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EXISTENCE THEOREMS FOR VECTOR VARIATIONAL INEQUALITIES

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Given two real Banach spaces X and Y, a closed convex subset K in X, a cone with nonempty interior C in Y and a multivalued operator from K to $2^{L(X,Y)}$, we prove theorems concerning the existence of solutions for the corresponding vector variational inequality problem, that is the existence of some $x_0 \in K$ such that for every $x \in K$ we have $A(x - x_0) \notin -\operatorname{int} C$ for some $A \in Tx_0$. These results correct previously published ones.

1. INTRODUCTION

Let X, Y be real Banach spaces, K be a closed, convex subset of X and $L(X, Y)$ be the set of all continuous linear operators from X to Y. Let further $T\colon K \to 2^{L(X,Y)} \setminus \{\emptyset\}$ be a multivalued operator and $C\colon K \to 2^Y$ be a multivalued mapping such that for each $x \in K$, $C(x)$ is a cone with nonempty interior $\operatorname{int} C(x)$. The purpose of this paper is to study the existence of solutions for the vector variational inequality problem (VVIP):

$$(1) \qquad \exists x_0 \in K\colon \forall x \in K, \exists A \in Tx \text{ such that } A(x - x_0) \notin -\operatorname{int} C(x_0).$$

In case $Y = \mathbf{R}$, $C(x) = \mathbf{R}^+$, the VVIP reduces to the well-known variational inequality problem [**13**]. The VVIP was introduced by Gianessi [**8**] for the case $Y = \mathbf{R}^n$ and was subsequently studied by many other authors [**2, 3, 4, 14, 17**] in connection with vector optimisation. Theorems asserting the existence of solutions of the VVIP are contained in [**3,** Theorem 2.1] for single-valued, monotone operators T, where Y has a constant cone C (that is, not depending on x), in [**2,** Theorem 2.1] for T a single-valued, monotone operator, where Y is equipped with a non-constant $C(x)$ and in [**14,** Theorem 2.1] for multivalued, pseudomonotone operators T, with $C(x)$ constant. However, the proofs of all these theorems contain a mistake: a certain set defined in these papers in asserted to be weakly compact, while this is not the case (see Remark 2 at the end of the present paper for details).

In the following paragraph we prove the existence of a solution of the VVIP for a multi-valued, monotone operator [**9**] with constant cone C (Theorem 3). We also prove

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the existence of solutions for multivalued, pseudomonotone or quasimonotone operators with values consisting of completely continuous operators.

We now recall some definitions and fix our notation. A cone C in Y is a non-empty, convex, proper subset of Y, such that for all $\lambda \geqslant 0$, $y \in C$, we have $\lambda y \in C$. The dual cone C^* of C is the set of all f in the dual space Y^* such that $f(y) \geqslant 0$, for all $y \in C$.

If C is closed, then

$$y \in C \Leftrightarrow f(y) \geqslant 0, \text{ for all } f \in C^*. \tag{2}$$

On the other hand, if $\operatorname{int} C \neq \emptyset$, then

$$y \in \operatorname{int} C \Leftrightarrow f(y) > 0, \text{ for all } f \in C^* \setminus \{0\}. \tag{3}$$

Note that in both cases we have $C^* \neq \{0\}$. We refer the reader to [**11**] for these and other properties of cones.

Now let $C \colon K \to 2^Y$ be a multivalued mapping such that for each $x \in K$, $C(x)$ is a cone with nonempty interior. A multivalued operator $T \colon K \to 2^{L(X,Y)} \setminus \{\emptyset\}$ is called:

(i) *monotone* [**9**], if for all $x, y \in K$ and all $A \in Tx$, $B \in Ty$ we have $(B - A)(y - x) \in C(x)$.

(ii) *(weakly) pseudomonotone* [**14**], if for all $x, y \in K$ and $A \in Tx$, $A(y - x) \notin -\operatorname{int} C(x)$ implies $B(y - x) \notin -\operatorname{int} C(x)$, for all (for some) $B \in Ty$.

(iii) *(weakly) quasimonotone*, if for all $x, y \in K$ and $A \in Tx$, $A(y - x) \notin -C(x)$ implies $B(y - x) \notin -\operatorname{int} C(x)$, for all (for some) $B \in Ty$.

It is obvious that (weak) quasimonotonicity is implied by (weak) pseudomonotonicity, which in turn, is implied by monotonicity. These notions generalise the well-known corresponding ones for the case $Y = \mathbf{R}$ [**12, 15**].

