Ergod. Th. & Dynam. Sys. (1986), **6**, 505–528
Printed in Great Britain

Restricted orbit changes of ergodic $\mathbb{Z}^d$-actions to achieve mixing and completely positive entropy

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(*Received* 17 *April* 1984 *and revised* 19 *November* 1985)

Abstract. We show that for every ergodic $\mathbb{Z}^d$-action T, there is a mixing $\mathbb{Z}^d$-action S which is orbit equivalent to T via an orbit equivalence that is a weak a-equivalence for all $a \geq 1$ and a strong b-equivalence for all $b \in (0, 1)$. If T has positive entropy, then S can be taken to have completely positive entropy. If the dimension d is greater than one, the orbit equivalence may be taken to be bounded and a strong b-equivalence for all $b > 0$.

Let T and T' be ergodic, measure-preserving $\mathbb{Z}^d$-actions on Lebesgue probability spaces $(\Omega, \mathcal{B}, \mu)$ and $(\Omega', \mathcal{B}', \mu')$. T and T' are said to be orbit equivalent if there exists a non-singular, bimeasurable map $\phi: \Omega \to \Omega'$, such that, for a.e. $\omega \in \Omega$, ϕ maps the T-orbit $O_T(\omega)$ of ω bijectively onto $O_{T'}(\phi(\omega))$. Equivalently, and more conveniently for our purposes, T and T' are orbit equivalent if there is a $\mathbb{Z}^d$-action S on Ω, isomorphic to T', having the same orbits as T. That is, for a.e. ω, $O_T(\omega) = O_S(\omega)$. We refer to such a pair (T, S) as an *orbit equivalence* between T and S.

Given an orbit equivalence between T and S, there is a measurable function $\alpha: \Omega \times \mathbb{Z}^d \to \mathbb{Z}^d$ given by, for all $v \in \mathbb{Z}^d$ and for a.e. ω, $T^v(\omega) = S^{\alpha(\omega,v)}(\omega)$, and satisfying (i) $\alpha(\omega, \cdot): \mathbb{Z}^d \to \mathbb{Z}^d$ is a bijection for a.e. ω and (ii) $\alpha(\omega, v+w) = \alpha(\omega, v) + \alpha(T^v\omega, w)$, for a.e. ω and all $v, w \in \mathbb{Z}^d$. We refer to α as the cocycle of the orbit equivalence (T, S). Conversely, given such a cocycle α for T (that is, a function α satisfying (i) and (ii)) we can define a function $\alpha^{-1}: \Omega \times \mathbb{Z}^d \to \mathbb{Z}^d$ by $\alpha(\omega, \alpha^{-1}(\omega, v)) = v$ and a $\mathbb{Z}^d$-action S by $S^v(\omega) = T^{\alpha^{-1}(\omega,v)}(\omega)$ so that (T, S) is an orbit equivalence with cocycle α. (Note that α^{-1} is the cocycle for (S, T).) Thus we can (informally) regard orbit equivalences and the associated cocycles as the same objects. We will indicate the above relationship between T and S by $S = T^{\alpha^{-1}}$.

We remark that a theorem of Dye [**2**] asserts that every two (ergodic, Lebesgue probability measure-preserving) $\mathbb{Z}^d$-actions are orbit equivalent. On the other extreme, we can describe isomorphism of $\mathbb{Z}^d$-actions by saying that T is isomorphic to S if and only if there is an orbit equivalence between T and S with cocycle α satisfying for all v, a.e. ω, $\alpha(\omega, v) = v$. This paper concerns the classification of $\mathbb{Z}^d$-actions up to orbit equivalences satisfying conditions less restrictive than isomorphism. The ideas here originated with observation of A. del Junco that one can characterize even Kakutani equivalence in terms of orbit equivalences. Rather than give his original characterization, we present one that comes out of his subsequent work with D. J. Rudolph [**1**].

THEOREM [**1**]. *Ergodic $\mathbb{Z}$-actions T and S are evenly Kakutani equivalent if and only if there is an orbit equivalence between T and S with cocycle α satisfying, for a.e. ω, there exists $I(\omega)\subset\mathbb{Z}$ of density* 1, *such that*

$$\lim_{\substack{n\to\infty\\ n\in I(\omega)}} \frac{(\alpha(\omega,n)-n)}{n}=0.$$

It is possible to give a similar characterization of uneven Kakutani equivalence but as we will not need it, we won't state it here. (See [**1**].)

An important consequence of these ideas is the development of the notion of Kakutani equivalence of $\mathbb{Z}^d$-actions ($d>1$) that is carried out in [**1**]. There one finds a family of relations, $\rightsquigarrow^M$, parametrized by $M\in \mathrm{GL}(n,\mathbb{R})$. The relation corresponding to the parameter $\mathrm{id}\in \mathrm{GL}(n,\mathbb{R})$ can be defined using the statement of the above theorem as a model.

Definition 1. Ergodic $\mathbb{Z}^d$-actions T and S are id-*Kakutani equivalent* ($T\rightsquigarrow^{\mathrm{id}} S$) if there is an orbit equivalence (T, S) with cocycle α satisfying, for a.e. ω, there exists $I(\omega)\subset\mathbb{Z}^d$ of density 1, such that

$$\lim_{\substack{\|v\|\to\infty\\ v\in I(\omega)}} \frac{\|\alpha(\omega,v)-v\|}{|v|}=0.$$

Again, this is not the definition given in [**1**], but is equivalent to it. We omit the definition of the corresponding relation for $M\in \mathrm{GL}(n,\mathbb{R})$, $M\neq \mathrm{id}$, but mention two facts proved in [**1**]. First, the parametrization of these relations satisfies: given $\mathbb{Z}^d$-actions T, S and U, with $T\rightsquigarrow^M S\rightsquigarrow^{M'} U$, we get $T\rightsquigarrow^{M'M} U$. Second, given T and M, there exists an action S with $T\rightsquigarrow^M S$.

It is natural to ask whether theorems known concerning Kakutani equivalence for $\mathbb{Z}$-actions extend to $\mathbb{Z}^d$-actions. We consider two such theorems and prove the corresponding extensions. Namely, Friedman and Ornstein proved [**3**] that every ergodic transformation T is Kakutani equivalent to a mixing one, and Ornstein and Smorodinsky [**5**] proved that every ergodic transformation of positive entropy is Kakutani equivalent to a K-automorphism.

A $\mathbb{Z}^d$-action T on $(\Omega, \mathcal{B}, \mu)$ is said to be mixing if for all $A, B\in\mathcal{B}$, $\varepsilon>0$ there exists M such that for all $v\in\mathbb{Z}^d$ with $|v|>M$, $|\mu(A\cap T^{-v}B)-\mu(A)\mu(B)|<\varepsilon$. T is called a *K-system* if it has no non-trivial factors of zero entropy.

We will show, in part, that for all ergodic $\mathbb{Z}^d$-actions T, and $M\in \mathrm{GL}(n,\mathbb{R})$, there exists a mixing action S with $T\rightsquigarrow^M S$, and similarly if T has positive entropy, we may choose S to be a K-system. We remark that in order to prove these results, it is sufficient to prove them with $M=\mathrm{id}$, as can be seen by applying the two facts from [**1**] mentioned above.

There is more to say, however, even in the 1-dimensional case. The orbit equivalence condition describing even Kakutani equivalence suggests that one consider other schemes of classification of ergodic actions, by varying the nature of the restriction on the desired orbit equivalence. We consider here two variants, both suggested by D. J. Rudolph.

In order to describe the first variant we need the following definition. A family of sets $\{I(\omega)\subset\mathbb{Z}^d\}_{\omega\in\Omega}$ is called a *sequence of measurable full density* if there exist an increasing sequence of measurable sets $A_i\subset\Omega$ and a sequence of integers M_i such that $\lim_{i\to\infty}\mu(A_i)=1$ and for all $\omega\in\Omega$, $I(\omega)=\bigcup_{i=1}^{\infty}\{v\in\mathbb{Z}^d \mid \|v\|_\infty > M_i$ and $T^v(\omega)\in A_i\}$. Let $a\geq 1$. We say that (T, S) is a *weak-a-equivalence* if there is a sequence $I(\omega)$ of measurable full density such that for a.e. ω,

$$\lim_{\substack{\|v\|\to\infty\\ v\in I(\omega)}}\frac{\|\alpha(\omega, v)-v\|^a}{\|v\|}=0.$$

The peculiar form of this definition is apparently necessary to make it correspond to the equivalence relations described in [**6**]. In that memoir, Rudolph shows that each of these (as well as other) equivalence relations admits an equivalence theorem analogous to Ornstein's isomorphism theorem for Bernoulli shifts. That is, there is for each a distinguished class of transformations, analogous to the finitely determined transformations, which are characterized up to the appropriate equivalence relation by their entropy. The equivalence theorem for even Kakutani equivalence is of course one of these.

For the second variant, we let $b\in(0, 1)$ and say that (T, S) is a *strong b-equivalence* if

$$\lim_{\|v\|\to\infty}\frac{\|\alpha(\omega, v)-v\|^b}{\min(\|v\|, \|\alpha(\omega, v)\|)}=0.$$

This condition does not give an equivalence relation on $\mathbb{Z}$-actions; it is not transitive. However, the condition that (T, S) is, for all $b\in(0, 1)$, a strong b-equivalence, does define an equivalence relation.

We will prove the following theorems.

THEOREM 1. *Let T be an ergodic (Lebesgue probability measure-preserving) $\mathbb{Z}^d$-action on $(\Omega, \mathcal{B}, \mu)$. There is a cocycle α for T giving a mixing action $S=T^{\alpha^{-1}}$, such that the orbit equivalence (T, S) is for all $a\geq 1$, a weak a-equivalence and for all $b\in(0, 1)$ a strong b-equivalence.*

THEOREM 2. *Let T be as in theorem 1 with $h(T)>0$. Then there is a cocycle α as in theorem 1, such that, in addition to the above, $S=T^{\alpha^{-1}}$ is a K-system.*

In dimension 1, if (T, S) is a strong b-equivalence for $b\geq 1$, then T is isomorphic to S by an isomorphism that preserves orbits. The same holds for bounded orbit equivalences; that is, orbit equivalences satisfying there exists $M\in\mathbb{R}$ such that for a.e. ω, and all $|v|=1$, $|\alpha(\omega, v)-v|<M$. In higher dimensions, however, this is not so, and one can prove theorems corresponding to theorems 1 and 2 for these more

restrictive conditions on α. These results will be presented following the proofs of theorems 1 and 2.

We remark that in Rudolph's memoir [6], he defines for $\mathbb{Z}$-actions a class of equivalence relations m_ψ parameterized by the class $\mathcal{F}_1$ of functions $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $\psi(0) = 0$, ψ is non decreasing, $\psi(x) \geq x$ and $\forall c > 0$

$$\lim_{x\to\infty} \frac{\psi(cx)}{\psi(x)} = C'(c) \qquad \text{where } \lim_{c\to 1} C'(c) = 1.$$

Two $\mathbb{Z}$-actions T and S are m_ψ-equivalent (he shows) if there is an orbit equivalence (T, S) with cocycle α such that for a sequence $I(\omega)$ of measurable full density,

$$\lim_{\substack{n\to\infty \\ n\in I(\omega)}} \frac{\psi(|\alpha(\omega, n) - n|)}{n} = 0.$$

These relations clearly generalize the notion of weak a-equivalence (for $\mathbb{Z}$-actions), but we can see that if, for all $a \geq 1$, T and S are weakly a-equivalent, then for all $\psi \in \mathcal{F}_1$, T and S are m_ψ-equivalent. Indeed, it is sufficient to show that for all $\psi \in \mathcal{F}_1$ there exists $m \in \mathbb{Z}$ and $x_0 \in \mathbb{R}$ such that for all $x > x_0$

$$x^m > \psi(x).$$

We prove this as follows: Fix $\psi \in \mathcal{F}_1$ and $c > 1$. Choose $x_0 > 1$ so that for all $x > x_0$

$$\psi(cx) < 2C'(c)\psi(x),$$

where $C'(c)$ is the function given by the definition of $\mathcal{F}_1$.

Choose m so that $c^m > 2C'(c)$ and $c^m > (2C'(c))^2\psi(x_0)/x_0^m$. Then, for all integers k,

$$(c^k x_0)^m > \psi(c^{k+1}x_0).$$

Indeed, $c^m x_0^m > (2C'(c))^2\psi(x_0) > \psi(c^2 x_0)$, so, for all k,

$$\begin{aligned}(c^k x_0)^m &= (c^m)^k x_0^m = (c^m)^{k-1} c^m x_0^m > (2C'(c))^{k-1}(2C'(c))^2\psi(x_0) \\ &= (2C'(c))^{k+1}\psi(x_0) > \psi(c^{k+1}x_0).\end{aligned}$$

Now suppose $x > x_0$. Then for some k, $x \in [c^k x_0, c^{k+1}x_0]$, so that

$$x^m \geq (c^k x_0)^m > \psi(c^{k+1}x_0) \geq \psi(x).$$

Rudolph also defines another class of equivalence relations, these parameterized by the class $\mathcal{F}_2$ of functions $\phi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $\phi(1) > 0$, ϕ is non-decreasing, and for all $c > 0$,

$$\lim_{x\to\infty} \frac{\phi(cx)}{\phi(x)} = 1.$$

We will not reproduce here the somewhat complicated definition of these equivalence relations. The reader familiar with them, however, will observe that our proofs of theorems 1 and 2 can be altered so that for every sequence $\{\phi_i\}_{i=1}^\infty \subset \mathcal{F}_2$, the orbit equivalence (T, S) can be constructed so that, in addition to the stated properties, (T, S) is, for each i, and m_{ϕ_i}-equivalence.

