RESTRICTIVE SEMIGROUPS OF CLOSED FUNCTIONS

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1. Introduction. *It is assumed that all topological spaces discussed in this paper are T_1 spaces.* A function f mapping a topological space X into itself is a closed function if $f[H]$ is closed for each closed subset H of S. The semigroup, under composition, of all closed functions mapping X into X is denoted by $\Gamma(X)$. These were among the semigroups under consideration in (**4**).

In the study of semigroups of functions, certain subsemigroups distinguish themselves quite naturally. One of these is the subsemigroup of $\Gamma(X)$ consisting of all those closed functions which map some non-empty subset Y of X into itself. Such a semigroup will be referred to as a restrictive semigroup of closed functions or, more simply, a restrictive semigroup since, in this paper, with the exception of §5, only semigroups of closed functions are considered. We state this formally as the following definition.

Definition (1.1). Let X be a topological space and Y a non-empty subspace of X. The semigroup, under composition, of all closed functions mapping X into X which also map Y into Y will be denoted by $\Gamma(X, Y)$ and will be referred to as a restrictive semigroup.

The results in §2 are concerned mainly with certain (two-sided) ideals of restrictive semigroups. Using these results, we are able to determine under what conditions $\Gamma(X, Y)$ can be isomorphic to some $\Gamma(Z)$. This happens only when $Y = X$. The analogous problem for $S(X, Y)$ (the semigroup of all continuous functions mapping X into X which also map Y into Y) was treated in (**5**).

In §3, isomorphisms between restrictive semigroups are investigated. Let ϕ be a bijection from $\Gamma(X, Y)$ onto $\Gamma(U, V)$. It is an easy matter to verify that in order for ϕ to be an isomorphism, it is sufficient that there exist a homeomorphism h from X onto U which carries Y onto V such that $\phi f = h \circ f \circ h^{-1}$ for each f in $\Gamma(X, Y)$. In Theorem (3.1) of §3 it is stated, among other things, that the condition is not only sufficient, but if Y has more than one point, then it is also necessary. This generalizes Theorem (2.10) of (**4**), where it is stated that a mapping ϕ from $\Gamma(X)$ onto $\Gamma(Y)$ is an isomorphism if and only if there exists a homeomorphism h from X onto Y such that $\phi f = h \circ f \circ h^{-1}$ for each f in $\Gamma(X)$.

Automorphisms of restrictive semigroups are studied in §4. It is shown that every automorphism of a restrictive semigroup $\Gamma(X, Y)$ is inner and furthermore, that the automorphism group of $\Gamma(X, Y)$ is isomorphic to the group,

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under composition, of all homeomorphisms mapping X onto X which also map Y onto Y. Some concluding remarks are given in §5.

2. Certain ideals of restrictive semigroups. Those functions mapping X into X whose ranges consist of a finite number of points are closed functions (since X is T_1) and they play a very important role in the subsequent discussion. Suppose that f is such a function and suppose, for example, that its range consists of three points x, y, and z. We find it useful to denote the function f by

$$\langle A, x; B, y; C, z\rangle, \tag{1}$$

where it is to be understood that A is the subset of X whose points are mapped by f into x, B is the subset whose points are mapped into y, and C is the subset whose points are mapped into z. In general, then, a function with finite range is given by "listing" the points of its range together with those subsets whose points are mapped into them. However, certain of these functions appear with enough frequency so that it is convenient to reserve special notation for them. For example, the functions of the form $\langle Y, y; X - Y, x\rangle$, $x \in X$ and $y \in Y$, play a particularly important role, and consequently, appear often. We shall simplify matters by letting

$$\langle Y, y; X - Y, x\rangle = [y, x]. \tag{2}$$

In other words, $[y, x]$ denotes that function which sends Y into y and the remainder of X into x. The function $\langle A_1, y; A_2, p\rangle$ also appears rather often, thus it is convenient to let

$$\langle A_1, x; A_2, p\rangle = A_{xp}. \tag{3}$$

Finally, we let

$$\langle A_1, x_1; A_2, x_2; \ldots ; A_n, x_n\rangle = A_x^n. \tag{4}$$

We note that A_x^n belongs to $\Gamma(X, Y)$ if and only if for some subset $\{i_j\}_{j=1}^M$ of $\{i\}_{i=1}^N$, $\{x_{i_j}\}_{j=1}^M \subset Y$ and $Y \subset \bigcup\{A_{i_j}\}_{j=1}^M$.

LEMMA (2.1). *A restrictive semigroup $\Gamma(X, Y)$ has a (two-sided) zero element if and only if Y consists of one point.*

Proof. Suppose that Y has at least two points, p and q. Then $\langle X, p\rangle$ and $\langle X, q\rangle$ are two distinct left zeros of $\Gamma(X, Y)$. This is incompatible with the assumption that $\Gamma(X, Y)$ has a zero. On the other hand, one easily checks that if $Y = \{p\}$, then $\langle X, p\rangle$ is a zero element of $\Gamma(X, Y)$.

The word ideal, when unmodified, will always mean a two-sided ideal.

Definition (2.2). Let I be an ideal of a semigroup S. An ideal J of S is said to be I-minimal if it properly contains I and there exists no ideal K distinct from both I and J such that $I \subset K \subset J$. If S contains a zero element 0 and $I = \{0\}$, we say that the ideal J is 0-minimal.

Suppose that Y consists of one point p. Then, according to Lemma (2.1), $\Gamma(X, \{p\})$ has a zero element 0 and, as pointed out in the proof of that lemma, $0 = \langle X, p\rangle$. Now let

$$I^* = \{0\} \cup \{A_{xp} = \langle A_1, x; A_2, p\rangle : x \in X - \{p\}, p \in A_2\}.$$

THEOREM (2.3). *Suppose that X has more than one point. Then I^* is a non-zero ideal which is contained in every non-zero ideal containing* 0. *Consequently, I^* is not only* 0*-minimal, but it is the only* 0*-minimal ideal of* $\Gamma(X, \{p\})$.

