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ON EVALUATION SUBGROUPS OF GENERALIZED HOMOTOPY GROUPS

BY
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ABSTRACT. $G(A, X)$ consists of all homotopy classes of cyclic maps from a space A to another space X . If A is an H-cogroup, then $G(A, X)$ is a group. $G(A, X)$ preserves products in the second variable and is a contravariant functor of A from the full subcategory of H-cogroups and maps into the category of abelian groups and homomorphisms. If X is an H-cogroup, then $G(X, X)$ is a ring.

1. **Introduction.** This work is a continuation of the study of the evaluation subgroups developed by Gottlieb [4] and [5] and Varadarajan [12], among others. Gottlieb studied the evaluation subgroups $G(X)$ and $G_n(X)$ extensively. Varadarajan generalized $G_n(X)$ to $G(A, X)$ and dualized. It is our purpose in this paper to make a further study of the evaluation subgroups in Varadarajan's general setting. Study of its dual will be discussed in a separate paper. Recall that $G(A, X)$ consists of all homotopy classes of cyclic maps [12] from A to X. In general, it is not a group but is known to be a subgroup of $[A, X]$ if A is an H-cogroup. In Section 3, we show that $\omega_{\#}([A, X^X]) = G(A, X)$. It is also shown that $G(A, X)$ preserves products in the second variable. In Section 4, a convenient subset $C(A, X)$ of $[A, X]$ (when A is a co-H-space) is introduced and some of its basic properties developed. As is well known, $G(A, X)$ is not a functor of X but is a contravariant functor of A from the subcategory of H-cogroups and co-H-maps into the category of groups and homomorphisms. We show in Section 5 that both $G(X, X)$ and $C(X, X)$ are rings if X is an H-cogroup. We also prove that $G(A, X)$ is a contravariant functor of A from the full subcategory of H-cogroups and maps (not necessarily co-H-maps) into the category of abelian groups and homomorphisms.

2. **Preliminary.** Unless otherwise stated, we shall work in the category of spaces with base points and having the homotopy type of locally finite CW-complexes. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by $*$. 1 (sometimes with decoration) will denote the identity

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function (resp. map) of a set or a group (resp. space) when it is clear from the context. For simplicity, we sometimes use the same symbol for a map and its homotopy class.

All function spaces will be endowed with the compact-open topology and, unless otherwise stated, the constant map will be taken to be the base point. X^X shall denote the space of free maps from X to X with 1_X as base point. The evaluation map $\omega : X^X \to X$ is defined to be $\omega(f) = f(*)$ for each $f \in X^X$.

The folding map $\nabla : X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. Frequently (not always) i and j will be reserved for the inclusion maps of the form $i_1 : X \to X \times Y$ or $i_2 : Y \to X \times Y$, and $j : X \vee Y \to X \times Y$ respectively. The projection is denoted by p with decoration.

ΣX and ΩX denote the reduced suspension and the loop space of X respectively. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ. The symbols e_A and e'_A denote $\tau^{-1}(1_{\Omega A})$ and $\tau(1_{\Sigma A})$ respectively, the subscript will be dropped if there is no danger of confusion.

Definition 2.1 ([12]). A map $f : A \to X$ is said to be cyclic if there exists a map $F : X \times A \to X$ such that the following diagram is homotopy commutative:

$$\begin{tikzcd} X \times A \arrow[r, "F"] & X \\ X \vee A \arrow[u, "j"] \arrow[ur, "\nabla(1 \vee f)"'] & \end{tikzcd}$$

that is, $Fj \simeq \nabla(1 \vee f)$. Since j is a cofibration, this is equivalent to saying that we can find a map $G : X \times A \to X$ such that $Gj = \nabla(1 \vee f)$. We call such a map G an associated map of f. The set of all homotopy classes of cyclic maps from A to X is denoted by $G(A, X)$ and is called the Gottlieb subset of $[A, X]$.

