks.** We have expressed the (weighted) number of solutions of (3) in three different ways so far, and the "reversal of roles" in Section 7 requires a fourth one. It is clear that we can do this in many other ways as well. The general shape of such an expression is

$$\begin{aligned}\sum_a\sum_b\sum_c \Lambda(a)\Lambda(b)\Lambda(c)\Lambda\Big(\frac{A_1a+B_1b}{D_1}\Big)\Lambda\Big(\frac{A_2a+C_2c}{D_2}\Big)\\ \times\Lambda\Big(\frac{A_3a+B_3b+C_3c}{D_3}\Big)\Lambda\Big(\frac{A_4a+B_4b+C_4c}{D_4}\Big),\end{aligned}\tag{33}$$

where a, b and c can denote any three of the seven variables, not only the three corners, in contrast to the former convention. This is, actually, the weighted number of the solutions of the system

$$\begin{cases} A_1a+B_1b+0c=D_1w\\ A_2a+0b+C_2c=D_2v\\ A_3a+B_3b+C_3c=D_3u\\ A_4a+B_4b+C_4c=D_4z.\end{cases}\tag{34}$$

In addition, remember that we did use very little about a, actually $\Lambda(a)$ in (33) can be changed to any function $\Lambda^*(a)$ with very mild growing and averaging conditions. The method works for any system having the above structure, that is if after diagonalization we ended up with at least two more zeros in different lines and columns. Note that the method of [B 1992] for the first three equations in six variables required only one additional zero.

We intend to detect the third equation by the circle method and approximate $\Lambda(z)$ by a sieve method. To this end we transform the last equation to

$$(A_4C_3-A_3C_4)a+(B_4C_3-B_3C_4)b+C_4D_3u=C_3D_4z.$$

Whether we have to use the method of Section 3 (and get a P_3) or Section 7 (and get a P_2) depends on how the condition $d|(A_4C_3-A_3C_4)a+(B_4C_3-B_3C_4)b+C_4D_3u$ affects the three generating functions, correspond to the variables b, c and u. If $B_4C_3-B_3C_4=0$ then the method of Section 3 is applicable as only u is affected by d. The method of Section 7 is, unfortunately, not applicable because (32) requires that the three generating functions are shifted to three different places by the rational numbers $\frac{h}{g}$. This is exactly the case when $B_4C_3-B_3C_4\neq 0$ (and $B_4C_3\neq 0$ is required anyway).

It turns out, that no matter how we write the number of solutions of (3) in the above form, if we intend to use a sieve method for the center (z formerly) or one of the corners

(a, b or c formerly) then $B_4C_3 - B_3C_4 = 0$ in the corresponding system and the divisibility condition affects two of the generating functions in an identical way. This is the symmetry which blocks the application of Heath-Brown's approach in those cases.

It is also worth mentioning that this is true on the other way around, if the divisibility condition affects the three generating functions in three different ways, that is $B_4C_3 - B_3C_4 \neq 0$ then we cannot use the (simpler, but less powerful) method of Section 3.

REFERENCES

[B 1990] A. Balog, *The prime k-tuplets conjecture on average*. Analytic Number Theory, (eds. B. Brendt, H. G. Diamond, H. Halberstam and A. Hildebrand), Birkhauser, 1990, 47–75.

[B 1992] ———, *Linear equations in primes*. Mathematika **39**(1992), 367–378.

[B,B 1995] A. Balog and J. Brüdern, *Sums of three cubes in three linked three-progressions*. J. Reine Angew. Math. **466**(1995), 45–85.

[C 1973] J.-R. Chen, *On the representation of a large even integer as the sum of a prime and a product of at most two primes*. Sci. Sinica **16**(1973), 157–176.

[H,R 1974] H. Halberstam and H. E. Richert, *Sieve methods*. Academic Press, London, 1974.

[H-B 1981] D. R. Heath-Brown, *Three primes and an almost-prime in arithmetic progression*. J. London Math. Soc. (2) **23**(1981), 396–414.

[I 1972] H. Iwaniec, *Primes of the type $\phi(x, y)+A$, where ϕ is a quadratic form*. Acta Arith. **21**(1972), 203–234.

[L 1961] A. F. Lavrik, *On the theory of distribution of primes based on I. M. Vinogradov's method of trigonometric sums*. Trudy Mat. Inst. Steklov **64**(1961), 90–125.

[vdC 1939] J. G. van der Corput, *Über Summen von Primzahlen und Primzahlquadraten*. Math. Ann. **116** (1939), 1–50.

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