Theorems 1 and 2 will be proved by constructing the desired cocycles quite explicitly. The formal presentation of the constructions, however, may obscure their simplicity, so we give here an informal description of the basic ideas.

For a.e. ω, the T-orbit $O_T(\omega)$ of ω is endowed by T with what we might call a $\mathbb{Z}^d$-affine structure. That is, a function $a: O_T(\omega)\times O_T(\omega)\to\mathbb{Z}^d$ given by $a(\omega_1,\omega_2)=v$ if $T^v\omega_1=\omega_2$ and satisfying

(i′) $(\forall\omega_1\in O_T(\omega))$, $a(\omega_1,\cdot)$ and $a(\cdot,\omega_1)$ are bijections between $O_T(\omega)$ and $\mathbb{Z}^d$; and

(ii′) $(\forall\omega_1,\omega_2,\omega_3\in O_T(\omega))$, $a(\omega_1,\omega_2)+a(\omega_2,\omega_3)=a(\omega_1,\omega_3)$.

We can visualize the elements of $O_T(\omega)$ as occupying the lattice points of $\mathbb{Z}^d$ where we suppress the $\mathbb{Z}^d$-coordinates and preserve only the relative positions (in $\mathbb{Z}^d$) of the lattice points.

If (T,T') is an orbit equivalence with cocycle α, then T' gives the orbits another affine structure a' related to a by

(1) $$a'(\omega_1,\omega_2)=\alpha(\omega_1,a(\omega_1,\omega_2)).$$

The function a', of course, determines T', so that T' can be viewed as having been obtained from T by a rearrangement of the elements of each orbit on the coordinate-free lattice $\mathbb{Z}^d$, corresponding to the change in affine structures.

Thus, to construct an orbit equivalence, we must rearrange the orbits of T to produce new affine structures and do so in such a way that the cocycle α defined by (1) is measurable.

The rearrangements we will construct have a simple form. They will be obtained by composing a sequence of rearrangements, each of which measurably selects disjoint rectangular blocks on each orbit, and permutes the points inside these blocks. By choosing the block sizes to grow fast enough, and the blocks to fill enough of the orbits, the compositions of these rearrangements will converge to limiting configurations on each orbit that give the desired affine structures.

To return to the language of cocycles, if (T_0,T_1) is an orbit equivalence with cocycle β_1 and (T_1,T_2) is an orbit equivalence with cocycle β_2, then the cocycle for (T_0,T_2) is the composition $\beta_2\circ\beta_1$, where $\beta_2\circ\beta_1(\omega,v)=\beta_2(\omega,\beta_1(\omega,v))$. We say that a sequence of cocycles $\{\alpha_i\}_{i=1}^\infty$ for T_0 *converge to* α if, for a.e. ω and all $v\in\mathbb{Z}^d$, $\lim_{i\to\infty}\alpha_i(\omega,v)=\alpha(\omega,v)$. (Note that such a limit α must satisfy the cocycle identity (ii), and $\alpha(\omega,\cdot)$ must be injective, but it need not be surjective.)

In the proofs of theorems 1 and 2, we will construct a sequence of cocycles $\{\alpha_i\}_{i=1}^\infty$ for a given $\mathbb{Z}^d$-action T_0, where $\alpha_i=\beta_i\circ\beta_{i-1}\circ\cdots\circ\beta_1$, and each β_i is a cocycle for $T^{(\alpha_{j-1})^{-1}}$ of the type we described informally above.

In order to describe more precisely the structure of the β_j, we introduce the following notation. For $L\in\mathbb{Z}^+$, we let

$$C_L=\{v=(v_1,\ldots,v_d)\in\mathbb{Z}^d\,|\,1\le v_i\le L,\ i=1,2,\ldots,d\}$$
$$\bar C_L=\{v=(v_1,\ldots,v_d)\in\mathbb{Z}^d\,|\,|v_i|\le L,\ i=1,2,\ldots,d\}.$$

For $J,K,L\in\mathbb{Z}^+$, we call a permutation π of C_L a (J,K,L)*-permutation* if there exists $u\in\mathbb{Z}^d$, and $v\in\bar C_K$ such that for all $w\in C_J+u$, $\pi(w)=w+v$. (See figure 1.) We refer to v as the *translation vector* of π and denote it by $tv(\pi)$. More generally, a permutation π of C_L+w, $w\in\mathbb{Z}^d$, is called a (J,K,L)*-permutation* if it is of the form $\pi=\pi_w\circ\pi'\circ\pi_w^{-1}$, where π_w is translation by w, and π' is a (J,K,L)-permutation of C_L.

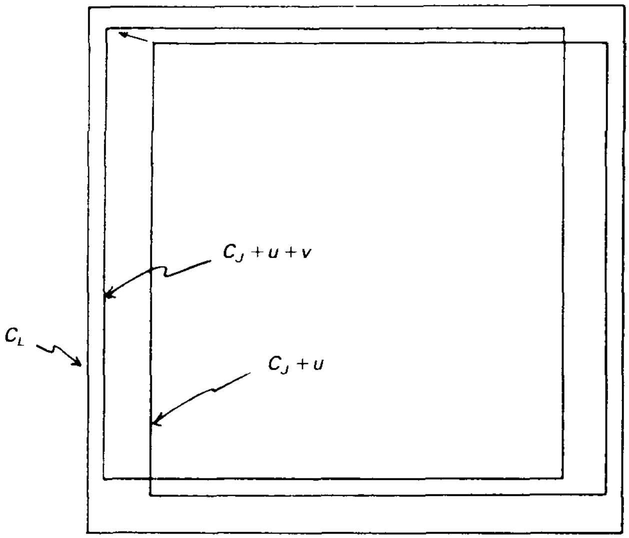

FIGURE 1. A (J, K, L)-permutation of C_L.

We say that a bijection π of $\mathbb{Z}^d$ is a (J, K, L)-blocked bijection if there is a collection $\{C_L+w_i\}$ of disjoint translates of C_L in $\mathbb{Z}^d$, on each of which π acts by a (J, K, L)-permutation, and off which π acts by the identity. If $C_J+u_i+w_i$ is the subset of C_L+w_i on which π acts by translation, we call the sets $\{C_L+w_i\}$, $\{C_J+u_i+w_i\}$, and $\{C_L+w_i \setminus C_J+u_i+w_i\}$ the *blocks*, *rigid blocks*, and *filler sets* of π, respectively.

We say that a cocycle α for a $\mathbb{Z}^d$-action T is a (J, K, L)-*blocked cocycle* if for a.e. ω, there is a (J, K, L)-blocked bijection π_ω of $\mathbb{Z}^d$ such that $\alpha(\omega, \cdot)=\pi_\omega-\pi_\omega(0)$, and the blocks, rigid blocks, and the permutations on them are measurably selected. That is, for all (J, K, L)-permutations π,

$$\{\omega \mid C_L \text{ is a block of } \pi_\omega \text{ and } \pi_\omega|_{C_L}=\pi\}$$

is measurable. The β_i in our construction will be (J, K, L)-blocked cocycles.

Given $S\subset\mathbb{Z}^d$, $\omega\in\Omega$, $A\subset\Omega$ and a $\mathbb{Z}^d$-action T on Ω, we let $T^S(\omega)=\{T^v\omega \mid v\in S\}$ and $T^S(A)=\{T^v\omega \mid \omega\in A, v\in S\}$. Suppose β is a (J, K, L)-blocked cocycle for T. By a block of β (respectively, a rigid block, or a filler set of β) we mean a set of the form $T^S(\omega)$, where S is a block (respectively, rigid block, filler set) of π_ω. Finally, we say a (J, K, L)-blocked cocycle β for T is a (J, K, L, ε)-*blocked cocycle* if $\mu\{\omega \mid \omega$ is in a block of $\beta\}>1-\varepsilon$.

The following lemma isolates the arguments needed in theorems 1 and 2 to construct cocycles that satisfy the weak a-equivalence and strong b-equivalence properties.

Informally, it says that these properties will be obtained if we compose a sequence of blocked cocycles whose block structures grow sufficiently rapidly.

LEMMA 1. *There is a sequence of functions* $\{f_i:\mathbb{N}^{i+1}\times(0,1)\to\mathbb{N}\}_{i=1}^\infty$ *such that if* $\{\varepsilon_i\}_{i=1}^\infty$ *is an arbitrary summable sequence of numbers in* $(0, 1)$ *and* $\{K_i\}_{i=1}^\infty$ *is an arbitrary sequence of positive integers, and if* $\{J_i\}_{i=1}^\infty$ *is a sequence of positive integers such that,* *for each i, $J_{i+1} > f_{i+1}(J_1, J_2, \ldots, J_i, K_{i+1}, d, \varepsilon_{i+1})$ and $\{\alpha_i\}_{i=0}^{\infty}$ is a sequence of cocycles for a $\mathbb{Z}^d$-action T_0 on $(\Omega, \mathcal{B}, \mu)$ (with $\alpha_0 =$ identity) such that the cocycles $\beta_i = \alpha_i \circ \alpha_{i-1}^{-1}$ of the action $T_{i-1} = T_0^{\alpha_{i-1}^{-1}}$ are $(J_i, K_i, L_i, \varepsilon_i)$-blocked cocycles, and $L_i < J_i + 4K_i$, then the α_i converge to a cocycle α which gives a weak a-equivalence for all $a > 1$ and a strong b-equivalence for all $b \in (0, 1)$, between T_0 and $S = T_0^{\alpha^{-1}}$.*

Proof. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of $(J_i, K_i, L_i, \varepsilon_i)$-blocked cocycles for T_0. We first consider the restrictions on the parameters sufficient to ensure convergence of the α_i. Note that, however the parameters are chosen, we have for a.e. ω and for all $v \in \mathbb{Z}^d$

$$|\alpha_i(\omega, v) - v| < B_i = 2d \sum_{j=1}^{i} L_j.$$

If J_{i+1} is chosen so that

(a) $K_{i+1}/J_{i+1} < \varepsilon_{i+1}$; and

(b) $(i + B_i)/J_{i+1} < \varepsilon_{i+1}$,

then setting

$$F_i = \{\omega \mid T_i^{\bar{C}_{i+B_i}}(\omega) \text{ is contained in a rigid block of } \beta_{i+1}\},$$

we get $\mu(\bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} F_j) = 1$. Let $\omega \in \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} F_j$ and $v \in \mathbb{Z}^d$. Choose i so that $|v| < i$ and $\omega \in \bigcap_{j=i}^{\infty} F_j$. Then for all $j > i$

$$\alpha_j(\omega, v) = \alpha_i(\omega, v) = \lim_{k \to \infty} \alpha_k(\omega, v) = \alpha(\omega, v).$$

As we remarked earlier, α necessarily satisfies the cocycle identity, and $\alpha(\omega, \cdot)$ is injective. We also have that $\alpha(\omega, \cdot)$ is surjective, for if $\omega \in \bigcap_{j=i}^{\infty} F_j$ and $w \in \mathbb{Z}^d$, then there exists j such that $j \geq i$ and $j + B_j > |w|$. Now since $\alpha_j(\omega, \cdot)$ is bijective, there exists v such that $\alpha_j(\omega, v) = w$, and since for all $k \geq j$, $\alpha_k(\omega, v) = \alpha_j(\omega, v)$, we get $\alpha(\omega, v) = w$.

We now see how to ensure that α gives a weak a-equivalence, for all $a \geq 1$. It will be somewhat easier to use the equivalent condition that α^{-1} gives a weak a-equivalence, for all $a \geq 1$.

Fix a sequence $a_i \uparrow \infty$, $a_i > 1$. Suppose, in addition to (a) and (b) above, each J_{i+1} is chosen so that there exists $R_i > 0$ such that

(c) $\varepsilon_{i+1} J_{i+1} > R_i > (|2d(\sum_{j=1}^{i} L_j + K_{i+1})|^{a_i})/\varepsilon_{i+1}$; and

(d) $K_{i+1}/R_{i+1} < \varepsilon_{i+1}$.

Let $G_i = \{\omega \mid T^{\bar{C}_{R_i}}(\omega)$ is contained in a rigid block of $\beta_{i+1}\}$. Then $\mu(\bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} G_j) = 1$.

Let $A_j = \bigcap_{k=j}^{\infty} \{\omega \mid \omega$ is not contained in the filler set of $\beta_j\}$. Then the A_j increase and $\mu(A_j) \to 1$. Let $\omega \in \bigcap_{j=i}^{\infty} G_j$, and define $I(\omega) \subset \mathbb{Z}^d$ by saying $v \in I(\omega)$ if $\|v\|_{\infty} > R_j$ and $S^v(\omega) \in A_j$. We now check that for all $a > 1$

$$\lim_{\substack{|v| \to \infty \\ v \in I(\omega)}} \frac{|\alpha^{-1}(\omega, v) - v|^a}{|v|} = 0.$$

Fix $a > 1$ and choose $j > i$ so that $a_j > a$. If $R_j \leq \|v\|_{\infty} \leq R_{j+1}$ and $v \in I(\omega)$, we get $S^v(\omega) = T_{j+1}^v(\omega)$ so that

$$\frac{|\alpha^{-1}(\omega, v) - v|^a}{|v|} = \frac{|\alpha_{j+1}^{-1}(\omega, v) - v|^a}{|v|} \leq \frac{|2d(\sum_{k=1}^{j} L_k + K_{j+1})|^{a_j}}{R_j} < \varepsilon_{j+1}.$$

Letting $j \to \infty$ gives the desired result.