The strong operator topology (SOT) on $L(X, Y)$ is the weakest topology for which the functions $L(X, Y) \ni A \to Ax \in Y$ are continuous, for every $x \in X$. The multivalued operator T is called upper hemicontinuous, if its restriction on line segments is SOT-upper semicontinuous. An operator $A \in L(X, Y)$ is called *completely continuous*, if it maps weakly convergent sequences to strongly convergent ones [**5**]. Any compact operator is completely continuous. The converse is not true, since the identity mapping in ℓ_1 is completely continuous without being compact [**6**]. If Y is finite-dimensional, all elements of $L(X, Y)$ are obviously completely continuous operators.

A point $x_0 \in K$ is called an *inner point* [**10**] or *relative quasi-interior point* [**1**] of K, if for all $f \in X^*$, we have

$$\forall x \in K,\ f(x - x_0) \geqslant 0 \Rightarrow \forall x \in K,\ f(x - x_0) = 0.$$

In other words, x_0 is an inner point of K if every closed hyperplane which supports K at x_0, necessarily contains K.

The set of inner points of K is denoted by $\operatorname{inn} K$. Note that interior points of K are also inner points, since in this case the above implication holds vacuously. In fact, whenever $\operatorname{int} K \neq \emptyset$, it can be shown that $\operatorname{int} K = \operatorname{inn} K$. However, for any separable K we have $\operatorname{inn} K \neq \emptyset$, even if $\operatorname{int} K = \emptyset$ [**1**, **10**]. In [**1**, **10**] it was also shown that $\operatorname{inn} K$ is *lineally full* in K, that is for every $x \in \operatorname{inn} K$ and every $y \in K$, we have $\{tx + (1-t)y : t \in (0, 1]\} \subseteq \operatorname{inn} K$.

For any $S \subseteq L(X, Y)$ and $x \in X$, $S(x)$ will denote the set $\{Ax \colon A \in S\}$.

2. The main results

In what follows, X and Y will be Banach spaces. Unless explicitly mentioned, we shall always consider the *weak* topology on X, the norm topology on Y and the strong operator topology on $L(X, Y)$. K will be a nonempty closed, convex subset of X and $C \colon K \to 2^Y$ a multifunction, such that $C(x)$ is a cone with nonempty interior for each $x \in K$. We set $D(x) = Y \setminus (-\operatorname{int} C(x))$ and for any operator $T \colon K \to 2^{L(X,Y)} \setminus \{\emptyset\}$ we define the multifunctions:

$$G(y) = \{x \in K \colon \exists A \in Tx \text{ such that } A(y - x) \in D(x)\} \tag{4}$$

$$F(y) = \{x \in K \colon \exists B \in Ty \text{ such that } B(y - x) \in D(x)\}. \tag{5}$$

Let S be the set of all $x \in K$ such that relation (1) holds, that is, S is the solution set of the VVIP. We note that $S = \bigcap_{y \in K} G(y)$.

We begin with some lemmas:

Lemma 1. *Let K be (weakly) compact. Then $\bigcap_{y \in K} \overline{G(y)} \neq \emptyset$.*

Proof: According to K. Fan's lemma [**7**], it is sufficient to show that for any $x = \sum_{i=1}^{n} \lambda_i x_i$, with $x_i \in G(x_i)$, $\lambda_i \in [0, 1]$, $\sum_{i=1}^{n} \lambda_i = 1$, we have $x \in \bigcup_{i=1}^{n} G(x_i)$. Indeed, were this not the case, we would have $x \notin G(x_i)$ for all i's, so for all $A \in Tx$ we would have $A(x_i - x) \in -\operatorname{int} C(x)$. Since $-\operatorname{int} C(x)$ is convex, this would imply $0 = \sum_{i=1}^{n} \lambda_i A(x_i - x) \in -\operatorname{int} C(x)$, a clear contradiction. ∎

Lemma 2. *Let T be upper hemicontinuous. Then $\bigcap_{y \in K} F(y) \subseteq \bigcap_{y \in K} G(y)$. If, in addition, $\operatorname{inn} K \neq \emptyset$ and T has compact values, then $\bigcap_{y \in K} F(y) = \bigcap_{y \in \operatorname{inn} K} F(y)$.*

Proof: Assume first that there exists $x \in \bigcap_{y \in K} F(y)$ such that $x \notin \bigcap_{y \in K} G(y)$. Then there would exist $y \in K$ such that $(Tx)(y - x) \subseteq -\operatorname{int} C(x)$. Set $x_t = ty +$

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$(1-t)x$, $t \in (0,1)$. Since $-\operatorname{int} C(x)$ is open and T is upper hemicontinuous, there exists $\delta > 0$ such that $(Tx_t)(y-x) \subseteq -\operatorname{int} C(x)$, for all $t \in (0,\delta)$. Since $t(y-x) = x_t - x$ and $-\operatorname{int} C(x)$ is a cone, we deduce that $(Tx_t)(x_t - x) \subseteq -\operatorname{int} C(x)$, that is $x \notin F(x_t)$, a contradiction. This proves the inclusion.