Proof. It is immediate that if X contains more than one point, then I^* consists of more than one element. For any f in $\Gamma(X, \{p\})$ and any $A_{xp} \in I^*$, $f \circ A_{xp} = 0$ if $f(x) = p$ and $f \circ A_{xp} = \langle A_1, f(x); A_2, p\rangle$ if $f(x) \neq p$. Futhermore, $A_{xp} \circ f = 0$ if $f^{-1}[A_1] = \emptyset$ and $A_{xp} \circ f = \langle f^{-1}[A_1], x; f^{-1}[A_2], p\rangle$ if $f^{-1}[A_1] \neq \emptyset$. This proves that I^* is an ideal of $\Gamma(X, \{p\})$. Now suppose that J is any non-zero ideal of $\Gamma(X, \{p\})$. Then J contains a function $f \neq 0$ which implies that there exists a point x in X such that $f(x) \neq p$. Therefore, for any $A_{yp} \in I^*$, we have that $A_{yp} = [p, y] \circ f \circ A_{xp}$ (recall that $[p, y]$ denotes that function which sends Y, in this case $\{p\}$, into p and the remainder of X into y) which implies that $A_{yp} \in J$. Thus $I^* \subset J$.

LEMMA (2.4). *Suppose that $A_{xp} = \langle A_1, x; A_2, p\rangle \in I^*$. Then $A_2 = \{p\}$ if and only if for every $f \neq 0$, we have that $A_{xp} \circ f \neq 0$.*

Proof. Suppose that $A_2 = \{p\}$ and $f \neq 0$. Then $f(y) \neq p$ for some $y \in X$ and $f^{-1}[A_1] = f^{-1}[X - \{p\}] \neq \emptyset$. This implies that

$$A_{xp} \circ f = \langle f^{-1}[A_1], x; f^{-1}[A_2], p\rangle \neq 0.$$

On the other hand, suppose that there exists an element z in $A_2 - \{p\}$. Then $[p, z] \neq 0$ but $A_{xp} \circ [p, z] = 0$.

If a semigroup has a minimal ideal, then that ideal is unique and is referred to as the kernel of the semigroup. The next result concerns the kernel of $\Gamma(X, Y)$. Its proof is straightforward and will not be given.

LEMMA (2.5). *$\{\langle X, y\rangle : y \in Y\}$ is the kernel of* $\Gamma(X, Y)$.

Definition (2.6). For each positive integer n, let

$$I_n = \{A_x^j \in \Gamma(X); 1 \leqq j \leqq n\}.$$

THEOREM (2.7). *I_1 is the kernel of* $\Gamma(X)$. *For each positive integer n which is less than the cardinality of X, I_{n+1} is an ideal which properly contains I_n and is contained in every other ideal which properly contains I_n. Thus, I_{n+1} is not only an I_n-minimal ideal of* $\Gamma(X)$, *but it is the only I_n-minimal ideal of* $\Gamma(X)$.

Proof. It follows from Lemma (2.5) by taking $Y = X$, that I_1 is the kernel of $\Gamma(X)$. One easily shows that each I_n is an ideal of $\Gamma(X)$ and it is evident that I_n is a proper subset of I_{n+1} if X has more than n elements. To complete the proof, we need only show that if J is any ideal of $\Gamma(X, Y)$ which properly contains I_n, then $I_{n+1} \subset J$. If J properly contains I_n, then J must contain a function f whose range contains at least $n + 1$ distinct elements $\{z_i\}_{i=1}^{n+1}$. Choose $y_i \in f^{-1}(z_i)$ for $1 \leqq i \leqq n + 1$ and let $H = X - \{z_i\}_{i=1}^{n}$. Then for any $A_x^{n+1} \in I_{n+1} - I_n$, we have that

$$A_x^{n+1} = \langle \{z_1\}, x_1; \ldots ; \{z_n\}, x_n; H, x_{n+1}\rangle \circ f \circ A_y^{n+1}.$$

Thus, $A_x^{n+1} \in J$ and it follows that $I_{n+1} \subset J$.

Definition (2.8.) We shall denote the kernel of $\Gamma(X, Y)$ by K_0, i.e., $K_0 = \{\langle X, y\rangle : y \in Y\}$. Furthermore, suppose that $Y \neq X$ and let

$$K_1 = K_0 \cup \{A_y^2 = \langle A_1, y_1; A_2, y_2\rangle : y_1, y_2 \in Y;\ Y \subset A_1\}.$$

THEOREM (2.9). *Suppose that $Y \neq X$ and Y consists of more than one point. Then K_1 is an ideal which properly contains K_0 and is contained in every other ideal properly containing K_0. Consequently, K_1 is not only a K_0-minimal ideal of $\Gamma(X, Y)$ but it is the only K_0-minimal ideal of $\Gamma(X, Y)$.*

Proof. For any f in $\Gamma(X, Y)$ and $A_y^2 = \langle A_1, y_1; A_2, y_2\rangle$ in $K_1 - K_0$, we have that

$$\begin{aligned}
f \circ A_y^2 &= \langle X, f(y_1)\rangle && \text{if } f(y_1) = f(y_2),\\
f \circ A_y^2 &= \langle A_1, f(y_1); A_2, f(y_2)\rangle && \text{if } f(y_1) \neq f(y_2),\\
A_y^2 \circ f &= \langle f^{-1}[A_1], y_1; f^{-1}[A_2], y_2\rangle && \text{if } f^{-1}[A_2] \neq \emptyset,\\
A_y^2 \circ f &= \langle X, y_1\rangle && \text{if } f^{-1}[A_2] = \emptyset.
\end{aligned}$$

It follows that K_1 is an ideal of $\Gamma(X, Y)$. Moreover, K_1 properly contains K_0 since Y has more than one point.

Now suppose that J is any ideal which properly contains K_0. This means that the range of f contains more than one point. Choose any $y \in Y$. It follows that $f(x) \neq f(y)$ for some $x \in X$. Let B_z^2 be any element in $K_1 - K_0$ and let $H = X - \{f(y)\}$. Then

$$B_z^2 = \langle \{f(y)\}, z_i; H, z_2\rangle \circ f \circ \langle B_1, y; B_2, x\rangle$$

which implies that $B_z^2 \in J$. Therefore $K_1 \subset J$ and the desired conclusion follows.