Finally, to ensure in addition that α be a strong b-equivalence for all $b \in (0, 1)$, fix a sequence $b_i \uparrow 1$, $b_i \in (0, 1)$. Suppose that each J_{i+1} is chosen so that there exists $R_i > 0$ such that

(e) $\varepsilon_{i+1} J_{i+1} > R_i > (2/\varepsilon_{i+1})(2d \sum_{j=1}^{i+1} L_j)^{b_i}$; and

(f) $R_i > 4d \sum_{j=1}^{i} L_j$.

Then if $H_i = \{\omega \mid T_i^{\bar{C}_{R_i}}(\omega)$ is contained in a rigid block of $\beta_{i+1}\}$, $\mu(\bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} H_j) = 1$.

On the basis of the foregoing argument, we can give explicit values for the desired functions f_i: set

$$f_{i+1}(J_1, J_2, \ldots, J_i, K_{i+1}, d, \varepsilon_{i+1}) = \max\left[\frac{K_{i+1}}{\varepsilon_{i+1}}, \frac{i+2\sum_{j=1}^{i} J_j}{\varepsilon_{i+1}}, \frac{(4d\sum_{j=1}^{i} J_j + K_{i+1})^{a_i}}{\varepsilon_{i+1}^2}, \left(\frac{2(8d)^{b_i}}{\varepsilon_{i+1}^2}\right)^{1/(1-b_i)}, \frac{4d\sum_{j=1}^{i} J_j}{\varepsilon_{i+1}}\right].$$

The reader may verify that if J_{i+1} is chosen greater than f_{i+1}, we obtain conditions (a)-(f) and hence the conclusion of the lemma. □

Proof of theorem 1. As we have already indicated, we will build a sequence of cocycles $\{\alpha_i\}_{i=1}^{\infty}$ for T such that for all i, $\beta_i = \alpha_i \circ \alpha_{i-1}^{-1}$ is a $(J_i, K_i, L_i, \varepsilon_i)$-blocked cocycle for $T_i = T^{\alpha_i^{-1}}$.

The limit α of the α_i will be the cocycle giving the desired mixing action $S = T^{\alpha^{-1}}$. We need only concern ourselves here with showing that S is mixing. The fact that α may be constructed to satisfy simultaneously the weak a-equivalence property and the strong b-equivalence property for all $a \geq 1$ and $b \in (0, 1)$ will follow from lemma 1 and the observation that the sequence $\{J_i\}$ may be chosen as it must for lemma 1 to apply, without interfering with the mixing properties of the construction.

We say that a $\mathbb{Z}^d$-process (T', P') is *ε-mixing between L and L'*, $L < L' \in \mathbb{Z}^+$ if for all $v \in \bar{C}_{L'} \setminus \bar{C}_L$:

$$|\text{dist}\,(P' \vee (T')^{-v} P') - (\text{dist}\, P' \times \text{dist}\, P')| < \varepsilon.$$

We fix for the remainder of the argument a refining sequence of (finite) partitions $\{P_i\}_{i=1}^{\infty}$ of Ω which increase to $\mathscr{B}$, and a sequence $\{\varepsilon_i\}_{i=1}^{\infty} \subset (0, 1)$ with $\varepsilon_i < \varepsilon_{i-1}/2$, all i.

The cocycles α_i that we construct will be such that, for all $1 \leq j \leq i \leq k$, the $\mathbb{Z}^d$-action $T_k = T^{\alpha_k^{-1}}$ will satisfy

(2) $\qquad (T_k, P_j)$ is $(\varepsilon_j/2^{i-j})$-mixing between L_i and L_{i+1}.

Once we establish (2), the proof will be complete, since (2) implies that for all $j \leq i$, (S, P_j) is $\varepsilon_j/2^{i-j}$-mixing between L_i and L_{i+1}, and hence (S, P_j) is mixing. Since the P_j increase to $\mathscr{B}$, we have that S is mixing.

The construction will be completed when we describe the $(J_i, K_i, L_i, \varepsilon_i)$-blocked cocycles β_i. In order to define β_i we need only choose the parameters J_i, K_i, L_i and ε_i and specify the blocks of β_i (in the orbits of T_{i-1}) and the (J_i, K_i, L_i) permutation associated with each block.

To do this for β_1, fix $\bar{\varepsilon}_1 < \varepsilon_1$, whose size will be determined by the argument to follow. Choose K_1 so that all ω in a set $E_1 \subset \Omega$ with $\mu(E_1) > 1 - \bar{\varepsilon}_1$ have a $(T, P_1 - C_{K_1})$-name with the distribution of P_1 within $\bar{\varepsilon}_1$ of dist (P_1). Choose $L_1 > K_1/\bar{\varepsilon}_1$. Fix $\bar{\varepsilon}_2 < \varepsilon_2$ (again, whose size is determined by the following) and choose $K_2 > L_1/\bar{\varepsilon}_2$

so that there exists $E_2 \subset \Omega$ with $\mu(E_2) > 1 - \bar{\varepsilon}_2$ and all $\omega \in E_2$ have $(T, P_2, -C_{K_2})$-names with P_2 distribution within $\bar{\varepsilon}_2$ of dist (P_2). Choose $L_2 > K_2/\bar{\varepsilon}_2$ and M_1, a multiple of L_1 with $M_1 > L_2/\bar{\varepsilon}_1$. Let B_1 be the base of a Rokhlin tower $\mathcal{T}_1 = \{T^v B_1\}_{v \in C_{M_1}}$ such that $\mu(T^{C_{M_1}} B_1) > 1 - \bar{\varepsilon}_1$ and

(3) $(\forall \omega \in B_1)(\exists S \subset C_{M_1})$ such that $|S| > (1 - \bar{\varepsilon}_1)|C_{M_1}|$ and $(\forall v \in S) T^v \omega \in E_1$.

For each $\omega \in B_1$, we can write $T^{C_{M_1}}(\omega)$ as a disjoint union

$$\bigcup_{w \in L_1 C_{M_1/L_1}} T^{C_{L_1}+w}(\omega).$$

The sets $T^{C_{L_1}+w}(\omega)$ will be the blocks of the cocycle β_1. To specify the associated permutations, we fix a set $\Pi_1 = \{\pi_1, \ldots, \pi_{K_1^d}\}$ of $(L_1 - K_1, K_1, L_1)$-permutations of C_{L_1} that act by translation on $C_{L_1 - K_1}$ and have pairwise distinct translation vectors in C_{K_1}. Using these we construct a family Σ_1 of $[K_1^d]^{(M_1/L_1)^d}$ permutations of C_{M_1} by selecting, in all possible ways, an element of Π_1 to act on each $C_{L_1} + w \subset C_{M_1}$, $w \in L_1 C_{M_1/L_1}$. (See figure 2.)

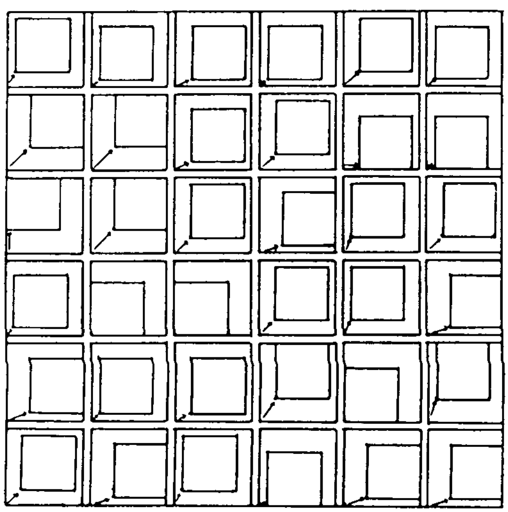

FIGURE 2. Schematic diagram of an element of Σ_1.

Now let $\sigma_1 : B_1 \to \Sigma_1$ be a measurable function such that for each atom A of $Q_1 = (T, P_1)_{C_{M_1}}|_{B_1}$, $\sigma_1|_A$ is uniformly distributed on Σ_1. (Here and hereafter, we use the notation $(T, P)_S$ to denote the partition $\bigvee_{v \in S} T^{-v} P$.) Thus σ_1 specifies the $(L_1 - K_1, K_1, L_1)$-permutation associated with each block of β_1.

We now verify that $(T_1, P_1) = (T^{\beta_1^{-1}}, P_1)$ is ε_1-mixing between L_1 and L_2. Fix $v \in \bar{C}_{L_2} \backslash \bar{C}_{L_1}$. We must show that

$$|\text{dist}\,(P_1 \vee T_1^{-v} P_1) - \text{dist}\, P_1 \times \text{dist}\, P_1| < \varepsilon_1.$$

We note that the tower $\mathcal{T}_1$ has become a tower $\bar{\mathcal{T}}_1$ for T_1 with base $\bar{B}_1 = \{T^w(\omega) \mid \omega \in B_1 \text{ and } \sigma_1(\omega)(w) = (1, \ldots, 1)\}$. We also get a partition $\bar{Q}_1$ of $\bar{B}_1$ whose atoms are the sets $\bar{A} = \{T^w(\omega) \mid \omega \in A \text{ and } \sigma_1(\omega)(w) = (1, \ldots, 1)\}$ where A is an atom

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of Q_1. Informally, we could describe the above by saying that each P_1-pure column of $\mathcal{T}_1$ has been permuted (in $|\Sigma_1|$ different ways) and the bases of these new columns are the atoms of the partition $\bar{Q}_1$ of $\bar{B}_1$.

Given $A \in Q_1$, we say that $u \in C_{M_1}$ is *a good position* for A if for all $\omega \in A$ both $T^{u-C_{K_1}}(\omega)$ and $T^{u+v-C_{K_1}}(\omega)$ are contained in rigid blocks of β_1, and

(4) $$|\text{dist}\,(P_1) - \text{dist}_{T^{u-C_{K_1}(A)}}(P_1)| < \bar{\varepsilon}_1$$

and

(5) $$|\text{dist}\,(P_1) - \text{dist}_{T^{u+v-C_{K_1}(A)}}(P_1)| < \bar{\varepsilon}_1.$$

We see that:

(6) a fraction greater than $1-(2\bar{\varepsilon}_1 + 4d\bar{\varepsilon}_1)$ of the $u \in C_{M_1}$ are good positions for A. This follows from the facts that $v \in \bar{C}_{L_2} \backslash \bar{C}_{L_1}$, $K_1/L_1 < \bar{\varepsilon}_1$, $L_2/M_1 < \bar{\varepsilon}_1$ and conditions (3).

We will show that, for all good positions u for A,

(7) $$|\text{dist}_{T_1^u\bar{A}}(P_1 \vee T_1^{-v}P_1) - \text{dist}\,P_1 \times \text{dist}\,P_1| < 2\bar{\varepsilon}_1.$$

From (7) and (6) we see that if $\bar{\varepsilon}_1$ is sufficiently small,

$$|\text{dist}_{T_1^{C_{M_1}}\bar{A}}(P_1 \vee T_1^{-v}P_1) - \text{dist}\,P_1 \times \text{dist}\,P_1| < \varepsilon_1/2,$$

and by arguing similarly on every atom of Q_1, we see that

$$|\text{dist}_{T_1^{C_{M_1}}\bar{B}_1}(P_1 \vee T_1^{-v}P_1) - \text{dist}\,P_1 \times \text{dist}\,P_1| < \varepsilon_1/2,$$

and since $\mu(T_1^{C_{M_1}}\bar{B}_1) > 1 - \bar{\varepsilon}_1$ we have

$$|\text{dist}\,(P_1 \vee T_1^{-v}P_1) - \text{dist}\,P_1 \times \text{dist}\,P_1| < \varepsilon_1 + 2\bar{\varepsilon}_1,$$

which we may assume is less than ε_1.

To establish (7), we let $(P_1)_A : C_{M_1} \to P_1$ denote the function determined by the (T, P_1, C_{M_1})-name of the points in A, and we also regard P_1 itself as a discrete random variable on Ω. Fix a good position u for A, and let $\pi_u : \bar{A} \to \Pi_1$ (respectively $\pi_{u+v} : \bar{A} \to \Pi_1$) satisfy for all $\omega \in \bar{A}$, $\pi_u(\omega)$ (resp. $\pi_{u+v}(\omega)$) is the element of Π_1 that was selected to act on the block of β_1 containing $T_1^u(u)$ (resp. $T_1^{u+w}(\omega)$).

For all $\omega \in \bar{A}$,

$$(P_1 \vee T_1^{-v}P_1)(T_1^u\omega) = [(P_1)_A(u - tv(\pi_u(\omega))), (P_1)_A(u+v-tv(\pi_{u+v}(\omega)))].$$

But since $v \notin \bar{C}_{L_1}$, $tv(\pi_u)$ and $tv(\pi_{u+v})$ are independent random variables, so

$$\underset{T_1^u\bar{A}}{\text{dist}}\,(P_1 \vee T_1^{-v}P_1) = \text{dist}\,(P_1)_A(u - tv(\pi_u)) \times \text{dist}\,(P_1)_A(u+v-tv(\pi_{u+v}))$$

$$= \underset{T^{u-C_{K_1(A)}}}{\text{dist}}(P_1) \times \underset{T^{u+v-C_{K_1(A)}}}{\text{dist}}(P_1).$$

Since u is a good position for A, conditions (4) and (5) imply (7).