Now suppose that $\operatorname{inn} K \neq \emptyset$. Suppose that there exists $x \in \bigcap_{y \in \operatorname{inn} K} F(y)$ such that $x \notin \bigcap_{y \in K} F(y)$. Then for some $y \in K$, we would have

$$(Ty)(y-x) \subseteq -\operatorname{int} C(x). \tag{6}$$

Since $(Ty)(y-x)$ is compact by assumption, relation (6) implies that there exists $\varepsilon > 0$ such that

$$(Ty)(y-x) + B_\varepsilon + B_\varepsilon \subseteq -\operatorname{int} C(x) \tag{7}$$

where $B_\varepsilon = \{x \in X : \|x\| \leqslant \varepsilon\}$.

We choose $z \in \operatorname{inn} K$ and set $y_t = tz + (1-t)y$, $t \in (0,1]$. Since $\operatorname{inn} K$ is lineally full, we have $y_t \in \operatorname{inn} K$, so $x \in F(y_t)$. We also have

$$(Ty_t)(y_t - x) \subseteq (Ty_t)(y-x) + (Ty_t)(y_t - y). \tag{8}$$

Upper hemicontinuity shows that for t sufficiently small, $(Ty_t)(y-x) \subseteq (Ty)(y-x) + B_\varepsilon$. On the other hand, since T has compact values and is upper hemicontinuous, the image of any line segment by T is compact; hence, for small t we have: $(Ty_t)(y_t - y) = t(Ty_t)(z-y) \subseteq B_\varepsilon$. Hence, relations (7) and (8) imply $(Ty_t)(y_t - x) \subseteq -\operatorname{int} C(x)$, that is, $x \notin F(y_t)$, a contradiction. This shows that $\bigcap_{y \in K} F(y) = \bigcap_{y \in \operatorname{inn} K} F(y)$. ∎

LEMMA 3. *Suppose that K is compact and for some $y \in K$, $T(y)$ is norm compact and its elements are completely continuous operators. Suppose further that the graph of D is sequentially closed in $X \times Y$. Then $F(y)$ is closed.*

PROOF: Let $x \in \overline{F(y)}$. By Eberlein's theorem, there exists a sequence $(x_n)_{n \in \mathbf{N}} \subset F(y)$ converging to x. Then for any $n \in \mathbf{N}$, there exists $B_n \in Ty$ such that $B_n(y - x_n) \in D(x_n)$. Since Ty is norm compact, we may assume with no loss of generality that $(B_n)_{n \in \mathbf{N}}$ norm-converges to some $B \in Ty$. Since B is completely continuous, we have $Bx_n \to Bx$, so using a standard argument, we conclude that $B_n(y - x_n) \to B(y-x)$. The sequential closedness of the graph of D implies that $B(y-x) \in D(x)$, that is $x \in F(y)$, so $F(y)$ is closed. ∎

LEMMA 4. *Suppose that T is weakly quasimonotone and upper hemicontinuous, with compact values. Then for all $y \in \operatorname{inn} K$ we have $G(y) \subseteq F(y) \cup S$.*

PROOF: Let $x \in G(y)$ be such that $x \notin F(y)$. We shall show that $x \in S$. The assumption on x implies that there exists $A \in Tx$ such that $A(y-x) \notin -\operatorname{int} C(x)$.

In addition, $A(y-x) \in -C(x)$, since otherwise the weak quasimonotonicity would imply that $x \in F(y)$. Hence $A(y-x)$ belongs to the boundary of $-C(x)$, so by the Hahn-Banach theorem there exists an $f \in Y^*$ such that $f(A(y-x)) \geqslant f(z)$, for all $z \in -C(x)$. Since $-C(x)$ is a cone containing $A(y-x)$, we easily deduce that

$$(f \circ A)(y-x) = 0 \geqslant f(z), \text{ for all } z \in -C(x) \tag{9}$$

so, in particular

$$(f \circ A)(y) = (f \circ A)(x).$$

We now show that

$$(f \circ A)(x) = (f \circ A)(y) \geqslant (f \circ A)(z), \quad \forall z \in K. \tag{10}$$

Indeed, suppose to the contrary, that $(f \circ A)(z) > (f \circ A)(x)$ for some $z \in K$. Set $y_t = tz + (1-t)y$, $t \in (0,1)$. Obviously $(f \circ A)(y_t - x) > 0$, for all $t \in (0,1)$, so (9) implies $A(y_t - x) \notin -C(x)$. Using the weak quasimonotonicity, we get

$$(Ty_t)(y_t - x) \cap D(x) \neq \emptyset. \tag{11}$$

On the other hand, $x \notin F(y)$, which means that $(Ty)(y-x) \subset -\operatorname{int} C(x)$. Using the same argument as in the second part of the proof of Lemma 2, we conclude that for t sufficiently small we have $x \notin F(y_t)$, a contradiction.