3. **Certain basic properties of** $G(A, X)$. The purpose of this section is to record two basic results. An application of the first result was already demonstrated in [9]. First we recall the following well-known lemma:

Lemma 3.1 ([8]). *Let X be a locally compact Hausdorff space, Z a Hausdorff space and Y any space. Then the function spaces $(Y^X)^Z$ and $Y^{X \times Z}$ are homeomorphic and a homeomorphism $H : (Y^X)^Z \to Y^{X \times Z}$ is given by $H(g)(x, z) = g(z)(x)$ for each $g : Z \to Y^X$, $x \in X$, $z \in Z$. Furthermore, $f \simeq g$ iff $H(f) \simeq H(g)$.*

Theorem 3.2. *Let X be a space having the homotopy type of a locally finite CW-complex and A any Hausdorff space. Suppose $\omega : X^X \to X$ is the evaluation map where X^X is the space of free maps from X to X with 1_X as base point. Then $\omega_\#([A, X^X]) = G(A, X)$ as sets, where $\omega_\#$ is the induced function of ω.*

Proof. We shall only outline the proof. Let $[g] \in [A, X^X]$. Let H be the homeomorphism given in Lemma 3.1. Then $H(g)j = \nabla(1 \vee \omega g)$ where $j : X \vee A \to X \times A$ is the inclusion. Thus $\omega_{\#}[g] \in G(A, X)$.

On the other hand, let $f \in G(A, X)$ and $F : X \times A \to X$ an associated map of f. By Lemma 3.1, we can find a map $f' : A \to X^X$ such that $F = H(f')$. Then $\omega_{\#}[f'] = [f]$. Thus $f \in \omega_{\#}([A, X^X])$.

Under the same hypotheses as the above theorem, if, in addition, A is an H-cogroup, then $\omega_{\#}([A, X^X]) = G(A, X)$ as groups. This justifies the term evaluation subgroup.

Remark 1. If $A = S^1$, then we have $G(X) = \omega_{\#}(\pi_1(X^X))$ which is Theorem III.1 of [4].

Remark 2. If $A = S^n$, then we have $G_n(X) = \omega_{\#}(\pi_n(X^X))$ which is Proposition 1.1 of [5].

Next we shall prove a product theorem which yields several corollaries, including another result of [5].

THEOREM 3.3. *Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a collection of spaces which have the homotopy type of* CW-*complexes and A any space. Then $G(A, \prod X_\alpha)$ and $\prod G(A, X_\alpha)$ are isomorphic as sets, where $\prod$ denotes the topological product or set product as the case may be.*

Proof. Let $f \in G(A, \prod X_\alpha)$. Then there exists a map $F : \prod X_\alpha \times A \to \prod X_\alpha$ such that $Fj = \nabla(1 \vee f)$ where j is the obvious inclusion. For each $\beta \in \Lambda$, let $f_\beta : A \to X_\beta$ be the map given by $f_\beta = p_\beta f$ where $p_\beta : \prod X_\alpha \to X_\beta$ is the obvious projection. We claim that $f_\beta \in G(A, X_\beta)$. To see this, let $F_\beta = p_\beta F\ (i_\beta \times 1) : X_\beta \times A \to X_\beta$ where $i_\beta : X_\beta \to \prod X_\alpha$ is the inclusion. If j_β denotes the inclusion $X_\beta \vee A \to X_\beta \times A$, then $F_\beta j_\beta = p_\beta F(i_\beta \times 1) j_\beta = p_\beta F j(i_\beta \vee 1) = p_\beta \nabla(1 \vee f)(i_\beta \vee 1) = \nabla(p_\beta \vee p_\beta)(i_\beta \vee f) = \nabla(p_\beta i_\beta \vee p_\beta f) = \nabla(1 \vee f_\beta)$. Hence $f_\beta \in G(A, X_\beta)$. We may therefore define a function $\Phi : G(A, \prod X_\alpha) \to \prod G(A, X_\alpha)$ as follows: for each $f \in G(A, \prod X_\alpha)$, let $\Phi(f) = \langle f_\alpha \rangle$ where $f_\alpha = p_\alpha f$ for each α.