We now describe the iterative procedure by which this construction is completed. At the ith stage, we have constructed T_{i-1}, and we have chosen $\{L_j\}_{j=1}^{i}$ and $\{K_j\}_{j=1}^{i}$ and $\{\bar{\varepsilon}_j\}_{j=1}^{i}$ and we know that for all $1 \le j \le k \le i-1$

(8) $$(T_{i-1}, P_j) \text{ is } (\varepsilon_j/2^{k-j})\text{-mixing between } L_k \text{ and } L_{k+1}.$$

In the previous stage of the construction, we chose K_i so that every ω in a set $E_i \subset \Omega$

with $\mu(E_i)>1-\bar{\varepsilon}_i$ has a $(T_{i-2}, P_i, -C_{K_i})$-name with distribution of P_i within $\bar{\varepsilon}_i$ of dist (P_i). We made T_{i-1} from T_{i-2} by an $(L_{i-1}-K_{i-1}, K_{i-1}, L_{i-1}, \bar{\varepsilon}_{i-1})$-blocked cocycle β_{i-1} and K_i was chosen so that $K_i > L_{i-1}/\bar{\varepsilon}_i$. Thus for all $\omega \in E_i$ the $(T_{i-1}, P_i, -C_{K_i})$-name of ω has P_i-distribution within $\bar{\varepsilon}_i + 2d\bar{\varepsilon}_i$ of dist (P_i).

We continue by choosing $\bar{\varepsilon}_{i+1} < \varepsilon_{i+1}$ of size to be specified later, and $K_{i+1} > L_i/\bar{\varepsilon}_{i+1}$ so that there exists $E_{i+1} \subset \Omega$ with $\mu(E_{i+1}) < \bar{\varepsilon}_{i+1}$ and every $\omega \in E_{i+1}$ has a $(T_{i-1}, P_{i+1}, -\bar{C}_{K_{i+1}})$-name with P_{i+1}-distribution within $\bar{\varepsilon}_{i+1}$ of dist (P_{i+1}). Choose $L_{i+1} > K_{i+1}/\bar{\varepsilon}_{i+1}$ and $M_i > L_{i+1}/\bar{\varepsilon}_i$, M_i a multiple of L_{i+1}, and M_i so large that we can build a (T_{i-1}, C_{M_i})-Rokhlin tower $\mathcal{T}_i$ with base B_i, $\mu(T^{C_{M_i}} B_i) > 1-\bar{\varepsilon}_i$ and for all $\omega \in B_i$ there exists $S \subset C_{M_i}$ such that $|S| > (1-\bar{\varepsilon}_i - 2d\bar{\varepsilon}_i)|C_{M_i}|$ and for all $v \in S$, $T^v(\omega) \in E_i$.

We can now construct a cocycle β_i for T_{i-1} and $T_i = T_{i-1}^{\beta_i^{-1}}$ exactly as in the first stage of the construction, and verify that (T_i, P_i) is ε_i-mixing between L_i and L_{i+1}. By the choice of ε_i and the fact that the sequence $\{P_i\}$ is increasing, we get for all $j \le i$, (T_i, P_j) is $\varepsilon_j/2^{i-j}$ mixing between L_i and L_{i+1}. It remains to show, however, that the mixing behaviour of T_{i-1} as in (8) is still exhibited by T_i.

For $j \le i-2$, we have that

$$\mu\{\omega | (\forall v \in \bar{C}_{L_{j+1}} \setminus \bar{C}_{L_j}) T_i^v(\omega) = T_j^v(\omega)\} > 1 - (\bar{\varepsilon}_i + 5d\bar{\varepsilon}_i),$$

so that if $\bar{\varepsilon}_i$ is sufficiently small, the mixing conditions satisfied by T_{i-1} between L_j and L_{j+1} are satisfied by T_i as well.

To preserve the mixing behaviour of T_{i-1} between L_{i-1} and L_i, fix $v \in \bar{C}_{L_i} \setminus \bar{C}_{L_{i-1}}$, and $u \in C_{M_i}$.

Case 1. As in the first stage of the construction, let $\bar{B}_i$ denote the base of the C_{M_i}-tower we construct for T_i. Suppose that u is such that for all $\omega \in \bar{B}_i$, $T_i^{u-C_{K_i}}(\omega)$ and $T_i^{u+v-C_{K_i}}(\omega)$ are contained in the same rigid block of β_i. Then

$$\begin{aligned}\operatorname*{dist}_{T_i^u \bar{B}_i}(P_{i-1} \vee T_i^{-v} P_{i-1}) &= \frac{1}{K_1^d} \sum_{w \in C_{K_i}} \operatorname*{dist}_{T_{i-1}^{u-w}}(P_{i-1} \vee T_{i-1}^{-v} P_{i-1}) \\ &= \operatorname{dist}(P_{i-1} \vee T_{i-1}^{-v} P_{i-1})\end{aligned}$$

so that

$$\left|\operatorname*{dist}_{T_i^u \bar{B}_i}(P_{i-1} \vee T_i^{-v} P_{i-1}) - \operatorname{dist}(P_{i-1}) \times \operatorname{dist}(P_{i-1})\right| < \varepsilon_{i-1}.$$

Case 2. Suppose that for all $\omega \in \bar{B}_i$, $T_i^{u-C_{K_1}}(\omega)$ and $T_i^{u+v-C_{K_i}}(\omega)$ are contained in different rigid blocks of β_i. Suppose also that for a set of atoms A of Q_i (notation as in the first part of the construction) of measure greater than $1-(2\bar{\varepsilon}_i + 4d\bar{\varepsilon}_i)^{1/2}$, u is a good position for A. For each such A,

$$\left|\operatorname*{dist}_{T_i^u \bar{A}}(P_{i-1} \vee T_i^{-v} P_{i-1}) - \operatorname{dist} P_{i-1} \times \operatorname{dist} P_{i-1}\right| < 2\bar{\varepsilon}_i,$$

so if $\bar{\varepsilon}_i$ is sufficiently small,

$$\left|\operatorname*{dist}_{T_1^u \bar{B}_i}(P_{i-1} \vee T_1^{-v} P_{i-1}) - \operatorname{dist} P_{i-1} \times \operatorname{dist} P_{i-1}\right| < \varepsilon_{i-1}.$$

These two cases account for a fraction greater than $1-[2\bar{\varepsilon}_i + 4d\bar{\varepsilon}_i]^{1/2}$ of the $u \in C_{\mu_i}$

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so since $\mu(T_i) > 1 - \bar{\varepsilon}_i$, we see that if $\bar{\varepsilon}_i$ is sufficiently small,

$$|\text{dist}\,(P_{i-1} \vee T_1^{-v} P_{i-1}) - \text{dist}\, P_{i-1} \times \text{dist}\, P_{i-1}| < \varepsilon_{i-1}. \qquad \square$$

In order to prove theorem 2, we will make use of a family of $\mathbb{Z}^d$-Bernoulli processes which we will describe as measures on $\{0,1\}^{\mathbb{Z}^d}$ constructed by the standard combinatorial device of concatenating finite blocks of symbols to specify the measures of cylinder sets. The blocks in question are functions $B_i^j : \bar{C}_{s_i} \to \{0,1\}$, $j \in \mathcal{J}_i$, $i = 0, 1, 2, \ldots$, and their inductive definition is governed by parameters $\{r_l\}_{l=1}^{\infty}$ and $\{m_l\}_{l=0}^{\infty}$ in $\mathbb{Z}^+$. Given the i-blocks B_i^j and $\{r_l\}_{l=1}^{i}$ and $\{m_l\}_{l=0}^{i-1}$ we construct the $(i+1)$-blocks in two stages. First we choose m_i and construct blocks $\bar{B}_{i+1}^k : \bar{C}_{\bar{s}_{i+1}} \to \{0,1\}$, $k \in \mathcal{K}_{i+1}$, where $\bar{s}_{i+1} = \frac{1}{2}((2m_i+1)(2s_i+1)-1)$ by concatenating i-blocks, $(2m_i+1)^d$ at a time, in all possible ways.

We then choose r_{i+1} and define the $(i+1)$-blocks $B_{i+1}^{k,v} : \bar{C}_{s_{i+1}} \to \{0,1\}$, $k \in \mathcal{K}_{i+1}$, $v \in \bar{C}_{r_{i+1}}$, where $s_{i+1} = \bar{s}_{i+1} + 2r_{i+1}$, by setting

$$B_{i+1}^{k,v}(w) = \begin{cases} 0 & \text{if } w \in \text{bdy}(\bar{C}_{s_{i+1}}) \\ \bar{B}_{i+1}^k(w - v) & \text{if } w \in \bar{C}_{\bar{s}_{i+1}} + v \\ 1 & \text{otherwise.} \end{cases}$$

We begin the construction by defining only one 0-block, namely $B_0 : \bar{C}_0 \to \{0\}$. We also stipulate that

$$\sum_{i=1}^{\infty} \frac{r_i}{\bar{s}_i} < \infty \tag{9}$$

and

$$\frac{r_{i+1}}{s_i} \uparrow \infty. \tag{10}$$

We then define a stationary measure μ on $\{0,1\}^{\mathbb{Z}^d}$ by setting for each cylinder γ, $\mu_i^j(\gamma)$ equal to the frequency of occurrence of γ in B_i^j and $\mu_i(\gamma) = (1/|\mathcal{J}_i|) \sum_{j \in \mathcal{J}_i} \mu_i^j(\gamma)$, and $\mu(\gamma) = \lim_{i \to \mathcal{J}} \mu_i(\gamma)$. Condition (9) insures the existence of this limit.

In order to show that the processes described are Bernoulli (that is, isomorphic to independent processes), one can verify a $\mathbb{Z}^d$-version of the very weak Bernoulli condition. These processes actually satisfy a condition stronger than what we need, but easier to state.

Definition. Given a $\mathbb{Z}^d$-action T on $(\Omega, \mathcal{B}, \mu)$ and a finite partition P, we say that the $\mathbb{Z}^d$-process (T, P) is *symmetrically very weak Bernoulli* if $(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)$ $(\forall m > n)(\exists \mathfrak{a} \subset (T, P)_{\bar{C}_m \setminus \bar{C}_n})$ such that $\mu(\bigcup \mathfrak{a}) > 1 - \varepsilon$ and $(\forall \alpha_1, \alpha_2 \in \mathfrak{a})$

$$\bar{d}[(T, P)_{\bar{C}_n}|_{\alpha_1}, (T, P)_{\bar{C}_n}|_{\alpha_2}] < \varepsilon.$$

The processes above can be shown to be symmetrically very weak Bernoulli by a nesting argument exactly like that of [**4**]. For a proof that this implies isomorphism to an independent process, we refer the reader to [**8**].

In our proof of theorem 2, it will be most convenient to use certain factors of the processes constructed above. Namely, if (T, P) is one of these processes (where P

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is the partition according to the coordinate at the origin) we consider the factor of (T, P) given by the partition B into the various types of 1-blocks and their complement. That is, if we let

$$b_1^v = \{\omega \mid u \mapsto P(T^u(\omega)) \text{ gives the 1-block } B_1^v \text{ on } T^{\bar{C}_{s_1}}(\omega)\},$$

we set

$$B(\omega) = \begin{cases} v & \text{if } \omega \in T^{\bar{C}_{s_1}}(b_1^v) \\ a & \text{(an arbitrary fixed symbol) otherwise.} \end{cases}$$

We will refer to the sets $T^{\bar{C}_{s_1}}(\omega)$, $\omega \in b_1^v$ as *blocks of type v*, and we let $\mathcal{T}_1$ denote $T^{\bar{C}_{s_1}}(\bigcup_{v \in \bar{C}_{r_1}} b_1^v)$, and we refer to the process (T, B) as a (J, K, L, ε)-blocked Bernoulli process, if $\bar{s}_1 = J$, $r_1 = K$, $s_1 = L$, and $\mu(\mathcal{T}_1^c) < \varepsilon$.

We will use blocked Bernoulli processes to direct the construction of the orbit equivalence in theorem 2. Of particular importance is the fact that for each blocked Bernoulli process (T, B), given that a block occurs at specified coordinates, the type of that block is independent of the name outside the block. More precisely, if $b = \{\omega \mid T^{\bar{C}_{s_1}}(\omega)$ is a block$\}$, then for all $m > s_1$, $B \perp (T, B)_{\bar{C}_m \setminus \bar{C}_{s_1}}$, conditioned on b.

We remark that we can easily construct blocked Bernoulli processes of arbitrarily small entropy.

Definition. A $\mathbb{Z}^d$-process (T, P) is *symmetrically K-mixing at* $\bar{C}_0$ if for all $\varepsilon > 0$ there exists N such that for all $m > N$

$$P \perp^{\varepsilon} (T, P)_{\bar{C}_m \setminus \bar{C}_N}.$$

We remind the reader that $P \perp^{\varepsilon} Q$ means that $(\exists \mathscr{E} \subset Q)$ with $\mu(\bigcup \mathscr{E}) > 1 - \varepsilon$, and for all $q \in \mathscr{E}$, $|\text{dist}_q P - \text{dist } P| < \varepsilon$.

LEMMA 2. *If a $\mathbb{Z}^d$-action T on $(\Omega, \mathscr{B}, \mu)$ admits a sequence of partitions P_i such that $P_i \uparrow \mathscr{B}$, and each (T, P_i) is symmetrically K-mixing at $\bar{C}_0$, then T is a K-system.*

Proof. Fix a finite partition P with $h(P) = \beta > 0$. Choose ε so that $|\bar{P} - P| < \varepsilon$ implies (for all $\mathbb{Z}^d$-actions S) $|h(S, P) - h(S, \bar{P})| < \beta/2$. Choose i so that for some $\bar{P} \subset P_i$, $|P - \bar{P}| < \varepsilon$. Since (T, P_i) is symmetrically K-mixing, it follows that $(T, \bar{P})$ is as well. For each $n \in \mathbb{Z}^+$, let T^n denote the $\mathbb{Z}^d$-action given by $(T^n)^v = T^{nv}$. Choose n so that $|h(T^n, \bar{P}) - h(\bar{P})| < \beta/2$. Then $h(T^n, P) > 0$, so that $h(T^n, (T, P_{C_{n+1}}) > 0$ and so $h(T, P) > 0$ as desired.