Hence (10) holds. Since $y \in \operatorname{inn} K$, we deduce that $(f \circ A)(x) = (f \circ A)(y) = (f \circ A)(z)$, $\forall z \in K$; that is, $(f \circ A)(z-x) = 0$, $\forall z \in K$. According to (9), f belongs to the polar cone of $C(x)$, hence relation (3) implies $A(z-x) \notin -\operatorname{int} C(x)$, for all $z \in K$, that is, $x \in S$. ∎

Theorem 1. *Suppose that T is upper hemicontinuous and for all $y \in K$, $T(y)$ is norm compact and its elements are completely continuous operators. Let the graph of D be sequentially closed in $X \times Y$ and K be compact. Then in each of the following cases, the VVIP has a solution:*

- **(α)** *T is weakly pseudomonotone,*
- **(β)** *T is weakly quasimonotone and $\operatorname{inn} K \neq \emptyset$.*

Proof: (α). If T is weakly pseudomonotone, then for all $y \in K$ we have: $G(y) \subseteq F(y)$, so invoking Lemma 3 we get $\overline{G(y)} \subseteq F(y)$. Combining now Lemmas 1 and 2 we get

$$\emptyset \neq \bigcap_{y \in K} \overline{G(y)} \subseteq \bigcap_{y \in K} F(y) \subseteq \bigcap_{y \in K} G(y) = S,$$

hence S is nonempty.

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(β). Let T be weakly quasimonotone. Suppose $S = \emptyset$. Then Lemmas 3 and 4 show that $\overline{G(y)} \subseteq F(y)$, for all $y \in \operatorname{inn} K$. Hence an application of Lemmas 1 and 2 gives

$$\emptyset \neq \bigcap_{y\in K} \overline{G(y)} \subseteq \bigcap_{y\in \operatorname{inn} K} \overline{G(y)} \subseteq \bigcap_{y\in \operatorname{inn} K} F(y) = \bigcap_{y\in K} F(y) \subseteq \bigcap_{y\in K} G(y) = S,$$

which is a contradiction. Thus $S \neq \emptyset$. ∎

Theorem 2 replaces the hypothesis of (weak!) compactness of K by a coercivity condition. We assume for simplicity that X is reflexive.

THEOREM 2. *Let X be a reflexive Banach space. The conclusion of the Theorem 1 still holds if the assumption "K is compact" is replaced by the following coercivity condition:*

"There exists an $R > 0$ such that for all $x \in K$, $\|x\| \geqslant R$, there exists a $z \in K$, $\|z\| < R$, such that $(Tx)(z - x) \subseteq -C(x)$."

PROOF: Define $K_1 = \{x \in K : \|x\| \leqslant R\}$. Then K_1 is a nonempty, convex, compact subset of X.

We consider two cases:

(α) If T is pseudomonotone, then by Theorem 1 the VVIP on K_1 has a solution x_0. By the coercivity condition, there exists a $z \in K$, $\|z\| < R$, such that

$$(Tx_0)(x_0 - z) \subseteq C(x_0) \tag{12}$$

(if $\|x_0\| < R$, we may take $z = x_0$). Now given $x \in K$, there exists $t \in (0, 1)$ such that $x_t = tz + (1 - t)x \in K_1$. By the definition of x_0, there exists $A \in Tx_0$, such that $A(x_t - x_0) \notin -\operatorname{int} C(x_0)$. Combining the latter with (12), we easily deduce that $tA(x_0 - z) + A(x_t - x_0) \notin -\operatorname{int} C(x_0)$, that is, $A(x - x_0) \notin -\operatorname{int} C(x_0)$. Hence x_0 is also a solution of the VVIP on K.