Conversely, for each $\beta \in \Lambda$, let $f_\beta \in G(A, X_\beta)$. Then we can find a map $F_\beta : X_\beta \times A \to X_\beta$ such that $F_\beta j_\beta = \nabla(1 \vee f_\beta)$ where j_β is the obvious inclusion. Define a map $f : A \to \prod X_\alpha$ by $f = (\langle f_\alpha \rangle)\Delta$. We claim that $f \in G(A, \prod X_\alpha)$. In fact, let $F : \prod X_\alpha \times A \to \prod X_\alpha$ be the map given by $F(\langle x_\alpha \rangle, a) = \langle F_\alpha(x_\alpha, a) \rangle$ for each $\langle x_\alpha \rangle \in \prod X_\alpha$ and $a \in A$. Then $Fj(\langle x_\alpha \rangle, *) = \langle F_\alpha(x_\alpha, *) \rangle = \langle x_\alpha \rangle = \nabla(1 \vee f)(\langle x_\alpha \rangle, *)$ and $Fj(*, a) = \langle F_\alpha(*, a) \rangle = \langle f_\alpha(a) \rangle = f(a) = \nabla(1 \vee f)(*, a)$. Thus $Fj = \nabla(1 \vee f)$ and hence $f \in G(A, \prod X_\alpha)$. We may therefore define a function $\Psi : \prod G(A, X_\alpha) \to G(A, \prod X_\alpha)$ as follows: for each $\langle f_\alpha \rangle \in \prod G(A, X_\alpha)$, let $\Psi(\langle f_\alpha \rangle) = (\langle f_\alpha \rangle)\Delta$. Moreover, it can be easily verified that the functions Φ and Ψ are inverse to each other and this establishes a one-to-one correspondence between the sets $G(A, \prod X_\alpha)$ and $\prod G(A, X_\alpha)$. The proof of the theorem is thus complete.

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An immediate consequence of the above theorem is the following corollary which includes Theorem 2.1 of [5] as a special case.

COROLLARY 3.4. *Let* $\{X_\alpha\}_{\alpha\in\Lambda}$ *be a collection of spaces which have the homotopy type of* CW-*complexes and* A *an* H-*cogroup. Then* $G(A, \prod X_\alpha) \simeq \oplus G(A, X_\alpha)$ *as groups, where* $\oplus$ *denotes the direct product.*

Proof. In view of the preceding theorem, it suffices to show that the function Φ defined in the proof is a homomorphism of groups. To do this, let $f, g \in G(A, \prod X_\alpha)$ and ϕ the given co-H-structure on A. Using the same symbol $+$ for the different group operations, we have

$$\begin{aligned}\Phi(f+g) &= \Phi(\nabla(f\vee g)\phi)\\ &= \langle p_\alpha \nabla(f\vee g)\phi\rangle\\ &= \langle \nabla(p_\alpha\vee p_\alpha)(f\vee g)\phi\rangle\\ &= \langle \nabla(p_\alpha f\vee p_\alpha g)\phi\rangle\\ &= \langle p_\alpha f + p_\alpha g\rangle\\ &= \Phi(f)+\Phi(g).\end{aligned}$$

Hence Φ provides the indicated isomorphism.

4. **Some basic properties of** $C(A, X)$. In this section we shall define a larger subset $C(A, X)$ of $[A, X]$ which includes $G(A, X)$ and shall study some of its basic properties. As it will be seen from the next section that the introduction of $C(A, X)$ will provide a new insight into $G(A, X)$, it deserves our attention here.

DEFINITION 4.1. Let A be a co-H-space (thus the function $\Omega:[A, X] \to [\Omega A, \Omega X]$, given by $f \mapsto \Omega f$, is injective). We define $C(A, X) \equiv \Omega^{-1}[\Omega A, \Omega X]_{C\Omega}$, where $[\Omega A, \Omega X]_{C\Omega}$ denotes the set of all homotopy classes of maps Ωf which are central [2].