LEMMA 3 (The relative Sinai theorem [7]). *Let T be an ergodic $\mathbb{Z}^d$-action on $(\Omega, \mathscr{B}, \mu)$ and P a finite partition such that $h(T, P) < h(T)$. Let $(\bar{T}, \bar{B})$ be a $\mathbb{Z}^d$-Bernoulli process on $(\bar{\Omega}, \bar{\mathscr{B}}, \bar{\mu})$ with $h(\bar{T}, \bar{B}) \leq h(T) - h(T, P)$. Then there exists a partition B on Ω such that (T, B) and $(\bar{T}, \bar{B})$ have the same finite distributions, and $(B)_T \perp (P)_T$.*

LEMMA 4. *Let T be a $\mathbb{Z}^d$-system on $(\Omega, \mathscr{B}, \mu)$ and P a partition such that $h(T, P) < h(T)$. Then for all partitions Q and $\varepsilon > 0$, there exists Q' such that $|Q' - Q| < \varepsilon$ and $h(T, P \vee Q') < h(T)$.*

This can be proved exactly as is lemma 2 of [**5**].

Proof of theorem 2. As in the proof of theorem 1, we will construct a cocycle α as the limit of cocycles α_i for T, where for all i, $\beta_i = \alpha_i \circ \alpha_{i-1}^{-1}$ is a $(2J_i + 1, K_i, 2L_i +$

$1, \varepsilon_i)$-blocked cocycle for $T_i = T^{\alpha_{i-1}^{-1}}$. At the ith stage of the construction, having made T_{i-1}, we will choose a $(J_i, K_i, L_i, \varepsilon_i)$-blocked Bernoulli factor (T_{i-1}, B_i) of T_{i-1} and use the blocks of this factor to specify β_i. Viewed as a rearrangement of the orbits of T_{i-1}, β_i will act by permuting each orbit within the blocks, leaving the rest of the orbit fixed. In order to define β_i, then, it is sufficient to specify the permutation associated with each type of block, and we will associate with the blocks of type v a permutation π_v of $\bar{C}_{L_i}$ that translates $\bar{C}_{L_i-2K_i}$ by v and is arbitrary (but fixed) on $\bar{C}_{L_i}\setminus\bar{C}_{L_i-2K_i}$.

We begin by choosing a sequence of finite partitions $P_i\uparrow\mathscr{B}$, and $\varepsilon_i\downarrow 0$. Choose $\bar{P}_1$ by lemma 4 so that $|\bar{P}_1 - P_1| < \varepsilon_1$ and $h(T, \bar{P}_1) < h(T)$. Choose $\bar{\varepsilon}_1 < \varepsilon_1$ with size to be determined by the argument to follow. Choose K_1 so that there exists $E_1 \subset \Omega$, with $\mu(E_1) > 1 - \bar{\varepsilon}_1$ and for all $\omega \in E_1$ the $(T, P_1, \bar{C}_{K_1})$-name of ω has $\bar{P}_1$ distribution within $\bar{\varepsilon}_1$ of $\operatorname{dist}(\bar{P}_1)$. Choose B_1 by lemma 3 so that (T, B_1) is an $(L_1 - 2K_1, K_1, L_1, \bar{\varepsilon}_1)$-blocked Bernoulli process with $h(T, \bar{P}_1 \vee B_1) < h(T)$ and $(T, B_1)_{\mathbf{Z}^d} \perp (T, \bar{P}_1)_{\mathbf{Z}^d}$.

Let β_1 be the cocycle for T determined by (T, B_1) as described above. We now show that $(\exists N_1)(\forall m > N_1)\bar{P}_1 \perp^{\varepsilon_1} (T_1, \bar{P}_1)_{\bar{C}_m \setminus \bar{C}_{N_1}} = Q_1$.

Let $N_1 = 3L_1$ and fix $m > N_1$. Let $R_1 = (T, \bar{P}_1 \vee B_1)_{\bar{C}_{m+L_1}\setminus \bar{C}_{N_1 - L_1}}$. Let $\mathfrak{A}$ denote the collection of sets $a(p \vee b, u, w)$ obtained by specifying $p \vee b \in R_1$, $u \in \bar{C}_{K_1}$, $w \in \bar{C}_{L_1}$ and setting

$$a(p \vee b, u, w) = \{\omega \mid T^{\bar{C}_{L_1}+w}(\omega) \text{ is a block of } \beta_1 \text{ of type } u,$$
$$\text{and } \omega \text{ is contained in the corresponding rigid block,}$$
$$\text{and the } R_1 \text{ name of } T^u(\omega) \text{ is } pvb.\}$$

We define an equivalence relation on $\mathfrak{A}$ by setting $a_1 = a((p \vee b)_1, u_1, w_1) \sim a((p \vee b)_2, u_2, w_2) = a_2$ if $u_2 - u_1 = w_2 - w_1$ and $(p \vee b)_1 = (p \vee b)_2$. Note that $a_1 \sim a_2 \Rightarrow \mu(a_1) = \mu(a_2)$ and a_1 and a_2 are contained in the same atom of Q_1. Also, if $R_1' = (T, \bar{P}_1)_{\bar{C}_{K_1}}$, then $a_1 \sim a_2 \Rightarrow \operatorname{dist} T^{-u_1} R_1'|_{a_1} = \operatorname{dist} T^{-u_2} R_1'|_{a_2}$. Let $[a]$ denote the equivalence class of $a \in \mathfrak{A}$. We say that $[a]$ is good if

(i) The parameters u_i corresponding to the elements of $[a]$ range over all of $\bar{C}_{K_1}$ (equivalently, $\#([a]) = \#(\bar{C}_{K_1})$); and

(ii) $(\forall a' \in [a])\mu(a' \cap T^{-u'}(E_1)) > (1 - \bar{\varepsilon}_1^{\frac{1}{2}})\mu(a')$, (equivalently, $(\exists a' \in [a])\mu(a' \cap T^{-u'}(E_1)) > (1 - \bar{\varepsilon}_1^{\frac{1}{2}})\mu(a')$, by the remark above).

Now if $T^{\bar{C}_{2K_1}}(\omega)$ is contained in a rigid block of β_1, then there exists $a \in \mathfrak{A}$ such that $\omega \in a$ and $[a]$ satisfies (i). Thus, if $\mu(\mathscr{T}_1^C)$ and K_1/L_1 are chosen to be sufficiently small, we have $\mu(\{a \in \mathfrak{A} \mid [a] \text{ satisfies (i)}\}) > 1 - 2\bar{\varepsilon}_1$.

Since $\mu(E_1) > 1 - \bar{\varepsilon}_1$, $\mu(\{a \in \mathfrak{A} \mid [a] \text{ satisfies (ii)}\}) > 1 - \bar{\varepsilon}_1^{\frac{1}{2}}$. Thus,

$$\mu\left(\bigcup_{[a]\text{good}} a\right) > 1 - 2\bar{\varepsilon}_1 - \bar{\varepsilon}_1^{\frac{1}{2}} > 1 - (4\bar{\varepsilon}_1)^{1/2}.$$

And therefore if

$$\mathscr{C} = \left\{\bar{p} \in Q_1 \,\middle|\, \mu\left(\bar{p} \cap \left(\bigcup_{[a]\text{good}} a\right)\right) > (1 - (4\bar{\varepsilon}_1)^{1/4})\mu(\bar{p})\right\},$$

$\mu(\bigcup \mathscr{C}) > 1 - (4\bar{\varepsilon}_1)^{1/4}$.

Fix $\bar{p} \in \mathscr{C}$. Since $\bar{p}$ is nearly covered by (complete) good equivalence classes $[a]$, it is sufficient to show that for such a class $[a]$, the distribution of $\bar{P}_1$ on $\bigcup_{a' \in [a]} a'$ is close to dist $(\bar{P}_1)$. Since $[a]$ is good, there is a family $\mathscr{D} \subset R'_1$ such that $\bigcup \mathscr{D} \subset E_1$, and for all $a_i = a((p \vee b)_i, u_i, w_i) \in [a]$,

$$\mu(a_i \cap T^{-u_i}(\bigcup \mathscr{D})) > (1 - \bar{\varepsilon}_1^{\frac{1}{2}})\mu(a_i).$$

Now for all $d \in \mathscr{D}$ the distribution $\bar{P}_1$ on $\bigcup_i (a_i \cap T^{-u_i} d)$ is exactly the distribution of $\bar{P}_1$ on the $\bar{C}_{K_1}$-coordinates of d. Since this distribution differs from dist $(\bar{P}_1)$ by less than $\bar{\varepsilon}_1$, we have $\bar{P}_1 \perp^{\varepsilon_1} Q_1$ as desired, provided $\bar{\varepsilon}_1$ is chosen sufficiently small.

Now $h(T_1) = h(T)$. In fact $T_1 \simeq T$ by an isomorphism that preserves the orbits of T and T_1. Namely, if we set

$$g(\omega) = \begin{cases} 0 & \text{if } \omega \in \mathscr{T}_1 \\ \pi(v) & \text{if } \omega = T^v \omega', \text{ where } T^{\bar{C}_{L_1}}(\omega') \text{ is a block of } \beta_1, \end{cases}$$

then $\omega \mapsto T^{g(\omega)}$ is an isomorphism between T_1 and T. (In other words, β_1 is within a coboundary of the identity.)

We can now see that $h(T_1, \bar{P}_1 \vee B_1) < h(T_1)$. We see that for all n and all $a \in (T, \bar{P}_1 \vee B_1)_{\bar{C}_{n+2L_1}}$, there exists $\bar{a} \in (T_1, \bar{P}_1 \vee B_1)_{\bar{C}_n}$ such that $a \subset \bar{a}$. By the Shannon-MacMillan theorem for all $\eta > 0$ there exists N such that for all $n \geq N$ a collection of fewer than $2^{(2(n+2L_1)+1)^d (h(T, \bar{P}_1 \vee B_1) + \eta)}$ atoms of $(T, \bar{P}_1 \vee B_1)_{\bar{C}_{n+2L_1}}$ cover a set of measure greater than $1 - \eta$.

Thus, a collection of fewer than $2^{(2(n+2L_1)+1)^d (h(T, \bar{P}_1 \vee B_1) + \eta)}$ atoms of $(T_1, \bar{P}_1 \vee B_1)_{\bar{C}_n}$ cover such a set. But for all $\bar{h} > h(T, \bar{P}_1 \vee B_1)$ and $\eta > 0$, if n is sufficiently large, then $2^{(2(n+2L_1)+1)^d (h(T, \bar{P}_1 \vee B_1) + \eta)} < 2^{(2n+1)^d (\bar{h} + \eta)}$, so that $h(T_1, \bar{P}_1 \vee B_1) \leq \bar{h}$, and hence

$$h(T_1, \bar{P}_1 \vee B_1) \leq h(T, \bar{P}_1 \vee B_1) < h(T) = h(T_1).$$

We now show how to continue the construction at this and every subsequent stage. At the ith stage, when we have constructed $(T_i, \bigvee_{j=1}^i (\bar{P}_j \vee B_j))$ so that $h(T_i, \bigvee_{j=1}^i (\bar{P}_j \vee B_j)) < h(T_i)$, we choose $\tilde{P}_{i+1}$ so that $|\tilde{P}_{i+1} - P_{i+1}| < \varepsilon_{i+1}$ and $h(T_i, \bigvee_{j=1}^i (\bar{P}_j \vee B_j) \vee \tilde{P}_{i+1}) < h(T_i)$, and then set $\bar{P}_{i+1} = \tilde{P}_{i+1} \vee \bar{P}_i$. We choose $\bar{\varepsilon}_{i+1} < \varepsilon_{i+1}$ and K_{i+1} so that there exists $E_{i+1} \subset \Omega$ with $\mu(E_{i+1}) > 1 - \bar{\varepsilon}_{i+1}$ and every $\omega \in E_{i+1}$ has a $(T_i, \bar{P}_{i+1})_{\bar{C}_{K_{i+1}}}$-name with $\bar{P}_{i+1}$-distribution within $\bar{\varepsilon}_{i+1}$ of dist $(\bar{P}_{i+1})$. We then choose L_{i+1} and using lemma 3, select a partition B_{i+1} so that (T_i, B_{i+1}) is an $(L_{i+1} - 2K_{i+1}, K_{i+1}, L_{i+1}, \bar{\varepsilon}_{i+1})$-blocked Bernoulli process such that $(T_i, B_{i+1})_{\mathbb{Z}^d} \perp (T_i, \bar{P}_{i+1} \vee \bigvee_{j=1}^i B_j)$ and $h(T_i, \bar{P}_{i+1} \vee \bigvee_{j=1}^{i+1} B_j) < h(T_i)$. We then form β^{i+1} and $T_{i+1} = T_i^{\beta_{i+1}^{-1}}$ as in the initial stage of the construction and argue that $h(T_{i+1}, \bar{P}_{i+1} \vee (\bigvee_{j=1}^{i+1} B_j)) < h(T_{i+1})$ and if $\bar{\varepsilon}_{i+1}$ and L_{i+1} are chosen appropriately, and $N_{i+1} = 3L_{i+1}$, then

$$(11) \qquad (\forall m > N_{i+1}) \qquad \bar{P}_{i+1} \perp^{\varepsilon_{i+1}} (T_{i+1}, \bar{P}_{i+1})_{\bar{C}_m \setminus \bar{C}_{N_{i+1}}}.$$

We must show, however, that we can preserve the approximate independence achieved at each stage.