Definition (2.10). Denote the following:

$$K_2^1 = K_1 \cup \{A_y^2 = \langle A_1, y_1; A_2, y_2\rangle : Y \subset A_1, y_1 \in Y, y_2 \in X\},$$

$$K_2^2 = K_1 \cup \{A_y^2 = \langle A_1, y_1; A_2, y_2\rangle : y_1, y_2 \in Y\},$$

$$K_2^3 = K_1 \cup \{A_y^3 = \langle A_1, y_1; A_2, y_2; A_3, y_3\rangle : Y \subset A_1; y_1, y_2, y_3 \in Y\}.$$

THEOREM (2.11). *Suppose that $Y \neq X$ and Y consists of more than one point. Then K_2^1 and K_2^2 are both K_1-minimal ideals. If Y has only two points, these*

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are the only K_1-minimal ideals. If Y has more than two points, $K_2{}^3$ is also a K_1-minimal ideal of $\Gamma(X, Y)$ and in this case $K_2{}^1$, $K_2{}^2$, and $K_2{}^3$ are the only K_1-minimal ideals of $\Gamma(X, Y)$.

Proof. One verifies that $K_2{}^1$, $K_2{}^2$, and $K_2{}^3$ are all ideals (see, for example, the verification of K_1 in the previous proof). Since $Y \neq X$, K_1 is properly contained in $K_2{}^1$. Now suppose that J is an ideal which properly contains K_1 and is contained in $K_2{}^1$ and let $A_y{}^2$ be any element in $K_2{}^1 - K_1$. Then $Y \subset A_1$ and $y_1 \in Y$. Since $K_1 \neq J$, there exists an element which belongs to J (and hence to $K_2{}^1$) but not to K_1. This element must then be of the form $B_z{}^2$, $Y \subset B_1$, $z_1 \in Y$. Moreover, since $B_z{}^2 \notin K_1$, it follows that $z_2 \notin Y$. Choose $x_1 \in Y$ and $x_2 \in B_2$. Then $A_y{}^2 = [y_1, y_2] \circ B_z{}^2 \circ A_x{}^2$ which implies that $A_y{}^2 \in J$. Thus $J = K_2{}^1$ which proves that $K_2{}^1$ is K_1-minimal.

Now we consider the ideal $K_2{}^2$. Here also, $K_2{}^2$ properly contains the ideal K_1. Suppose that J is an ideal which properly contains K_1 and is contained in $K_2{}^2$ and let $A_y{}^2$ be any element of $K_2{}^2 - K_1$. Then both y_1 and y_2 belong to Y and $A_1 \cap Y \neq \emptyset \neq A_2 \cap Y$. Since $K_1 \neq J$, there exists an element $B_z{}^2 \in J - K_1$, where z_1, $z_2 \in Y$ and $B_1 \cap Y \neq \emptyset \neq B_2 \cap Y$. Choose $x_1 \in B_1 \cap Y$, $x_2 \in B_2 \cap Y$ and let $H = X - \{z_1\}$. Then,

$$A_y{}^2 = \langle \{z_1\}, y_1; H, y_2 \rangle \circ B_z{}^2 \circ A_x{}^2$$

which implies that $A_y{}^2 \in J$. Thus, $K_2{}^2 = J$ and it follows that $K_2{}^2$ is K_1-minimal.

Now suppose that Y contains at least three points. Then $K_2{}^3$ is an ideal which properly contains K_1. Let J be an ideal which properly contains K_1 and is contained in $K_2{}^3$ and let $A_y{}^3$ be an element in $K_2{}^3 - K_1$. Then $Y \subset A_1$ and $y_1, y_2, y_3 \in Y$. Choose $B_2{}^3 \in J - K_1$. Then $Y \subset B_1$ and $z_1, z_2, z_3 \in Y$. Choose $x_1 \in Y$, $x_2 \in B_2$, $x_3 \in B_3$ and let $H = X - \{z_1, z_2\}$. Then

$$A_y{}^3 = \langle \{z_1\}, y_1; \{z_2\}, y_2; H, y_3 \rangle \circ B_z{}^3 \circ A_x{}^3$$

which implies that $A_y{}^3 \in J$. Thus $J = K_2{}^3$, which proves that $K_2{}^3$ is K_1-minimal.

Now we show that if Y consists of two points, y_1 and y_2, then $K_2{}^1$ and $K_2{}^2$ are the only two K_1-minimal ideals of $\Gamma(X, Y)$. Let J be any K_1-minimal ideal of $\Gamma(X, Y)$. Then J contains an element f which does not belong to K_1. We consider two cases:

(i) $$f[X] \subset Y$$

and

(ii) $$f[X] \not\subset Y.$$

Suppose that case (i) holds. Since $f \notin K_0$, we have that $f[X] = \{y_1, y_2\}$. Thus, $f \in K_2{}^2 - K_1$ which implies that $K_2{}^2 \cap J$ properly contains K_1. Since both $K_2{}^2$ and J are K_1-minimal, this implies that $J = K_2{}^2$.

Now assume that case (ii) holds. Then $f(x) \notin Y$ for some $x \in X - Y$. Then $f \circ [y_1, x] = [f(y_1), f(x)]$ belongs to $K_2{}^1 - K_1$. Thus $K_2{}^1 \cap J$ properly contains K_1 and since both $K_2{}^1$ and J are K_1-minimal, it follows that $J = K_2{}^1$. This proves that when Y consists of two elements, $K_2{}^1$ and $K_2{}^2$ are the only K_1-minimal ideals of $\Gamma(X, Y)$.

Now we consider the case where Y has at least three points and we show that $K_2{}^1$, $K_2{}^2$, and $K_2{}^3$ are the only K_1-minimal ideals of $\Gamma(X, Y)$. Let J be any K_1-minimal ideal of $\Gamma(X, Y)$ and let f be any element which belongs to J but not to K_1. We consider three possibilities:

(i) $f[X] \subset Y$ and $f[X]$ consists of two points,

(ii) $f[X] \subset Y$ and $f[X]$ contains more than two points,

(iii) $f[X] \not\subset Y$.