LEMMA 4.2 ([3]). *Let* $X \flat A \xrightarrow{L} X \vee A \to X \times A$ *be a fibration. Then* $\nabla(1\vee f)L = *$ *iff* Ωf *is central.*

PROPOSITION 4.3. *If* A *is a co-*H*-space, then* $G(A, X) \subset C(A, X)$, *and* $G(A, \Sigma X) = C(A, \Sigma X)$ *for all spaces* X.

Proof. We need only show that $C(A, \Sigma X) \subset G(A, \Sigma X)$ since by Proposition 4.13 of [9], we have $G(A, X) \subset C(A, X)$ for all spaces X. We first consider the case where A is a suspension, say $A = \Sigma B$. Thus suppose $f:\Sigma B \to \Sigma X$ is in $C(\Sigma B, \Sigma X)$, that is, $\Omega f:\Omega\Sigma B \to \Omega\Sigma X$ is central. By Lemma 4.2, we have $\nabla(1\vee f)L \simeq *$ where $L:\Sigma X \flat \Sigma B \to \Sigma X \vee \Sigma B$ is the fibre of the inclusion $\Sigma X \vee \Sigma B \to \Sigma X \times \Sigma B$. Let $i_1:\Sigma X \to \Sigma X \vee \Sigma B$ and $i_2:\Sigma B \to \Sigma X \vee \Sigma B$ be the obvious inclusions. Then we can form the generalized Whitehead product

$[i_1, i_2]: \Sigma(X \wedge B) \to \Sigma X \vee \Sigma B$. Since $[i_1, i_2]$ coclassifies $\Sigma X \times \Sigma B$, it follows that $[i_1, i_2]$ factors through L, that is, we can find a map $g: \Sigma(X \wedge B) \to \Sigma X \flat \Sigma B$ such that $Lg \simeq [i_1, i_2]$. Thus $\nabla(1 \vee f)\ [i_1, i_2] \simeq \nabla(1 \vee f)Lg \simeq *$, and hence we can find a map $h: \Sigma X \times \Sigma B \to \Sigma X$ which extends $\nabla(1 \vee f)$. Thus $f \in G(\Sigma B, \Sigma X)$. In the general case where A is a co-H-space, let $e: \Sigma\Omega A \to A$ be the adjoint of the identity map $\Omega A \to \Omega A$. Consider the map $fe: \Sigma\Omega A \to \Sigma X$. Since Ωf is central, so is $\Omega(fe) = (\Omega f)(\Omega e)$ by [2]. Since the domain of fe is a suspension, by the first part of the proof, it follows that fe is cyclic. Since A is a co-H-space, we can find a map $s: A \to \Sigma\Omega A$ such that $es \simeq 1$. Hence $fes \simeq f$ is cyclic.

Remark. In the case that A is a suspension, this proposition has been proved by Hoo [7] in the following form.

COROLLARY 4.4. *Let $f: \Sigma B \to \Sigma X$ be a map. Then the following are equivalent*:

(a) *f is cyclic.*

(b) *f maps $\Omega\Sigma B$ into the center of $\Omega\Sigma X$.*

(c) $[1_{\Sigma X}, f] = 0$, *where* $[,]$ *denotes the generalized Whitehead product.*

Remark. Note that condition (b) simply means that Ωf is central.

We shall show in the next section that $C(A, X)$ is a subgroup contained in the center of $[A, X]$ if A is a co-H-space with a right homotopy inverse and X is any space.

DEFINITION 4.5.

$$W(\Sigma A, X) \equiv \{\alpha \in [\Sigma A, X] \mid [\alpha, \beta] = 0 \text{ for all } \beta \in [\Sigma B, X] \text{ for all } B\}$$

$$P(\Sigma A, X) \equiv \{a \in [\Sigma A, X] \mid [\alpha, \beta] = 0 \text{ for all } \beta \in [\Sigma^l A, X] \text{ and for all } l \geq 1\}.$$

Here $[\alpha, \beta]$, as usual, denotes the generalized Whitehead product of α and β.

