A REMARK ON CRITICAL GROUPS

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Problem 24 of Hanna Neumann's book [3] reads: Does there exist, for a given integer $n > 0$, a Cross variety that is generated by its k-generator groups and contains $(k+n)$-generator critical groups? In such a variety, is every critical group that needs more than k generators a factor of a k-generator critical group, or at least of the free group of rank k? In a recent paper [1], R. G. Burns pointed out that the answer to the first question is an easy affirmative, and asked instead the question which presumably was intended: Given two positive integers k, l, does there exist a variety $\mathfrak{V}$ generated by k-generator groups and also by a set S of critical groups such that S contains a group G minimally generated by $k+l$ elements and $S\setminus\{G\}$ does not generate $\mathfrak{V}$? The purpose of this note is to record a simple example which shows that the answer to the question of Burns is affirmative at least for $k = 2$, $l = 1$, and also that the answer to the second question of Hanna Neumann's Problem 24 is negative.

Let $\mathfrak{V}$ be the variety defined by the law $x^9[x, y, u]^3[x, y, u, v]$. It can be read off from Bjarni Jónsson's description [2] of the lattice of nilpotent varieties of class at most 3 that $\mathfrak{V}$ has precisely two maximal subvarieties: the subvariety $\mathfrak{U}$ defined by the additional law $[x, y, y]$, and the subvariety $\mathfrak{W}$ defined by the additional law $[x, y]^3$; moreover, $\mathfrak{U}$ is certainly not of class 2. Since $\mathfrak{W}$ is defined (within $\mathfrak{V}$) by a two-variable law, it cannot contain the $\mathfrak{V}$-free group F of rank 2; nor can F be contained in $\mathfrak{U}$, for the two-generator groups of $\mathfrak{U}$ are all of class at most 2 while $\mathfrak{V}$ contains the wreath product of two cyclic groups of order 3, a two-generator group of class 3. Thus F generates $\mathfrak{V}$. A direct calculation shows that the proper subgroups of F are all of class at most 2: this, and a similar calculation below, is somewhat simplified by the observation that the Frattini subgroup of any group in $\mathfrak{V}$ is contained in the second term of the upper central series of the group.

Next, consider the $\mathfrak{U}$-free group H of rank 3, on the free generators a, b, c. As the only relators of these generators are the laws of $\mathfrak{U}$, neither $[a, b]^3$ nor $[a, b, c]$ can be 1. On the other hand, as $\mathfrak{U}$ has class 3 and $[[x, y]^3, u]$ is a law even in $\mathfrak{V}$, the element $[a, b]^3[a, b, c]$ is central in H. Let N be maximal among the normal subgroups of H which contain $[a, b]^3[a, b, c]$ but not $[a, b, c]$, and put $H/N = G$. By construction, G is

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monolithic, has class precisely 3, and does not belong to $\mathfrak{W}$. As $G \in \mathfrak{U}$ and the two-generator groups of $\mathfrak{U}$ are all of class at most 2, it follows that G cannot be generated by two elements. It is now easy to see that G cannot be isomorphic to any factor of F, and that no proper factor of G can have class greater than 2: in particular, G is critical. If S is the set consisting of G and of the critical groups of $\mathfrak{W}$, then S generates $\mathfrak{V}$ but $S \setminus \{G\}$ does not.

References

[1] R. G. Burns, 'Verbal wreath products and certain product varieties of groups', *J. Austral. Math. Soc.* 7 (1967), 356–374.
[2] Bjarni Jónsson, 'Varieties of groups of nilpotency 3', *Notices Amer. Math. Soc.* 13 (1966), 488.
[3] Hanna Neumann, *Varieties of groups* (Springer, Berlin etc., 1967).

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