If (i) holds, then $f \in K_2{}^2 - K_1$ and $K_2{}^2 \cap J$ properly contains K_1. This implies that $J = K_2{}^2$ since both J and $K_2{}^2$ are K_1-minimal. If the second possibility occurs, then there exists points $x_1 \in Y$, $x_2, x_3 \in X$ such that $f(x_1)$, $f(x_2)$, and $f(x_3)$ are all distinct. If $X - Y$ has more than one point, we decompose $X - Y$ into two non-empty disjoint subsets H_2 and H_3. Then

$$f \circ \langle Y, x_1; H_2, x_2; H_3, x_3\rangle = \langle Y, f(x_1); H_2, f(x_2); H_3, f(x_3)\rangle \in K_2{}^3 - K_1.$$

Thus, $K_2{}^3 \cap J$ properly contains K_1 which implies that $J = K_2{}^3$. If, however, $X - Y$ consists of only one point, then one of x_2 and x_3, say x_2, must belong to Y. Decompose X into two non-empty disjoint subsets G_1 and G_2 such that $G_1 \cap Y \neq \emptyset \neq G_2 \cap Y$. Then

$$f \circ \langle G_1, x_1; G_2, x_2\rangle = \langle G_1, f(x_1); G_2, f(x_2)\rangle \in K_2{}^2 - K_1$$

which implies that $K_2{}^2 \cap J$ properly contains K_1. Thus, $J = K_2{}^2$ in this instance.

Suppose that the third possibility occurs. Then there exists a point x_2 in X such that $f(x_2) \notin Y$. This, of course, implies that $x_2 \notin Y$. Let x_1 be any element of Y and it follows that

$$f \circ [x_1, x_2] = [f(x_1), f(x_2)] \in K_2{}^1 - K_1.$$

Thus $K_2{}^1 \cap J$ properly contains K_1 and it follows that $J = K_2{}^1$. This proves that if Y contains more than two elements, then $K_2{}^1$, $K_2{}^2$, and $K_2{}^3$ are the only K_1-minimal ideals of $\Gamma(X, Y)$.

The following lemma shows, among other things, that $K_2{}^1$ is distinguished algebraically from the other two (or one, if Y has only two elements) K_1-minimal ideals.

LEMMA (2.12). *Suppose that $Y \neq X$ and Y consists of more than one point. Then $K_1[K_2{}^1 - K_1] \not\subset K_0$ while $K_1[K_2{}^2 - K_1] \subset K_0$ and (if Y has more than two elements) $K_1[K_2{}^3 - K_1] \subset K_0$.*

Proof. Choose $x \in X - Y$ and two distinct points $y_1, y_2 \in Y$. Then $[y_1, x] \in K_2{}^1 - K_1$ and $[y_1, y_2] \in K_1$, but $[y_1, y_2] = [y_1, y_2] \circ [y_1, x] \notin K_0$.

In order to prove the two inclusions, it is sufficient to note that if $A_y^2 \in K_1 - K_0$, and $f \in K_2^2 - K_1$ or $f \in K_2^3 - K_1$, then $Y \subset A_1$ and $A_y^2 \circ f = \langle X, y_1 \rangle \in K_0$.

Definition (2.13). Suppose that $Y \neq X$ and Y consists of more than one point. The set of all elements in K_2^1 which are of the form $[x_1, x_2]$; $x_1 \neq x_2$, $x_1 \in Y$, $x_2 \in X$, will be denoted by the letter L.

LEMMA (2.14). *Suppose that $Y \neq X$ and Y consists of more than one point. An element f in K_2^1 belongs to L if and only if $f \circ g \notin K_0$ for each $g \in K_2^1 - K_1$.*

Proof. Suppose that $f \in L$ and $g \in K_2^1 - K_1$. Then $f = [y_1, y_2]$; $y_1 \in Y$, $y_2 \in X$, and $g = \langle A_1, x_1; A_2, x_2 \rangle$; $Y \subset A_1, x_1 \in Y, x_2 \in X - Y$. Thus,

$$f \circ g = \langle A_1, y_1; A_2, y_2 \rangle \notin K_0.$$

Now suppose that $f \in K_2^1 - L$. Then $f = \langle A_1, x_1; A_2, x_2 \rangle$, where $x_1 \in Y$, $x_2 \in X$ and A_1 properly contains Y. Choose $z \in A_1 - Y$. Then

$$[x_1, z] \in K_2^1 - K_1$$

and

$$f \circ [x_1, z] = \langle X, x_1 \rangle \in K_0.$$

Consequently, if $f \in K_2^1$ and $f \circ g \notin K_0$ for each $g \in K_2^1 - K_1$, then f must belong to L.

LEMMA (2.15). *Suppose that $Y \neq X$ and Y consists of more than one point. Let $[y_1, x_1]$ and $[y_2, x_2]$ be two elements of $L - K_1$. Then $x_1 = x_2$ if and only if for every element $f \in K_2^1 - K_1$, $f \circ [y_1, x_1] \in K_0$ implies $f \circ [y_2, x_2] \in K_0$.*

Proof. Suppose that $x_1 = x_2$ and $f \in K_2^1 - K_1$. Then $f = \langle A, y_3; B, x_3 \rangle$; $Y \subset A$, $y_3 \in Y$, $x_3 \in X - Y$. If $f \circ [y_1, x_1] \in K_0$, then $x_1 \in A$ and since $x_1 = x_2, f \circ [y_2, x_2] = \langle X, y_3 \rangle \in K_0$. On the other hand, suppose that $x_1 \neq x_2$ and let $H = X - \{x_2\}$. Then

$$f = \langle H, y_1; \{x_2\}, x_2 \rangle \in K_2^1 - K_1$$

since $x_2 \in X - Y$. It follows that

$$f \circ [y_1, x_1] = \langle X, y_1 \rangle \in K_0$$

while

$$f \circ [y_2, x_2] = [y_1, x_2] \notin K_0.$$

LEMMA (2.16). *Suppose that $Y \neq X$ and Y consists of more than one point. Let $[y_1, x_1] \in L$ and $\langle X, y_2 \rangle \in K_0$. Then $y_1 = y_2$ if and only if*

$$[y_1, x_1] \circ \langle X, y_2 \rangle = \langle X, y_2 \rangle.$$

Furthermore, $x_1 = y_2$ if and only if $y_1 \neq y_2$ and $f \circ [y_1, x_1] \neq \langle X, y_2 \rangle$ for every f in $\Gamma(X, Y)$ with the property that $f \circ g \neq g$ for each $g \in K_0$.

Proof. First of all, $[y_1, x_1] \circ \langle X, y_2 \rangle = \langle X, y_1 \rangle$, thus it is immediate that $[y_1, x_1] \circ \langle X, y_2 \rangle = \langle X, y_2 \rangle$ if and only if $y_1 = y_2$.

Now suppose that $x_1 = y_2$. Then $y_1 \neq y_2$ since y_1 and x_1 must be distinct. Furthermore, if $f \circ g \neq g$ for each $g \in K_0$, then $f(y) \neq y$ for each $y \in Y$ and it follows readily that $f \circ [y_1, x_1] = f \circ [y_1, y_2] \neq \langle X, y_2 \rangle$.