In fact, at the jth stage, when we construct β_j and T_j, we make the construction so that

$$(12) \qquad (\forall i < j)(\forall m > N_i) \qquad \bar{P}_i \perp^{(9\varepsilon_i)^{\frac{1}{2}}} (T_j, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_i}}.$$

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We do this by requiring that for all k we build (T_k, B_{k+1}) so that

$$\mu(\{\omega \mid T_k^{\bar{C}_{N_k}}(\omega) \text{ is contained in a rigid block of } \beta_{k+1}\}) > 1 - \frac{\varepsilon_k}{2^k}.$$

Since the ε_k decrease and the N_k increase, this gives that for all $i < j$, if $G_{ij} = \{\omega \mid (\forall k \in [i+1, j])\ T_i^{\bar{C}_{N_i}}(\omega)$ is contained in a rigid block of $\beta_k\}$, then

$$\mu(G_{ij}) > \left(1 - \sum_{k=i+1}^{j} \frac{\varepsilon_i}{2^k}\right) > 1 - \varepsilon_i.$$

Now to establish (12), we fix $m > N_i$, and choose m' so that if $a \in R = (T_i, \bar{P}_i)_{\bar{C}_{m'}/\bar{C}_{N_i}} \vee \bigvee_{k=i}^{j-1} (T_k, B_{k+1})_{\bar{C}_m}$, and $a \subset G_{i,j}$, then there is a (unique) $\bar{p} \in (T_j, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_i}}$ containing a.

From (11) we know that there is a collection of atoms $\mathscr{C} \subset (T_i, \bar{P}_i)_{\bar{C}_{m'} \setminus \bar{C}_{N_i}}$ with $\mu(\bigcup \mathscr{C}) > 1 - \varepsilon_i$ such that for all $c \in \mathscr{C}$, $|\text{dist}_c \bar{P}_i - \text{dist}\, \bar{P}_i| < \varepsilon_i$. Since $(T_i, \bar{P}_i)_{\bar{C}_{m'}} \perp \bigvee_{k=i}^{j-1} (T_k, B_{k+1})_{\bar{C}_{m'}}$, we have for all $c \in \mathscr{C}$,

$$\mu(\bigcup \{a \in R \mid a \subset (c \cap G_{ij})\}) > (1 - \varepsilon_i)\mu(c),$$

and for each such $a \in R$,

$$|\text{dist}_a\, \bar{P}_i - \text{dist}\, \bar{P}_i| = |\text{dist}_c\, \bar{P}_i - \text{dist}\, \bar{P}_i| < \varepsilon_i.$$

Thus,

$$\mu(\bigcup \{a \in R \mid a \subset G_{i,j} \text{ and } |\text{dist}_a\, \bar{P}_i - \text{dist}\, \bar{P}_i| < \varepsilon_i\}) > (1 - \varepsilon_i)^2 > 1 - 3\varepsilon_i$$

(providing $\varepsilon_i^2 < \varepsilon_i$). Let Q denote the partition consisting of the elements of R contained in G_{ij} and the complement of their union, and let $\bar{Q} = Q \vee (T_j, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_i}}$. Then $\bar{Q} \supset (T_j, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_i}}$ and $\bar{P}_i \perp^{3\varepsilon_i} \bar{Q}$, so $\bar{P}_i \perp^{(9\varepsilon_i)^{\frac{1}{2}}} (T_j, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_i}}$, as desired.

Now, as in the proof of theorem 1, if the parameters J_i are chosen to grow rapidly enough, then the cocycles $\alpha_i = \beta_i \circ \beta_{i-1} \circ \cdots \circ \beta_1$ converge to a cocycle α that satisfies the weak a-equivalence and strong b-equivalence properties for all $a \geq 1$ and $b \in (0, 1)$. To verify that $S = T^{\alpha^{-1}}$ is a K system we first note that condition (12) holds with S in place of T_j. Now we can show that for all i, $(S, \bar{P}_i)$ is symmetrically K-mixing at $\bar{C}_0$. For given $\varepsilon > 0$, we can choose $j > i$ and N_j so that $\varepsilon_j < \varepsilon$ and for all $m > N_j$, $\bar{P}_j \perp^{\varepsilon_j} (S, \bar{P}_j)_{\bar{C}_m \setminus \bar{C}_{N_j}}$. If ε_j is sufficiently small (say $\varepsilon_j < \varepsilon^8/3^7$) then $\bar{P}_i \perp (S, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_j}}$. The theorem now follows from lemma 2. □

We now proceed to show that in dimensions $d \geq 2$, we can construct bounded and strong b-equivalences ($b \geq 1$) of ergodic $\mathbb{Z}^d$-actions to achieve mixing and K-mixing. More precisely, we prove the following.

THEOREM 3. *Let T be an ergodic $\mathbb{Z}^d$-action $(d \geq 2)$ on $(\Omega, \mathscr{B}, \mu)$. Then there is a cocycle α for T giving a mixing action $S = T^{\alpha^{-1}}$ such that the orbit equivalence (T, S) is, for all $b \geq 1$, a strong b-equivalence, and both (T, S) and (S, T) are bounded orbit equivalences.*

THEOREM 4. *Let T be as above with $h(T) > 0$. Then α may be chosen so that in addition to the above, $S = T^{\alpha^{-1}}$ is a K-system.*

Because of the similarity of the proofs of these theorems to those of theorems 1 and 2, we will confine ourselves largely to pointing out the novel aspects of these arguments.

Given T, we will construct α as before, as a limit of cocycles α_i of T, each of which gives a $\mathbb{Z}^d$-action $T_i = T^{\alpha_i^{-1}}$. Each α_{i+1} will be obtained as $\beta_{i+1} \circ \alpha_i$, where β_{i+1} is a cocycle for T_i. The construction of β_{i+1} will also be of the same character as before. Namely, we will choose a Rokhlin tower $\tau = T_i^{\bar{C}_t} B$ for T_i with base B and choose a measurable function π mapping B into a set of permutations of $\bar{C}_t$. We then let $g : \Omega \to \mathbb{Z}^d$ be given by

$$g(\omega) = \begin{cases} \pi(T_i^{-u}(\omega))(u) - u, & \text{if } \omega \in T_i^u B \text{ and } u \in \bar{C}_t \\ 0, & \text{if } \omega \notin \tau \end{cases} \tag{13}$$

and set

$$\beta_{i+1}(\omega, v) = v + g(T_i^v(\omega)) - g(\omega). \tag{14}$$

(Informally, we permute each orbit block $T_i^{\bar{C}_t}(\omega)$, where $\omega \in B$, by $\pi(\omega)$.)

The main difference between these arguments and the previous ones is in the choice of the permutations assigned to the orbit blocks by the function π. Here we use a somewhat more complicated family of permutations, not available in one dimension. In order to simplify our discussion, we will describe these permutations and present our arguments in dimension two and then indicate briefly how one may extend these ideas to higher dimensions.

Let k and r be integers with $r > 2|k|$. We define a permutation $\pi_{k,r}$ of $\bar{C}_r$ as follows. Let $S_l = \{v \in \mathbb{Z}^2 \mid \|v\|_\infty = l\}$. Then if $k \geq 0$, $\pi_{k,r}$ leaves each $S_l \subset \bar{C}_r$ invariant, rotating the points of S_l counterclockwise by

$$\begin{cases} l \text{ units}, & \text{if } 0 \leq l \leq k \\ k \text{ units}, & \text{if } k \leq l \leq r-k \\ r-l \text{ units}, & \text{if } r-k \leq l \leq r \end{cases}$$

For $k < 0$, we let $\pi_{k,r}$ be the inverse of $\pi_{-k,r}$. We refer to $\pi_{k,r}$ as a ***k-twist*** of $\bar{C}_r$. (See figure 3.) Similarly, by a k-twist of a translate $\bar{C}_r + v$ of $\bar{C}_r$, we mean a

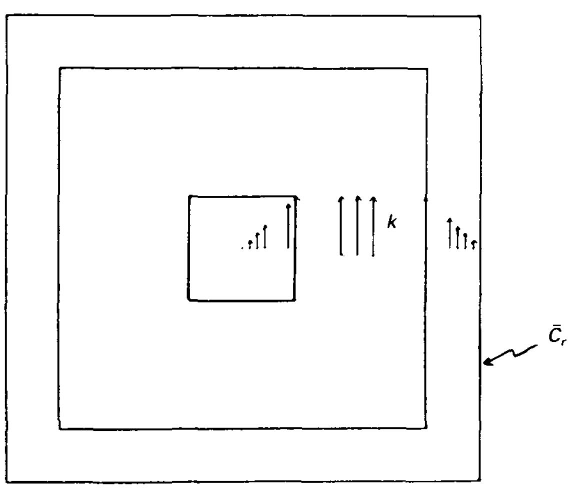

FIGURE 3. A k-twist of $\bar{C}_r$.

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permutation of the form $\pi_v \pi_{k,r} \pi_{-v}$, where π_v denotes translation by v. If π is a k-twist of $\bar{C}_r + v$, then we refer to the four sets $\{w \in \bar{C}_r + v \mid \pi(w) - w = (k, 0)$ (respectively, $(0, k), (-k, 0), (0, -k))\}$ as the *rigid sectors* of π.

We now concatenate k-twists on blocks congruent to $\bar{C}_r$ to produce permutations of large blocks. Specifically, we choose $K > 0$, $r > 2K$ and $s > 0$, and let

$$\bar{C}_{\bar{t}} = \bigcup_{v \in (2r+1)\bar{C}_s} (\bar{C}_r + v)$$

(so that $\bar{t} = \frac{1}{2}[(2r+1)(2s+1)-1]$). We now construct a permutation σ of $\bar{C}_{\bar{t}}$ by choosing, for each $v \in (2r+1)\bar{C}_s$, an integer k_r, $|k_r| \le K$, and applying a k_r-twist to $\bar{C}_r + v$. Similarly, we construct a permutation ρ of $\bar{C}_{\bar{t}}$ by choosing, for each $v \in (2r+1)\bar{C}_s$ such that $\bar{C}_r + v + (r, r) \subset \bar{C}_{\bar{t}}$, an integer l_v, $|l_v| \le K$, and applying an l_v-twist to $\bar{C}_r + v + (r, r)$, and by setting ρ equal to the identity elsewhere. We now form the composition $\rho \circ \sigma$, and extend this to a permutation π of $\bar{C}_t$, where $t = t(r, s) = \bar{t} + 10$, by setting π equal to the identity on $\bar{C}_t \setminus \bar{C}_{\bar{t}}$. We call such a permutation π a (K, r, s)*-twist* on $\bar{C}_t$, and let $\Pi_{K,r,s}$ denote the set of all such. (See figures 4(a), 4(b).) We refer to (K, r, s)-twists of a translate $\bar{C}_t + v$ of $\bar{C}_t$, with the obvious meaning, and identify the set of such with $\Pi_{K,r,s}$. By a rigid sector of $\pi \in \Pi_{K,r,s}$, we mean the intersection of rigid sectors of component k_v-twists and l_v-twists of π. By a rigid (K, r, s)-sector of $\bar{C}_t$ $(t = t(r, s))$, we mean an intersection, over all $\pi \in \Pi_{K,r,s}$, of rigid sectors of π. Finally, given a $\mathbb{Z}^2$-action T, integers K, r, s, and $t = t(r, s)$, and a Rohklin tower $\tau = T^{\bar{C}_t}B$ with base B, then by a rigid (K, r, s)-sector of τ we mean a set of the form $T^A(\omega)$, where $\omega \in B$ and A is a rigid (K, r, s)-sector of $\bar{C}_t$.

The (K, r, s)-twists will play the role in the present constructions that the (J, K, L)-permutations played in the previous arguments. In particular, in the above description of the cocycles β_{i+1} in our construction, the function π will take values in $\Pi_{K,r,s}$, for some choice of K, r, and s. We will refer to a cocycle so constructed as a (K, r, s)-twist cocycle.

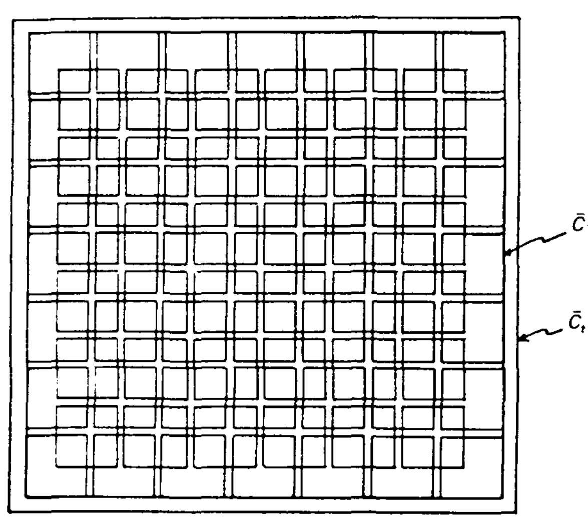

FIGURE 4(a). Schematic diagram of a (K, r, s)-twist on $\bar{C}_t$.

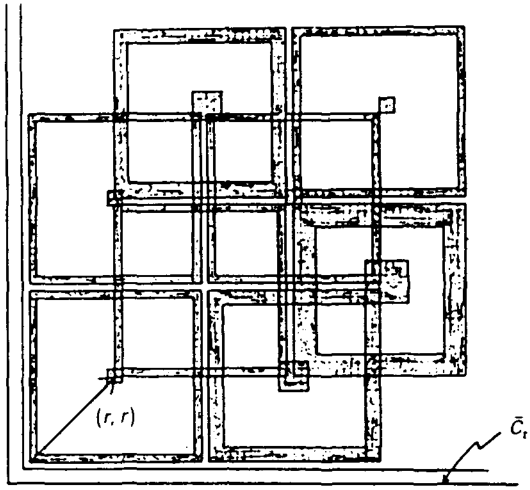

FIGURE 4(b). Detail of a (K, r, s)-twist on $\bar{C}_t$.