(β) Let T be quasimonotone and $\operatorname{inn} K \neq \emptyset$. Since $\operatorname{inn} K$ is lineally full, there exists $z \in \operatorname{inn} K$ such that $\|z\| < R$. Then it is easy to prove that $z \in \operatorname{inn} K_1$ (see also the proof of Theorem 3.1 in [**10**]), so $\operatorname{inn} K_1 \neq \emptyset$. Hence, by Theorem 1, the VVIP on K_1 has a solution x_0, which is in fact, as in the previous case, a solution on K. ∎

Note that for a pseudomonotone operator T, the assumption of the norm compactness of Ty may be replaced by that of compactness. Indeed, if the latter is the case, we set

$$F_1(y) = \{x \in K : (Ty)(y - x) \subseteq D(x)\},\ y \in K.$$

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Then obviously

$$F_1(y) \subseteq F(y),\ \forall y \in K.$$

Hence Lemma 2 gives

$$\bigcap_{y\in K} F_1(y) \subseteq \bigcap_{y\in K} G(y).$$

An analogous proof to that of Lemma 3 shows that $F_1(y)$ is closed for all $y \in K$. Finally, the proof of Theorem 1 goes through if we consider $F_1(y)$ instead of $F(y)$.

If the cone C does not depend on x and T is monotone, then the existence of solutions for the VVIP is a trivial consequence of the analogous theorem for the (scalar) variational inequality problem, as the following shows:

Theorem 3. *Let $T\colon K \to 2^{L(X,Y)} \setminus \{\emptyset\}$ be a monotone, upper hemicontinuous operator with compact values and let C be a cone with nonempty interior in Y. Suppose that K is compact or that X is reflexive and T satisfies the coercivity condition of Theorem 2. Then the VVIP*

$$\forall y \in K,\ \exists A \in Tx \text{ such that } A(y-x) \notin -\operatorname{int} C$$

has a solution x on K.

Proof: Choose $f \in C^* \setminus \{0\}$. Then the operator $f \circ T\colon K \to 2^{X^*} \setminus \{\emptyset\}$ is obviously monotone, upper hemicontinuous with w^*-compact values, so there exists a solution $x \in K$ of the variational inequality

$$\forall y \in K,\ \exists u \in (f \circ T)(x)\colon (u,\, y-x) \geqslant 0$$

(see, for instance, [**16**]). Obviously, $u = f \circ A$ for some $A \in Tx$ and this according to relation (3) shows that $A(y-x) \notin -\operatorname{int} C$, that is, x is also a solution for the VVIP. ∎

Remark 1. In the case $Y = \mathbf{R}$, the set of solutions for the (scalar) V.I.P. of the pseudomonotone operator is known to be convex. This does not hold for the VVIP even if the operator T is constant, as the following example shows: Let $X = Y = \mathbf{R}^2$, $C(x) = C = \mathbf{R}^2_+$, $K = \{x \in \mathbf{R}^2 \colon \|x\|_2 \leqslant 1\}$ and Tx be the identity operator for all $x \in K$. Then $x_1 = (0,-1)$ and $x_2 = (-1,0)$ are solutions for the VVIP while all convex combinations of them are not.

Remark 2. The set $F(y)$ defined by relation (5) is not compact under the assumptions of Theorem 3, as it is asserted to be in the proof of Theorem 2.1 in [**3, 2, 14**] (where it is denoted by $F_2(y)$). Here is a counterexample: Let $X = Y = \ell_2$ and let B be the closed unit ball. Let $(e_n)_{n\in\mathbf{N}}$ be the canonical basis of ℓ_2 and $K = e_1 + B$. For each

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$x \in K$, let $C(x) = C$, where C is the cone $\bigcup_{\lambda \geqslant 0} \lambda(e_1 + (1/4)B)$. Note that $\operatorname{int} C \neq \emptyset$. For any y, z in B the scalar product $\langle e_1 + y/4, e_1 + z/4 \rangle$ is positive; it follows that the scalar product of any two elements of C is nonnegative. Hence, $C \subseteq C^*$, so in particular $\operatorname{int} C^* \neq \emptyset$. (This was an additional assumption in [**3**, Theorem 2.1]). Finally, let $T\colon K \to 2^{L(\ell_2, \ell_2)}$ be such that Tx is the identity operator on ℓ_2 for each $x \in K$. Then T is of course single-valued and monotone. One may immediately check that $F(0) = K \setminus \operatorname{int} C$. It follows that for all $n > 1$ we have $e_1 + e_n \in F(0)$ (indeed, otherwise we would have $e_1 + e_n = \lambda(e_1 + z/4)$ for some $z \in B$; this is impossible, since the norm of $(1 - \lambda)e_1 + e_n$ is easily seen to be greater than $\lambda/4$). However, e_1 is the weak limit of $e_1 + e_n$; on the other hand, since $e_1 \in \operatorname{int} C$, we have $e_1 \notin F(0)$, that is, $F(0)$ is not weakly closed.

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