It remains to verify that if $y_1 \neq y_2$ and $f \circ [y_1, x_1] \neq \langle X, y_2 \rangle$ for every f in $\Gamma(X, Y)$ such that $f \circ g \neq g$ for each $g \in K_0$, then $x_1 = y_2$. Suppose, on the contrary, that $x_1 \neq y_2$. Define a function f by $f(x) = y_2$ for $x \neq y_2$ and $f(y_2) = y$, where y is any point of Y distinct from y_2. Then $f \circ g \neq g$ for each $g \in K_0$, but $f \circ [y_1, x_1] = \langle X, y_2 \rangle$ which is a contradiction.

We conclude this section with a result that answers the following question: "Under what conditions does there exist a space Z such that $\Gamma(X, Y)$ is isomorphic to $\Gamma(Z)$?". The analogous problem for the semigroups $S(X, Y)$ and $S(Z)$ was treated in (**5**). $S(Z)$, we recall, is the semigroup, under composition, of all continuous functions mapping Z into Z and $S(X, Y)$ is the subsemigroup of $S(X)$ consisting of all those functions in $S(X)$ which also map Y into Y. It is stated in (**5**, Theorem 1) that if $X = \beta Y$ (the Stone-Čech compactification of Y), then there does exist a space Z such that $S(X, Y)$ is isomorphic to $S(Z)$. Indeed, in this particular case, $S(X, Y)$ is isomorphic to $S(Y)$. An example is given to show that $S(X, Y)$ may be isomorphic to $S(Y)$ without having $X = \beta Y$ even though X is compact. However, it is stated in (**5**, Theorem 2) that if X is a compact Hausdorff space and the subspace Y contains an arc, then an isomorphism between $S(X, Y)$ and $S(Z)$ does guarantee that $X = \beta Y$ if Z satisfies certain conditions.

The situation for the semigroup $\Gamma(X, Y)$ is quite different. This next result shows that $\Gamma(X, Y)$ is isomorphic to a $\Gamma(Z)$ only in the trivial case, $Y = X$.

THEOREM (2.17). *$\Gamma(X, Y)$ is isomorphic to $\Gamma(Z)$ for some space Z if and only if $X = Y$.*

Proof. The non-trivial portion of the proof consists of showing that if $Y \neq X$, then $\Gamma(X, Y)$ is not isomorphic to any $\Gamma(Z)$. We first consider the case where Y consists of one point. Suppose, in this case, that $\Gamma(X, Y)$ is isomorphic to $\Gamma(Z)$. According to Lemma (2.1), $\Gamma(X, Y)$ has a zero element. Therefore $\Gamma(Z)$ has a zero element and since $\Gamma(Z) = \Gamma(Z, Z)$, Lemma (2.1) implies that Z consists of one point which, in turn, implies that $\Gamma(Z)$ consists of one element. This, however, is a contradiction since $Y \neq X$ implies that $\Gamma(X, Y)$ has more than one element.

Now let us consider the case where Y contains more than one point. By Theorem (2.9), there exists exactly one K_0-minimal ideal, K_1, of $\Gamma(X, Y)$ and by Theorem (2.11), there are at least two K_1-minimal ideals. Let I_1 denote the kernel of $\Gamma(Z)$. In Theorem (2.7) it is stated that there is at most one I_1-minimal ideal, I_2, of $\Gamma(Z)$, and if I_2 exists, there is at most one I_2-minimal ideal of $\Gamma(Z)$. From these facts, it follows that $\Gamma(X, Y)$ and $\Gamma(Z)$ cannot be isomorphic.

3. Isomorphisms between restrictive semigroups.

THEOREM (3.1). *Let ϕ be any isomorphism from the restrictive semigroup $\Gamma(X, Y)$ onto the restrictive semigroup $\Gamma(U, V)$. Then there exists a unique bijection h from X onto U which maps Y onto V such that $\phi f = h \circ f \circ h^{-1}$ for each f in $\Gamma(X, Y)$. In fact, h is a homeomorphism provided only that Y has more than one point. If $Y = \{p\}$, then $V = \{q\}$. In this case, h carries every closed subset of X containing p into a closed subset of Y and h^{-1} carries every closed subset of V containing q into a closed subset of X.*

Proof. We first prove the existence of the bijection h and, in doing so, we consider two cases:

(i) Y consists of one point,

(ii) Y consists of more than one point.

Suppose that case (i) holds and ϕ is an isomorphism from

$$\Gamma(X, Y) = \Gamma(X, \{p\})$$

onto $\Gamma(U, V)$. By Lemma (2.1), $\Gamma(X, \{p\})$ has a zero element. Thus $\Gamma(U, V)$ has a zero element and that same lemma implies that $\Gamma(U, V) = \Gamma(U, \{q\})$ for some $q \in U$. It follows from Theorem (2.3) and Lemma (2.4) that the isomorphism ϕ must map $\{[p, x]: x \in X - \{p\}\}$ bijectively onto

$$\{[q, u]: u \in U - \{q\}\}.$$

Therefore, for each $x \in X - \{p\}$, there exists a unique $u \in U - \{q\}$ such that $\phi[p, x] = [q, u]$. Define $h(x) = u$ and $h(p) = q$. The function h is a bijection from X onto U and for every $x \in X - \{p\}$, $\phi[p, x] = [q, h(x)]$. Let f be any element in $\Gamma(X, \{p\})$. It is immediate that $\phi f(q) = h \circ f \circ h^{-1}(q)$. For $u \in U - \{q\}$, we let $h^{-1}(u) = x$ and we get

$$h \circ f \circ h^{-1}(u) = h \circ f(x) = [q, h \circ f(x)](u) = \phi[p, f(x)](u) =$$

$$\phi(f \circ [p, x])(u) = \phi f \circ \phi[p, x](u) = \phi f \circ [q, u](u) = \phi f(u).$$

Therefore $\phi f = h \circ f \circ h^{-1}$.