It will be convenient in implementing our construction to use sequences of Rokhlin towers that are nested in a special way. Given a $\mathbb{Z}^2$-action T and integers K_i, r_i, s_i, $t_i = t(r_i, s_i)$ and a sequence of Rokhlin towers $\tau_i = T^{\bar{C}_{t_i}}(B_i)$ in bases B_i, we say that these towers are well-nested (with respect to the parameters K_i, r_i, and s_i) if for all $i<j$ and all $\omega \in B_i$, $T^{\bar{C}_{t_i}}(\omega)$ is contained in a rigid (K_j, r_j, s_j)-sector of τ_j or in τ_j^c.

Given a well-nested sequence of towers τ_i for T as above, and measurable functions $\pi_i : B_i \to \Pi_{K_i, r_i, s_i}$, we obtain cocycles β_i for T. Note, however, that we may equally well regard β_i as a cocycle for $T_{i-1} = T^{\alpha_{i-1}^{-1}}$, where $\alpha_{i-1} = \beta_{i-1} \circ \cdots \circ \beta_1$. Indeed, B_i is the base of the Rokhlin tower $\hat{\tau}_i$ for T_{i-1} which coincides with τ_i, so that π_i determines a cocycle $\hat{\beta}_i$ for T_{i-1}. In fact, if g_i is the solution (13) to the coboundary equation (14) for β_i and T, and $\hat{g}_i$ is the corresponding function for $\hat{\beta}_i$ and T_{i-1}, then $g_i = \hat{g}_i$, as a result of the well-nested property. Thus, in our construction we can refer without confusion to twist cocycles arising from well-nested towers, without specifying whether the towers and cocycles are to be associated with T or with the constructed actions T_i.

The following is the counterpart to lemma 1 in the present setting.

LEMMA 5. *There is a sequence of functions $\{f_i : N^i \to N\}_{i=1}^{\infty}$ such that if $\{K_i\}_{i=1}^{\infty}$ is a sequence of positive integers and if $\{\alpha_i\}_{i=0}^{\infty}$ is a sequence of cocycles for a $\mathbb{Z}^d$-action T_0 on $(\Omega, \mathcal{B}, \mu)$ (with $\alpha_0 =$ identity) such that the cocycles $\beta_i = \alpha_i \circ \alpha_{i-1}^{-1}$ of the action $T_{i-1} = T_0^{\alpha_{i-1}^{-1}}$ are (K_i, r_i, s_i)-twist cocycles, whose corresponding towers τ_i are well-nested and, for each i,* $\min\{r_i/K_i, s_i/r_i\} > f_i(K_1, K_2, \ldots, K_i)$, *then the cocycles α_i converge pointwise to a cocycle α which gives a $\mathbb{Z}^d$-action $S = T^{\alpha^{-1}}$, such that (T, S) is, for all $b \geq 1$, a strong b-equivalence, and both (T, S) and (S, T) are bounded orbit equivalences.*

Proof. Fix a summable sequence $\{\varepsilon_i > 0\}_{i=1}^{\infty}$. To ensure that the α_i converge to a (bijective) cocycle, we let $M_i = i + 2\sum_{j=1}^{i} K_j$ and choose $f_i(K_1, K_2, \ldots, K_i)$ so large that if $\min\{r_i/K_i, s_i/r_i\} > f_i(K_1, K_2, \ldots, K_i)$ and we set

$$F_i = \{\omega \mid T_{i-1}^{\bar{C}_{M_{i-1}}}(\omega) \text{ is contained in a rigid sector of } \tau_i \text{ or in } \tau_i^c\}_\infty$$

then $\mu(F_i) > 1 - \varepsilon_i$, so that $\mu(\bigcup_{j=1}^\infty \bigcap_{i=j}^\infty F_i) = 1$. We then argue as in the proof of lemma 1.

In order to show that α and α^{-1} will be bounded, we need the following facts about twist permutations, which the reader can verify. Let π be a (K, r, s)-twist of $\bar{C}_t$. Then for all $v, w \in \bar{C}_t$, $\|v - w\| = 1$ implies $\|\pi(v) - \pi(w)\| < 10$, and $\|v - w\| \geq 10$ implies $\|\pi(v) - \pi(w)\| > 1$. Now suppose that $\|v\| = 1$, and observe that for all ω, there is at most one i such that $\alpha_{i+1}(\omega, v) \neq \alpha_i(\omega, v)$. Indeed, given ω, if i is the first integer such that $\alpha_{i+1}(\omega, v) \neq \alpha_i(\omega, v)$, then ω and $T_i^v(\omega)$ $(= T^v(\omega))$ are contained in an orbit block $T_i^{\bar{C}_{t_{i+1}}}(\omega')$, $\omega' \in B_{i+1}$, and since the towers are well-nested, $\alpha_j(\omega, v)$ remains constant for $j \geq i+1$. Hence, by the above remarks, $\|\alpha(\omega, v)\| = \|\alpha_{i+1}(\omega, v)\| = \|\beta_{i+1}(\omega, v)\| < 10$, so that α is bounded.

To argue that α^{-1} is bounded as well, suppose that $\|v\| = 1$ and $\alpha(\omega, w) = v$. Then we claim that $\|w\| < 10$. In fact, for all i, $\|\alpha_i(\omega, v)\| < 10$. For if, for some i, $\|\alpha_i(\omega, v)\| \geq 10$, then one of two cases obtains. Either ω and $T_i^w(\omega)$ are not in a single orbit block of τ_{i+1}, in which case $\|\alpha_{i+1}(\omega, v)\| \geq 10$ (using the fact that (K, r, s)-twists leave a boundary of width 10 fixed), or ω and $T_i^w(\omega)$ are in a single orbit block of τ_{i+1}, in which case $\|\alpha_{i+1}(\omega, v)\| > 1$, and since the towers are well-nested, $\alpha_j(\omega, v) = \alpha_{i+1}(\omega, v)$ for all $j > i+1$. Thus, $\|\alpha(\omega, w)\| = \lim_{j\to\infty} \|\alpha_j(\omega, w)\| > 1$, a contradiction.

Finally, we show that we can ensure that α^{-1} gives a strong b-equivalence, for all $b \geq 1$. (This is equivalent to showing that α does so.) For each i, choose $f_i(K_i \cdots K_i)$ so large that if $\min\{r_i/K_i, s_i/r_i\} > f_i(K_1, \ldots, K_i)$ then there is an increasing sequence of positive integers N_i such that $(2\sum_{j=1}^i K_j)^i / 2N_i < \varepsilon_i$, and if

$$G_i = \{\omega \mid T_{i-1}^{\bar{C}_{N_i}}(\omega) \text{ is contained in a rigid sector of } \tau_i \text{ or } \tau_i^c\},$$

then $\mu(G_i) > 1 - \varepsilon_i$, so that $\mu(\bigcup_{j=1}^\infty \bigcap_{i=j}^\infty G_i) = 1$.

Then we can argue that for all $\omega \in \bigcup_{j=1}^\infty \bigcap_{i=j}^\infty G_i$ and all $b \geq 1$,

$$\lim_{v\to\infty} \|\alpha^{-1}(\omega, v) - v\|^b / \|v\| = 0.$$

For suppose $\omega \in \bigcap_{i=j}^\infty G_i$ and $b \geq 1$ is given. If $k \geq j$ and $b_k \geq b$, then for all $v \in \bar{C}_{N_{k+1}} \setminus \bar{C}_{N_k}$,

$$\frac{\|\alpha^{-1}(\omega, v) - v\|^b}{\|v\|} = \frac{\|\alpha_k^{-1}(\omega, v) - v\|^b}{\|v\|} \leq \frac{(2\sum_{l=1}^k K_l)^{b_k}}{2N_k} < \varepsilon_k,$$

which establishes the desired limit. □

The following is the basic fact about (K, r, s)-twists that is required to prove theorems 3 and 4. We leave the proof to the reader.

LEMMA 6. *Fix $\varepsilon > 0$ and K. If r/K and s/r are sufficiently large, and $t = t(r, s)$, then there exists $A \subset \bar{C}_t$ with $|A|/|\bar{C}_t| > 1 - \varepsilon$ such that for all $v \in A$ and all $f: \bar{C}_t \to P$, a finite set,*

$$\frac{1}{|\Pi_{K,r,s}|} \sum_{\pi \in \Pi_{K,r,s}} (\operatorname{dist}(f \circ \pi^{-1})|_v) = \frac{1}{|\bar{C}_K|} \sum_{w \in \bar{C}_K} (\operatorname{dist} f|_{v+w}).$$

In other words, by permuting f by the elements of $\Pi_{K,v,s}$, and weighting each $f\circ\pi^{-1}$ equally, we obtain a distribution on $P^{\bar{C}_t}$ whose marginal distribution on most coordinates v coincides with the distribution of f itself, restricted to the coordinates $v+\bar{C}_K$.

Proof of theorem 3. The proof is essentially the same as that of theorem 1, with the exception that we must now use well-nested towers to build our sequence of cocycles. Given T on $(\Omega, \mathcal{B}, \mu)$, we choose finite partitions $P_i\uparrow\mathcal{B}$, $\{\varepsilon_i\}_{i=1}^{\infty}\downarrow 0$, and sequences $\{K_i\}_{i=1}^{\infty}$, $\{r_i\}_{i=1}^{\infty}$, and $\{s_i\}_{i=1}^{\infty}$ increasing as required by lemma 5. The K_i are chosen so that (for suitable $\bar{\varepsilon}_i<\varepsilon_i$) there exists $E_i\subset\Omega$ with $\mu(E_i)>1-\bar{\varepsilon}_i$ and for all $\omega\in E_i$ the distribution of P_i on $T^{\bar{C}_{K_i}}(\omega)$ is within $\bar{\varepsilon}_i$ of dist(P_i). The K_i are also chosen to increase so rapidly that if $T_i=T^{\alpha_i^{-1}}$ and for all $v\in\mathbb{Z}^2$, $\omega\in\Omega$, $\|\alpha_i(\omega, v)-v\|<M_i$ then for all $\omega\in E_{i+1}$ the distribution of P_{i+1} on $T_i^{\bar{C}_{K_{i+1}}}(w)$ is within $2\bar{\varepsilon}_{i+1}$ of dist (P_{i+1}). (It would be sufficient that $2\sum_{j=1}^{i}K_j/K_{i+1}<\bar{\varepsilon}_{i+1}/4$). Finally, we choose a sequence $\{L_i\}_i\uparrow\infty$ and let $\bar{C}_{u_i}=\bigcup_{v\in(2t_i+1)\bar{C}_{L_i}}(\bar{C}_{t_i}+v)$, and we build a sequence of towers $\tau_i=T^{\bar{C}_{u_i}}(B_i)$ with bases B_i and $\mu(\tau_i)>1-\bar{\varepsilon}_i$.

Note that each τ_i can be regarded as a $\bar{C}_{t_i}$-tower τ_i' by taking as the base $B_i'=\bigcup_{v\in(2t_i+1)\bar{C}_{L_i}}T^vB_i$. We can make the τ_i' well-nested by simply deleting every orbit block $T^{\bar{C}_{t_i}}(\omega')$, $\omega'\in B_i'$, which for some $j>i$ is not contained in a rigid sector of τ_j' and is not contained in $(\tau_j')^c$. If the $\{r_i/K_i\}_{i=1}^{\infty}$ and $\{s_i/r_i\}_{i=1}^{\infty}$ grow sufficiently rapidly, this entails deleting less than $\bar{\varepsilon}_i$ of the measure of τ_i'. Let τ_i'' denote the $\bar{C}_{t_i}$-tower that remains and B_i'' its base.

We now construct the permutation-valued function π_i'' needed to specify the cocycles of our construction. They will be defined so as to satisfy certain inductive conditions with respect to an auxiliary sequence of partitions $\{R_i\}_{i=0}^{\infty}$. Specifically, we let R_0 be the trivial partition $(\Omega, \varnothing)$, and given $R_0, R_1\cdots R_{i-1}$, we define a function $\bar{\pi}_i$ mapping B_i into permutations of $\bar{C}_{u_i}$ in such a way that for distinct v and w in $(2t_i+1)\bar{C}_{L_i}$, the maps $\omega\mapsto\bar{\pi}_i(\omega)|_{\bar{C}_{t_i}+v}$ and $\omega\mapsto\bar{\pi}_i(\omega)|_{\bar{C}_{t_i}+w}$ and the partition $(T, P_i\vee\bigvee_{j=0}^{i-1}R_j)_{\bar{C}_{u_i}}|_{B_i}$ are jointly independent, and the above maps take their values in and are uniformly distributed on Π_{K_i,r_i,s_i}. We then modify $\bar{\pi}_i$ to π_i by setting $\pi_i(\omega)|_{\bar{C}_{t_i}+v}=\text{id}$ when $T^{\bar{C}_{t_i}+v}(\omega)$ is a deleted block. R_i is defined to be the partition obtained by labelling the points of an orbit block $T^{\bar{C}_{u_i}}(\omega)$, $\omega\in B_i$, by $\pi_i(\omega)$, and labelling τ_i^c by a fixed symbol. Each π_i then gives a Π_{K_i,r_i,s_i}-valued function π_i'' on B_i'' by setting $\pi_i''(\omega'')=\pi_i(\omega)|_{\bar{C}_{t_i}+v}$, when $\omega\in B_i$ and $\omega''=T^v(\omega)\in B_i''$. We then argue as in the proof of theorem 1 that if the sequences $\{L_i\}_{i=1}^{\infty}$, $\{r_i/K_i\}_{i=1}^{\infty}$, $\{s_i/r_i\}_{i=1}^{\infty}$ and $\{1/\bar{\varepsilon}_i\}_{i=1}^{\infty}$ grow sufficiently rapidly (as required by lemma 6) the (K_i, r_i, s_i)-twist cocycles β_i determined by τ_i'' and π_i'' give cocycles $\alpha_i=\beta_i\circ\cdots\circ\beta_1$ converging to α so that the orbit equivalence between T and $S=T^{\alpha^{-1}}$ is of the desired type and for all $j\le i$, (S, P_j) is $\varepsilon_j/2^{i-j}$-mixing between $3t_i$ and $3t_{i+1}$, so that S is mixing. □

Proof of theorem 4. We begin by devising the family of Bernoulli processes we use to direct our construction. Let K, r, s and $t=t(r, s)$ be given (with $r>2K$) and let (T, B_1) be a (two-dimensional) blocked Bernoulli process, where the parameter s_1 of (T, B_1) equals t. Let $\bar{B}_1$ denote the two-set partition which distinguishes blocks of (T, B_1) from non-blocks. Thus, (T, B_1) can be viewed as the extension of $(T, \bar{B}_1)$

obtained by labelling each block of $(T, \bar{B}_1)$ by an element of $\bar{C}_{r_1}$ in such a way that for every finite $(T, \bar{B}_1)$-name a, all the names that can be obtained from a by labelling its blocks as above are equiprobable. We wish to consider the extension (T, B) of $(T, \bar{B}_1)$ obtained by labelling $(T, \bar{B}_1)$-blocks in this manner by elements of $\Pi_{K,r,s}$ rather than by elements of $\bar{C}_{r_1}$. As before, such a process can be shown to be Bernoulli by a nesting argument, and we refer to it as a (K, r, s)-*Bernoulli process.* We note that for all $\varepsilon > 0$ and all $M > 0$ there is a (K, r, s)-Bernoulli process with entropy less than ε and K, r/K, and s/r greater than M.