Clearly $W(\Sigma A, X) \subset P(\Sigma A, X)$. It is shown in [12] that $P(\Sigma A, X)$ is a subgroup of $[\Sigma A, X]$. We now relate $C(\Sigma A, X)$ and $W(\Sigma A, X)$ for any spaces A and X.

PROPOSITION 4.6. *Let A, B and X be spaces. If $f \in C(\Sigma A, X)$, then $[f, g] = 0$ for all $g \in [\Sigma B, X]$.*

Proof. Let $q: A \times B \to A \wedge B$ be the quotient map, and let $p_1: A \times B \to A$, $p_2: A \times B \to B$ be the usual projections. Then according to [1], we have

$$[f, g]\Sigma q = f\Sigma p_1 + g\Sigma p_2 - f\Sigma p_1 - g\Sigma p_2.$$

Taking adjoints, we obtain the equation

$$\tau([f, g])q = (\Omega f)e_1' p_1 + (\Omega g)e_2' p_2 - (\Omega f)e_1' p_1 - (\Omega g)e_2' p_2$$

where $e_1': A \to \Omega\Sigma A$, $e_2': B \to \Omega\Sigma B$ are the adjoints of the obvious identity

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maps. Since Ωf is central, it follows that $\tau([f, g])q = 0$. Now $q^{\#}$ is a monomorphism, and τ, the operation of taking adjoints, is an isomorphism. Hence $[f, g] = 0$.

COROLLARY 4.7. *For all spaces A and X, we have*

$$G(\Sigma A, X) \subset C(\Sigma A, X) \subset W(\Sigma A, X) \subset P(\Sigma A, X) \subset [\Sigma A, X].$$

PROPOSITION 4.8. *For all spaces A and X, we have*

$$G(\Sigma A, \Sigma X) = C(\Sigma A, \Sigma X) = W(\Sigma A, \Sigma X).$$

Proof. We need only show that $W(\Sigma A, \Sigma X) \subset G(\Sigma A, \Sigma X)$. Let $f \in W(\Sigma A, \Sigma X)$. Then $[f, 1_{\Sigma X}] = 0$ by Definition 4.5, so that $[1_{\Sigma X}, f] = 0$. According to Corollary 4.4, $f \in G(\Sigma A, \Sigma X)$.

THEOREM 4.9. *For any space X, we have*

$$G(\Sigma X, \Sigma X) = C(\Sigma X, \Sigma X) = W(\Sigma X, \Sigma X) = P(\Sigma X, \Sigma X),$$

(see Example 3 in the next section).

Proof. It is obvious that $W(\Sigma X, \Sigma X) = P(\Sigma X, \Sigma X)$.

As an application of Proposition 4.6, we have the following result.

PROPOSITION 4.10. *Let G be a topological group and H a closed subgroup. Let $p : G \to G/H$ be the natural map onto the space of left cosets. If A is such that $p_{\#} : [\Sigma A, G] \to [\Sigma A, G/H]$ is onto, then for all α in $[\Sigma A, G/H]$ and all β in $[\Sigma B, G/H]$ where B is any space, we have $[\alpha, \beta] = 0$.*

Proof. Since $p_{\#}$ is onto, we can find a map $\gamma : \Sigma A \to G$ such that $p\gamma = \alpha$. Then α is cyclic, and the assertion follows from Proposition 4.6.

Remark. The above proposition says that

$$W(\Sigma A, G/H) = [\Sigma A, G/H].$$

5. $G(X, X)$ **and** $C(X, X)$ **as Rings.** We have now come to the central part of our paper. Our main object here is to show that both $G(X, X)$ and $C(X, X)$ are rings if X is an H-cogroup. In the course of achieving our aim, we show that for a fixed space X both $G(-, X)$ and $C(-, X)$ turn out to be contravariant functors from the full subcategory of H-cogroups and maps (not necessarily co-H-maps) into the category of abelian groups and homomorphisms.

We shall first prove the following proposition.