Now suppose that case (ii) holds. By Lemma (2.1), $\Gamma(X, Y)$ has no zero element and therefore $\Gamma(U, V)$ has no zero element since $\Gamma(X, Y)$ and $\Gamma(U, V)$ are isomorphic. Thus, Lemma (2.1) implies that V has more than one point. Let $K_0(X)$ and $K_0(U)$ denote the kernels of $\Gamma(X, Y)$ and $\Gamma(U, V)$, respectively. Since the isomorphism ϕ maps $K_0(X)$ bijectively onto $K_0(U)$, it follows that for each $y \in Y$, $\phi\langle X, y\rangle = \langle U, v\rangle$ for some $v \in V$. We define $h(y) = v$. Then h is a bijection from Y onto V such that

$$\phi\langle X, y\rangle = \langle U, h(y)\rangle \tag{3.1.1}$$

for each $\langle X, y\rangle \in K_0(X)$. For any function f in $\Gamma(X, Y)$ and any point $v \in V$ we get

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$$h \circ f \circ h^{-1}(v) = h \circ f(y) = \langle U, h \circ f(y)\rangle(v) = \phi\langle X, f(y)\rangle(v) = \tag{3.1.2}$$
$$\phi(f \circ \langle X, y\rangle)(v) = \phi f \circ \phi\langle X, y\rangle(v) = \phi f \circ \langle U, v\rangle(v) = \phi f(v).$$

If $Y = X$, then it follows from Theorem (2.17) that $V = U$. In this case, h maps X bijectively onto U and it follows from (3.1.2) that $\phi f = h \circ f \circ h^{-1}$. The remaining case is where $Y \neq X$ and, consequently (by Theorem (2.17)), $V \neq U$. Let

$$K_1(X) = K_0(X) \cup \{A_y{}^2 \colon y_1, y_2 \in Y;\ Y \subset A_1\},$$
$$K_2{}^1(X) = K_1(X) \cup \{A_y{}^2 \colon y_1 \in Y, y_2 \in X,\ Y \subset A_1\},$$
$$L(X) = \{[x_1, x_2] \colon x_1 \neq x_2, x_1 \in Y, x_2 \in X\}.$$

Define $K_1(U)$, $K_2{}^1(U)$, and $L(U)$ analogously. It follows from Theorems (2.9) and (2.11) and Lemmas (2.12) and (2.14) that ϕ maps $L(X) - K_1(X)$ (i.e., those functions $[x_1, x_2]$ in $L(X)$ for which $x_2 \in X - Y$) bijectively onto $L(U) - K_1(U)$. This fact allows us to extend the function h to a bijection from X onto U as follows: for any $x \in X - Y$, choose $y \in Y$ and get $\phi[y, x] = [v, u]$. It follows from Lemma (2.15) that u does not depend upon the choice of y but only upon the point x. We define $h(x) = u$. It follows from Lemma (2.16) that $v = h(y)$. Hence, for any $[y, x] \in L(X) - K_1(X)$, we have that $\phi[y, x] = [h(y), h(x)]$. Now suppose that $[y, x] \in L(X) \cap K_1(X)$, i.e., both x and y belong to Y. In this case, $\phi[y, x] = [v, u] \in L(U) \cap K_1(U)$. Using Lemma (2.16) once again we see that $v = h(y)$ and $u = h(x)$. Thus, $\phi[y, x] = [h(y), h(x)]$ for $[y, x] \in L(X) \cap K_1(X)$. Therefore, we can state that for any $[y, x] \in L(X)$,

$$\phi[y, x] = [h(y), h(x)]. \tag{3.1.3}$$

Now let f be any element in $\Gamma(X, Y)$ and let u be any element in $U - V$. Choose any $v \in V$ and let $x = h^{-1}(u)$ and $y = h^{-1}(v)$. Then using (3.1.3), we get

$$\phi f(u) = \phi f \circ [v, u](u) = \phi f \circ \phi[y, x](u) = \phi(f \circ [y, x])(u).$$

If $f(y) = f(x)$, then $f \circ [y, x] = \langle X, f(x)\rangle$ and by (3.1.1),

$$\phi\langle X, f(x)\rangle(u) = \langle U, h \circ f(x)\rangle(u) = h \circ f(x) = h \circ f \circ h^{-1}(u).$$

If $f(y) \neq f(x)$, then $f \circ [y, x] = [f(y), f(x)]$ and by (3.1.3),

$$\phi[f(y), f(x)](u) = [h \circ f(y), h \circ f(x)](u) = h \circ f(x) = h \circ f \circ h^{-1}(u).$$

In either event, $\phi f(u) = h \circ f \circ h^{-1}(u)$. This, together with (3.1.2), implies that

$$\phi f = h \circ f \circ h^{-1} \tag{3.1.4}$$

for each $f \in \Gamma(X, Y)$.

Now we prove that the bijection h is unique. Let k be any bijection from X

onto U which also maps Y onto V such that $\phi f = k \circ f \circ k^{-1}$ for each $f \in \Gamma(X, Y)$. In the event $Y = X$, we choose any $x \in X$ and get

$$\langle U, h(x)\rangle = h \circ \langle X, x\rangle \circ h^{-1} = \phi\langle X, x\rangle = k \circ \langle X, x\rangle \circ k^{-1} = \langle U, k(x)\rangle.$$

Consequently, $h(x) = k(x)$. In case $Y \neq X$, choose any $x \in X - Y$ and $y \in Y$ and get

$$[h(y), h(x)] = h \circ [y, x] \circ h^{-1} = \phi[y, x] = k \circ [y, x] \circ k^{-1} = [k(y), k(x)].$$

Thus, $h(y) = k(y)$ and $h(x) = k(x)$. In either event, $h = k$.

Now let H be any closed subset of X which has a non-empty intersection with Y and choose $y \in H \cap Y$. Define a function f mapping X into X by

$$f(x) = x \quad \text{for } x \in H,$$

$$f(x) = y \quad \text{for } x \in X - H.$$

For any closed subset K of X, we have that

$$f[K] = f[K \cap H] \cup f[K - H] = [K \cap H] \cup f[K - H].$$

The set $K \cap H$ is closed and $f[K - H]$ is either empty or consists of the single point y. In either event, $f[K]$ is closed. Since f maps Y into Y, it follows that $f \in \Gamma(X, Y)$. Using (3.1.4) and the fact that $f[X] = H$, we get

$$h[H] = h \circ f[X] = h \circ f \circ h^{-1}[U] = \phi f[U].$$

This proves that if H is any closed subset of X which has a non-empty intersection with Y, then $h[H]$ is a closed subset of U. In a similar manner, one shows that if H is any closed subset of U which has a non-empty intersection with V, then $h^{-1}[H]$ is a closed subset of X. It follows from Lemma (2.1) that Y consists of one point if and only if V consists of one point. In view of this, the proof will be complete when we show that if Y has more than one point, then h is a homeomorphism. For this it is sufficient to show that for any space Z and any subspace W with more than one point, the family of all those closed subsets of which have a non-empty intersection with W is a basis for the family of all closed subsets of Z. This is accomplished by noticing that for any closed subset H of Z and any pair x, y of distinct points of W, one has $H = (H \cup \{x\}) \cap (H \cup \{y\})$.