Now given T of positive entropy on $(\Omega, \mathscr{B}, \mu)$ we choose finite partitions $P_i \uparrow \mathscr{B}$ and $\varepsilon_i \downarrow 0$ and proceed as in the proof of theorem 2 to obtain partitions $\{\bar{P}_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ such that for all i, $|\bar{P}_i - P_i| < \varepsilon_i$, $\bar{P}_{i+1} \supset \bar{P}_i$, $(T, \bar{P}_i) \perp (T, B_i)$ and (T, B_i) is a (K_i, r_i, s_i)-Bernoulli process. Let $b_i = \{\omega \mid T^{\bar{C}_{t_i}}(\omega)$ forms a block of $(T, B_i)\}$, and let $\tau_i = T^{\bar{C}_{t_i}}(b_i)$ denote the $\bar{C}_{t_i}$-Rokhlin tower for T with base b_i. The τ_i are not well-nested. but we can make them so by simply deleting from each τ_i the sets $T^{\bar{C}_{t_i}}(\omega)$ where $\omega \in b_i$ and for some $j > i$, $T^{\bar{C}_{t_i}}(\omega)$ is not contained in a rigid sector of τ_j and $T^{\bar{C}_{t_i}}(\omega)$ is not contained in τ_j^c. We let $\bar{\tau}_i = T^{\bar{C}_{t_i}}(\bar{b}_i)$ denote the tower with base $\bar{b}_i$ that remains after this deletion, and we let $\bar{B}_i$ denote the partition obtained from B_i by including the deleted set in the set that formerly consisted of τ_i^c. Let $\pi_i : b_i \to \Pi_{K_i, r_i, s_i}$ be given by $\pi_i(\omega) = \bar{B}_i(\omega)$. We are free to do all of the above in such a way that the parameter sequences $\{K_i\}_{i=1}^{\infty}\{r_i/K_i\}_{i=1}^{\infty}$ and $\{s_i/r_i\}_{i=1}^{\infty}$ increase as fast as we wish.

We must show that if the parameters increase fast enough, then the cocycles β_i determined by the towers τ_i and the functions π_i give cocycles $\alpha_i = \beta_i \circ \cdots \circ \beta_1$ converging to a cocycle α for T of the desired type. We will argue, in fact, that the parameters can be chosen so that (letting $T_j = T^{\alpha_j^{-1}}$)

$$(15) \qquad (\forall i)(\exists N_i)(\forall j \geq i)(\forall m > N_i) \qquad \bar{P}_i \perp^{\varepsilon_i} (T_j, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_i}}.$$

Once this is established, the proof can be completed exactly as was the proof of theorem 2.

In the proof theorem 2, this was accomplished in two stages, first showing how to get ε_i-independence for $(T_i, \bar{P}_i)$, and second, showing how to preserve this behaviour for subsequent T_j. Here we will do this in one step. Additional care must be taken here since we no longer have all the independence properties of the factors $(T, \bar{B}_i)$ that specify the orbit changes.

Fix $i \leq j$, and let $N_i = 10t_i$. Fix $m > N_i$. The outline of our argument to establish (15) is similar to that of theorem 2 in that we produce a partition refining $(T_j, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_i}}$ that is $\bar{\varepsilon}_i$-independent of $\bar{P}_i$, for sufficiently small $\bar{\varepsilon}_i$ to imply (15).

To do this, assume the parameters have been chosen so that $3\sum_{k=1}^{i-1} t_k < t_i$ and let $\hat{N}_i = N_i - 3\sum_{k=1}^{i-1} t_k$. Let $\hat{m} > m + 3\sum_{k=1}^{j} t_k$. Let $Q = (T, \bar{P}_i)_{\bar{C}_{\hat{m}} \setminus \bar{C}_{\hat{N}_i}} \vee \bigvee_{k=1}^{j} (T, \bar{B}_k)$. Write $q_1 \sim q_2$ (for $q_i \in Q$) if either (i) there exists $v \in \bar{C}_{t_i}$ such that for all $\omega, \omega' \in q_1 \cup q_2$ the points $T^{\bar{C}_{t_i}+v}(\omega)$ form a $(T, \bar{B}_i)$-block and the points $T^{\bar{C}_{10K_i}}(\omega)$ are contained in a rigid sector of that block, and the Q-names of ω and ω' agree in every respect except possibly for the $\bar{B}_i$-name of the above block, and $T^{\bar{C}_{N_i}+v}(\omega)$ is contained in

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a rigid sector of a $(T, \bar{B}_{i+1})$-block, or (ii) q_1 and q_2 are not in any of the equivalence classes described in (i).

Let $\bar{Q}$ denote the partition consisting of the unions of these equivalence classes. Note that we can make the atom q_0 of $\bar{Q}$ corresponding to condition (ii) arbitrarily small by choosing the parameters r_k/K_k, s_k/r_k and $1/\mu(\bar{\tau}_k^c)$, $k = i, i+1, \ldots, j$, sufficiently large.

For each k, let g_k denote the solution to the coboundary equation (14) with respect to β_k and T such that $g_k(\omega) \equiv 0$ on $\bar{\tau}_k^c$. For $u \in \bar{C}_{K_i}$, $q \in \bar{Q}$, $q \neq q_0$, let

$$a(u, q) = \{\omega \mid g_i(\omega) = u, \text{ and } T^{\sum_{k=1}^{i} g_k(\omega)}(\omega) \in q\}.$$

(Note that $\sum_{k=1}^{i} g_k$ is, in fact, constant across $a(u, q)$.) Write $a(u, q) \sim a'(u', q')$ if $q = q'$. Let $[a]$ denote the equivalence class (so defined) of such a set a.

We make the following observations about these equivalence classes. First, if r_i/K_i, s_i/r_i and $1/\mu(\bar{\tau}_1^c)$ are sufficiently large, then the subset of Ω not contained in the union of the above equivalence classes is arbitrarily small. Second, if $a \sim a'$ then $\mu(a) = \mu(a')$. Third, if $a \sim a'$ and v (resp. v') $= \sum_{k=1}^{i} g_k(\omega)$, for all $\omega \in a$ (resp. a') and $R = (T, \bar{P}_i)_{\bar{C}_{K_i}}$, then $\text{dist}_a T^{-v}R = \text{dist}_{a'} T^{-v'}R$. (Denote this common distribution by $\delta([a])$). To verify the second and third observations, we use the fact that, conditioning on the set of points with a $(T, \bar{B}_i)$-block at specified coordinates, the $\bar{B}_i$-name of that block is independent of the process $(T, \bar{P}_i \vee \bigvee_{k \neq i} \bar{B}_k)$ and of the process $(T, \bar{B}_i)$ outside that block. Fourth, the sets $a(u, q)$, and hence the equivalence classes $[a]$, are pairwise disjoint. Fifth, if $a \sim a'$, then a and a' are contained in the same atom of $(T_j, \bar{P}_i)_{\bar{C}_m \setminus \bar{C}_{N_i}}$.

Now suppose that K_i was chosen so that $(\sum_{k=1}^{i-1} t_k)/K_i$ is very small compared to ε_i, and so that for all ω outside a set of measure much less than ε_i, the distribution of $\bar{P}_i$ on $T^{\bar{C}_{K_i}}(\omega)$ differs from dist$(\bar{P}_i)$ by much less than ε_i.

Fix an equivalence class $[a(u, q)]$. Then there exists $w \in \bar{C}_{t_i}$ and there exists a permutation π of $\bar{C}_{t_i}$ such that for all $a' \in [a]$ and all $\omega' \in a'$, if $v' = \sum_{k=1}^{i} g_k(\omega')$, then $T^{v' + \bar{C}_{t_i} - w}(\omega')$ is a $(T, \bar{B}_i)$-block, and the cocycle $\alpha_{i-1} = \beta_{i-1} \circ \cdots \circ \beta_1$ permutes the points of this block by π. Observe that for all $\omega \in \bigcup [a]$, $T^{\sum_{k=1}^{i} g_k(\omega)}(\omega)$ occupies position w in its $(T, \bar{B}_i)$-block and $\pi^{-1}(w - u')$ is the position each point of $a(u', q) \in [a]$ occupies in the $(T, \bar{B}_i)$-block containing it. Furthermore, the values $\pi^{-1}(w - u')$ are pairwise distinct, for distinct u', and are contained in $w + \bar{C}_{K_i} + \sum_{k=1}^{i-1} t_k$.

Thus, since the random variable u on $\bigcup [a](u\colon \omega \mapsto u'$ if $\omega \in a(u', q))$ is uniformly distributed on $\bar{C}_{K_1}$, the random variable $w - \sum_{k=1}^{i} g_k(\omega)$, which equals $\pi^{-1}(w - u')$ for $\omega \in a(u', q)$, is nearly uniformly distributed on $\bar{C}_{K_1}$. Consequently, the distribution of $\bar{P}_i$ on $\bigcup [a]$ is nearly the average of the marginal distributions of $\delta([a])$ at each of its coordinates. For most $[a]$, this average nearly equals dist$(\bar{P}_i)$, and this establishes (15). □

In order to prove theorems 3 and 4 in dimension $d > 2$, we only have to exhibit a suitable family of permutations to play the role of the (K, r, s)-twists.

Fix $r > 2|k|$ and $1 \leq i < j \leq d$. By a *$k_{i,j}$-twist of $\bar{C}_r \subset \mathbb{Z}^d$*, we mean a permutation obtained by choosing a two-dimensional k-twist π and applying π to each of the

https://doi.org/10.1017/S0143385700003667 Published online by Cambridge University Press

two-dimensional cubes of radius r in $\bar{C}_r$ parallel to the i, j-plane. Similarly, we define a $k_{i,j}$-twist of a translate $\bar{C}_r + v$ of $\bar{C}_r$.

Given $r > 2K > 0$, $s > 0$, $t = t(r, s)$, and $1 \leq i < j \leq d$, we define a *$(K, r, s)_{i,j}$-twist of $\bar{C}_t$* to be a permutation obtained by applying a $k_{i,j}$-twist to each block $\bar{C}_r + v$, $v \in (2r+1)\bar{C}_s$, the choice of k varying with v, and then applying a $k_{i,j}$-twist to each $\bar{C}_r + v + (r, \ldots, r)$ contained in $\bar{C}_{\bar{t}} = \bigcup_{v \in (2r+1)\bar{C}_s} (\bar{C}_r + v)$, again, k varying with v, and finally, fixing the points $w \in \bar{C}_t$ such that $\|w\|_\infty > t - 10$. We then let $\Pi^{i,j}_{K,r,s}$ denote the set of $(K, r, s)_{i,j}$-twists, and $\Pi_{K,r,s} = \bigcup_{1 \leq i < j \leq d} \Pi^{i,j}_{K,r,s}$. The reader can verify that the families $\Pi_{K,r,s}$ satisfy the key property needed to prove theorems 3 and 4, namely the counterpart to lemma 6. Given this, all the arguments go through as before.

REFERENCES

[1] A. del Junco & D. J. Rudolph. Kakutani equivalence of ergodic $\mathbb{Z}^n$-actions. *Ergod. Th. & Dynam. Sys.*

[2] H. A. Dye. On groups of measure-preserving transformations, I., *Amer. J. Math.* **81** (1959).

[3] N. A. Friedman & D. S. Ornstein. Ergodic transformations induce mixing transformations. *Adv. in Math.* **10** (1973) 147–163.

[4] D.S. Ornstein. Factors of Bernoulli shifts, *Israel J. Math.* **21** (1975), 145–153.

[5] D. S. Ornstein & M. Smorodinsky. Ergodic flows of positive entropy can be time changed to become K-flows. *Israel J. Math.* **26** (1977) 75–83.

[6] D. J. Rudolph. Restricted orbit equivalence, *Mem. Amer. Math. Soc.* **54** (1985), No. 323.

[7] J.-P. Thouvenot. Quelques propriétés des systèmes dynamiques qui se décomposent en un produit de deux systèmes dont l'un est un schème de Bernoulli. *Israel J. Math.* **21** (1975), 177–203.

[8] B. Weiss, General theory of $\mathbb{Z}^d$-actions. Unpublished notes.