PROPOSITION 5.1. *Let A be a co-H-space with a right homotopy inverse ν, and let X be a space. Then $C(A, X)$ and $[\Omega A, \Omega X]_{C\Omega}$ are subgroups contained in the centers of $[A, X]$ and $[\Omega A, \Omega X]$ respectively, and $\Omega : C(A, X) \to [\Omega A, \Omega X]_{C\Omega}$ is an isomorphism of abelian groups.*

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Consider a co-H-space A with co-H-structure $\phi : A \to A \vee A$. Applying the co-Hopf construction to ϕ, we obtain a map $H(\phi) : \Omega A \to \Omega(A \flat A)$. Let $f, g : A \to X$ be maps. Let $L : A \flat A \to A \vee A$ be the fibre of $A \vee A \to A \times A$. Then we can form $\Omega\{\nabla(f \vee g)L\}H(\phi) : \Omega A \to \Omega X$. To show the proposition, we have to appeal to the following result of Hoo [6]:

LEMMA 5.2. *Let A be a co-H-space with co-H-structure $\phi : A \to A \vee A$. Let $f, g : A \to X$ be maps. Then $\Omega(f+g) = \Omega\{\nabla(f \vee g)L\}H(\phi) + \Omega f + \Omega g$.*

Proof of proposition. According to Lemma 4.2, if f or h is in $C(A, X)$ then $\nabla(f \vee h)L \simeq *$ and hence $\Omega(f+h) = \Omega f + \Omega h$. Let $\Omega f, \Omega g \in [\Omega A, \Omega X]_{C\Omega}$. Then $\Omega(f+g) = \Omega f + \Omega g$. By [2], both $(\Omega f)p_2$ and $(\Omega g)p_2$ lie in the center of $[\Omega X \times \Omega A, \Omega X]$, so that $(p_1, (\Omega f + \Omega g)p_2) = (p_1, (\Omega f)p_2 + (\Omega g)p_2) = 0$. Thus $\Omega f + \Omega g$ is central by [2] again. Let $\mu : \Omega X \to \Omega X$ be the loop inverse. We shall show that $-\Omega f = \mu \Omega f$ is central. In fact, since $0 = \Omega(f + f\nu) = \Omega f + \Omega(f\nu)$, $-\Omega f = (\Omega f)(\Omega \nu)$ is central. Thus $[\Omega A, \Omega X]_{C\Omega}$ is a subgroup of $[\Omega A, \Omega X]$. That it is contained in the center is clear. To see that $C(A, X)$ is contained in the center of $[A, X]$, let $h \in [A, X]$. Then $\Omega f + \Omega h = \Omega h + \Omega f$ since $\Omega f \in [\Omega A, \Omega X]_{C\Omega}$. Thus $\Omega(f+h) = \Omega(h+f)$ and hence $f+h = h+f$ since $\Omega : [A, X] \to [\Omega A, \Omega X]$ is injective. Hence $C(A, X)$ and $[\Omega A, \Omega X]_{C\Omega}$ are subgroups contained in the centers of $[A, X]$ and $[\Omega A, \Omega X]$ respectively, and $\Omega : C(A, X) \to [\Omega A, \Omega X]_{C\Omega}$ is an isomorphism of abelian groups. This completes the proof of the proposition.

Remarks. It follows that if A is a co-H-space with a right homotopy inverse, then for every space X, $G(A, X) \subset C(A, X) \subset$ center of $[A, X]$ as subgroups. This generalizes Gottlieb's result [4] that $G(X)$ lies in the center of $\pi_1(X)$.

We shall now proceed to establish the right distributive law. Suppose that $f : A \to B$ is a map from a homotopy associative co-H-space A to co-H-space B. Then we can find a co-H-map $s : A \to \Sigma\Omega A$ such that $es \simeq 1$ where $e : \Sigma\Omega A \to A$ is the usual map. Let $g_1, g_2 : B \to Y$ be maps where Y is any space. We can form $(g_1 + g_2)f : A \to Y$. In general $(g_1 + g_2)f \neq g_1 f + g_2 f$. A suitable distributive law would compensate for this by providing a correction term. For our purposes, the correction term would have to be such that it vanishes in case g_1 or g_2 is in $C(B, Y)$.