By taking $Y = X$ and $V = U$ in Theorem (3.1), we obtain Theorem (2.10) of (**4**) which states that a mapping ϕ from $\Gamma(X)$ onto $\Gamma(U)$ is an isomorphism if and only if there exists a homeomorphism h from X onto U such that $\phi f = h \circ f \circ h^{-1}$ for each f in $\Gamma(X)$.

The following example shows that if Y consists of one point, then the bijection h in Theorem (3.1) need not be a homeomorphism.

Example (3.2). Let X denote the one-point compactification of the natural numbers and let p denote the limit point of X. Let U denote the discrete space whose points are the natural numbers and some point, $q \notin X$. It follows that

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$\Gamma(X, \{p\})$ consists of all functions f mapping X into X such that $f(p) = p$ and $\Gamma(U, \{q\})$ consists of all functions f mapping U into U such that $f(q) = q$. It is a routine matter to show that the mapping ϕ from $\Gamma(X, \{p\})$ onto $\Gamma(U, \{q\})$ defined by

$$\begin{aligned} \phi f(q) &= q, \\ \phi f(x) &= q \qquad \text{if } x \in U - \{q\} \text{ and } f(x) = p, \\ \phi f(x) &= f(x) \quad \text{if } x \in U - \{q\} \text{ and } f(x) \neq p \end{aligned}$$

is an isomorphism. According to Theorem (3.1), the isomorphism uniquely determines a bijection h from X onto U such that $\phi f = h \circ f \circ h^{-1}$ for each $f \in \Gamma(X, \{p\})$. The function h is defined by $h(p) = q$ and $h(x) = x$ for $x \neq p$ and is evidently not a homeomorphism.

4. Automorphisms of restrictive semigroups.

We define an automorphism ϕ of a semigroup S to be inner if there exist elements $a, b \in S$ such that $\phi(x) = axb$ for all $x \in S$. M. L. Vitanza has shown in (**6**) that if ϕ is such an automorphism, then the semigroup must contain an identity and a and b must be inverses of each other.

THEOREM (4.1). *Every automorphism of a restrictive semigroup is an inner automorphism.*

Proof. We observe that the units of a restrictive semigroup $\Gamma(X, Y)$ are the homeomorphisms from X onto X which also map Y onto Y. If Y contains more than one point, it follows immediately from Theorem (3.1) that every automorphism of $\Gamma(X, Y)$ is inner. Therefore, we need only consider the case where $Y = \{p\}$. Suppose that ϕ is an automorphism $\Gamma(X, \{p\})$. According to Theorem (3.1), there exists a bijection h from X onto X such that $h(p) = p$, $\phi f = h \circ f \circ h^{-1}$ for each $f \in \Gamma(X, \{p\})$, and both $h[H]$ and $h^{-1}[H]$ are closed if $p \in H$. The proof will be complete when we show that the bijection h is a homeomorphism. Suppose that H is a closed subset of X. As we observed previously, if $p \in H$, then $h[H]$ is closed. Let us consider the case where $p \notin H$. Define a function f mapping X into X as follows:

$$\begin{aligned} f(x) &= h(x) \quad \text{for } x \in h^{-1}[H], \\ f(x) &= p \qquad\;\; \text{for } x \notin h^{-1}[H]. \end{aligned}$$

For any closed subset K of X, we have that

$$f[K] = f[K \cap h^{-1}[H]] \cup f[K - h^{-1}[H]] = h[K \cap h^{-1}[H]] \cup \{p\} = [h[K] \cap H] \cup \{p\}.$$

Now, $K \cup \{p\}$ is closed and therefore

$$h[K \cup \{p\}] = h[K] \cup \{h(p)\} = h[K] \cup \{p\}$$

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is closed. Thus, $[h[K] \cup \{p\}] \cap H = h[K] \cap H$ is closed which implies that $[h[K] \cap H] \cup \{p\}$ is closed. Therefore, $f \in \Gamma(X, \{p\})$ and hence, $\phi f[H]$ is a closed subset of X. But

$$\phi f[H] = h \circ f \circ h^{-1}[H] = h \circ h \circ h^{-1}[H] = h[H].$$

In a similar manner, one shows that $h^{-1}[H]$ is closed for any closed subset H of X. This completes the proof.

Definition (4.2). We shall denote the group, under composition, of all homeomorphisms mapping the topological space X into itself by $G(X)$. For any non-empty subspace Y of X, the subgroup of $G(X)$ consisting of all those homeomorphisms which map Y onto Y will be denoted by $G(X, Y)$. Finally, the automorphism group of the restrictive semigroup $\Gamma(X, Y)$ will be denoted by $A(X, Y)$ and the automorphism group of $\Gamma(X)$ will be denoted by $A(X)$.

COROLLARY (4.3). *For any restrictive semigroup* $\Gamma(X, Y)$, $A(X, Y)$ *is isomorphic to* $G(X, Y)$.

Proof. For each ϕ in $A(X, Y)$, there exists, according to Theorem (4.1), a homeomorphism h in $G(X, Y)$ such that $\phi f = h \circ f \circ h^{-1}$ for each f in $\Gamma(X, Y)$. The function h is uniquely determined by ϕ and we define $\Phi(\phi) = h$. One shows, in a straightforward manner, that Φ is a homomorphism from $A(X, Y)$ onto $G(X, Y)$. Furthermore, if $\Phi(\phi) = i$, the identity mapping of $G(X, Y)$, then $\phi f = i \circ f \circ i^{-1} = f$ for each f in $\Gamma(X, Y)$, i.e., ϕ is the identity automorphism. Thus, the kernel of Φ consists of the identity and it follows that Φ is an isomorphism from $A(X, Y)$ onto $G(X, Y)$.

For any $f \in G(X, Y)$, the restriction, f/Y, of f to Y is an element of $G(Y)$. The mapping ϕ which takes f onto f/Y is a homomorphism from $G(X, Y)$ into $G(Y)$. If X is a Hausdorff space and Y is a dense subspace of X, the homomorphism ϕ is actually a monomorphism since any two functions mapping X into X which agree on Y must be identical. These observations, together with Corollary (4.3), combine to give the following corollary.

COROLLARY (4.4). *Let* $\Gamma(X, Y)$ *be any restrictive semigroup. Then* $A(X, Y)$ *is isomorphic to a subgroup of* $A(X)$. *If* X *is Hausdorff and* Y *is dense in* X, *then* $A(X, Y)$ *is also isomorphic to a subgroup of* $A(Y)$.

In view of the latter result, it is reasonable to ask if there exist instances where $A(X, Y)$ is isomorphic to $A(X)$ or $A(Y)$ other than in the trivial case $Y = X$. The following result answers this affirmatively.

THEOREM (4.5). *Suppose that* X *is a completely regular Hausdorff space with the property that each point has a countable base and let* βX *denote the Stone-Čech compactification of* X. *Then the automorphism groups of the semigroups* $\Gamma(\beta X, X)$, $\Gamma(\beta X)$, *and* $\Gamma(X)$ *are all isomorphic.*

Proof. By Corollary (4.3), $A(\beta X, X)$ is isomorphic to $G(\beta X, X)$, $A(\beta X)$ is isomorphic to $G(\beta X)$, and $A(X)$ is isomorphic to $G(X)$. The proof will be complete when we show that $G(\beta X, X) = G(\beta X)$ and that $G(X)$ is isomorphic to $G(\beta X)$. Let $h \in G(X)$ be given. Then h can be regarded as a continuous mapping from X into βX and, as such, has a unique continuous extension h^E over βX. The same is true for $k = h^{-1}$. Since both $h^E \circ k^E$ and $k^E \circ h^E$ agree with the identity mapping on the dense subset X, they are equal to the identity mapping. This implies that both h^E and k^E belong to $G(\beta X)$. We define a mapping ϕ from $G(X)$ into $G(\beta X)$ by $\phi(h) = h^E$. The mapping ϕ is injective. Moreover, since $g^E \circ f^E$ and $(g \circ f)^E$ agree on the dense subset X, it follows that $g^E \circ f^E = (g \circ f)^E$ and hence, that ϕ is a monomorphism. Now suppose that h is any element of $G(\beta X)$. Since each point of X has a countable base in X, each point of X also has a countable base in βX (if $\{H_n\}_{n=1}^{\infty}$ is a countable base for x in X, then $\{\mathrm{int}_{\beta X}(\mathrm{cl}_{\beta X} H_n)\}_{n=1}^{\infty}$ is a countable base for x in βX). However, no point of $\beta X - X$ has a countable base in βX. In fact, no point of $\beta X - X$ is even a G_δ (**2**, Corollary 9.6, p. 132) in βX. From this, it follows that h must map X onto X and $\beta X - X$ onto $\beta X - X$. Thus, $h/X \in G(X)$ and $\phi(h/X) = (h/X)^E = h$. This proves that ϕ is an isomorphism from $G(X)$ onto $G(\beta X)$. Finally, since any function $h \in G(\beta X)$ must map points of X onto points of X, it follows that $h \in G(\beta X, X)$. Therefore, $G(\beta X) = G(\beta X, X)$.

5. Some concluding remarks. We close with a few remarks about the semigroup $S(X, Y)$ of all continuous functions mapping X into X which also map Y into Y. It follows from Theorem (3.1) of this paper that if Y and V each contain more than one point, then the restrictive semigroups $\Gamma(X, Y)$ and $\Gamma(U, V)$ are isomorphic if and only if there exists a homeomorphism from X onto U which also maps Y onto V. The following example shows that it is not possible to prove such a result for $S(X, Y)$ and $S(U, V)$.

Let W denote the space of all ordinals less than the first uncountable ordinal. For a nice discussion of some properties of spaces of ordinals, see (**2**, pp. 72–76). Let Y be any infinite set which is disjoint from W and let $X = Y \cup W$. Topologize Z by defining a subset H of X to be open if and only if $H \cap W$ is an open subset of W. Now W is a closed subspace of X which is not realcompact (**2**, p. 114). Therefore, by (**2**, Theorem 8.10, p. 119), X is not realcompact. Hence X is a proper subspace of its Hewitt realcompactification, vX. For a discussion of the space vX, see (**2**, pp. 116–119). As is customary, we let $C(X)$ denote the ring of all real-valued continuous functions on X. There exist unbounded functions in $C(X)$. To see this, choose any countable subset $\{y_n\}_{n=1}^{\infty} \subset Y$ and define a function f by

$$f(y_n) = n \text{ for each positive integer } n,$$

$$f(x) = 0 \text{ for each } x \in X - \{y_n\}_{n=1}^{\infty}.$$

It easily follows that f is continuous. We recall, once again, that βX denotes the Stone-Čech compactification of X. Since X is not compact, X is a proper subset of βX. Furthermore, since $C(X)$ contains unbounded functions and every function in $C(X)$ can be continuously extended over υX, it follows that υX cannot be compact and therefore cannot be homeomorphic to βX. We shall show, however, that $S(\upsilon X, X)$ and $S(\beta X, X)$ are isomorphic semigroups. Let $f \in S(\upsilon X, X)$. Then the restriction, f/X of f to X can be regarded as a continuous mapping from X into the compact space βX and, as such, has a continuous extension to a function, $(f/X)^E$, in $S(\beta X, X)$. We define a mapping ϕ from $S(\upsilon X, X)$ into $S(\beta X, X)$ by $\phi f = (f/X)^E$. If two elements of $S(\beta X, X)$ agree on the dense subspace X, they must be identical. Using this fact, one can show that ϕ is a homomorphism and, moreover, that ϕ is injective. Now let g be any function in $S(\beta X, X)$. Then g/X can be regarded as a continuous mapping from X into the realcompact space υX and, consequently, has a continuous extension to a function f in $S(\upsilon X, X)$. Since $(f/X)^E$ and g agree on X, it follows that $\phi f = g$. Thus ϕ is an isomorphism from $S(\upsilon X, X)$ onto $S(\beta X, X)$.

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