Consider the map $\phi f : A \to B \vee B$ where ϕ is the co-H-structure on B. Applying the co-Hopf construction to this map, we obtain a map $H(\phi f) : \Omega A \to \Omega(B \flat B)$. Taking adjoint, we obtain $\tau^{-1}\{H(\phi f)\} : \Sigma\Omega A \to B \flat B$. Let $L : B \flat B \to B \vee B$ be the fibre of $B \vee B \to B \times B$. Then we can form $L\tau^{-1}\{H(\phi f)\}s : A \to B \vee B$. The following lemma is also due to Hoo [6]:

LEMMA 5.3. *Let $f : A \to B$ be a map from a homotopy associative co-H-space A to a co-H-space B, and let $s : A \to \Sigma\Omega A$ be a co-H-map such that $es \simeq 1 : A \to A$. Let $\phi : B \to B \vee B$ be the co-H-structure on B. Let $g_1, g_2 : B \to Y$ be* *maps, where Y is any space. Then*

$$(g_1+g_2)f=\nabla(g_1\vee g_2)L\tau^{-1}\{H(\phi f)\}s+g_1f+g_2f.$$

PROPOSITION 5.4. *Let $f:A\to B$ be a map from a homotopy associative co-H-space A to a co-H-space B. Let g_1, $g_2:B\to Y$ be maps such that either g_1 or g_2 is in $C(B, Y)$, where Y is any space. Then $(g_1+g_2)f=g_1f+g_2f$.*

Proof. According to Lemma 4.2, $\nabla(g_1\vee g_2)L\simeq *$ and hence the relation in Lemma 5.3 reduces to $(g_1+g_2)f=g_1f+g_2f$ as asserted.

Remark. If f is a co-H-map, the the above proposition is trivial.

Proposition 5.4 yields the following corollaries.

COROLLARY 5.5 *Let $f:A\to B$ be a map from a homotopy associative co-H-space A to a co-H-space B. Then $f^{\#}:C(B,X)\to C(A,X)$ and $f^{\#}:G(B,X)\to G(A,X)$ are homomorphisms for any space X.*

Proof. The first part follows directly from Proposition 5.4. Restriction to $G(B,X)$ gives the other result.

COROLLARY 5.6. *If $f:\Sigma A\to B$ is a map where B is a co-H-space, then $f^{\#}:C(B,X)\to C(\Sigma A,X)$ and $f^{\#}:G(B,X)\to G(\Sigma A,X)$ are homomorphisms for any space X.*

EXAMPLE 1. Let $f:\Sigma A\to\Sigma B$ be any map. Then $f^{\#}:C(\Sigma B,X)\to C(\Sigma A,X)$ and $f^{\#}:G(\Sigma B,X)\to G(\Sigma A,X)$ are group homomorphisms for any space X.

EXAMPLE 2. Let $f:S^n\to S^r$ be any map. Then $f^{\#}:C(S^r,X)\to C(S^n,X)$ and $f^{\#}:G_r(X)\to G_n(X)$ are group homomorphisms for any space X.

In view of Propositions 5.1 and 5.4, we conclude that both $G(-,X)$ and $C(-,X)$ are contravariant functors from the full subcategory of H-cogroups and maps into the category of abelian groups and homomorphisms.

Remark. Without Proposition 5.1 and 5.4 the above observation would be by no means trivial although it is evident from the remark following Proposition 5.4 that $G(-,X)$ is a contravariant functor from the subcategory of H-cogroups and co-H-maps into the category of groups and homomorphisms.

Our main theorem is now clear.

THEOREM 5.7. *For any H-cogroup X, $G(X,X)$ and $C(X,X)$ are rings.*

EXAMPLE 3. For any space X.

$$G(\Sigma X,\Sigma X)=C(\Sigma X,\Sigma X)=W(\Sigma X,\Sigma X)=P(\Sigma X,\Sigma X)$$

as rings (see Theorem 4.9).

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