FINITE TOPOLOGICAL SPACES AND QUASI-UNIFORM STRUCTURES

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1. Introduction. In [6], H. Sharp gives a matrix characterization of each topology on a finite set $X = \{x_1, x_2, \ldots, x_n\}$. The study of quasi-uniform spaces provides a more natural and obviously equivalent characterization of finite topological spaces. With this alternate characterization, results of quasi-uniform theory can be used to obtain simple proofs of some of the major theorems of [1], [3] and [6]. Moreover, the class of finite topological spaces has a quasi-uniform property which is of interest in its own right. All facts concerning quasi-uniform spaces which are used in this paper can be found in [4].

2. Preliminaries.

DEFINITION. Let X be a non-empty set and let $\mathfrak{U}$ be a filter base on $X \times X$ such that

i) each element of $\mathfrak{U}$ is a reflexive relation on X ;

ii) if $U \in \mathfrak{U}$ there exists $W \in \mathfrak{U}$ such that $W \circ W \subset U$.

Then $\mathfrak{U}$ is a quasi-uniformity on X . If $\mathfrak{U}$ is a filter, then $\mathfrak{U}$ is called a quasi-uniform structure.

DEFINITION. Let X be a set and let $\mathfrak{U}$ be a quasi-uniformity on X. Let $\mathfrak{T}_{\mathfrak{U}} = \{A \subset X$: if $a \in A$ then there exists $U \in \mathfrak{U}$ such that $U(a) \subset A\}$. Then $\mathfrak{T}_{\mathfrak{U}}$ is the quasi-uniform topology on X generated by $\mathfrak{U}$.

DEFINITION. Let $(X, \mathfrak{T})$ be a topological space and let $\mathfrak{U}$ be a quasi-uniformity on X. Then $\mathfrak{U}$ is compatible if $\mathfrak{T} = \mathfrak{T}_{\mathfrak{U}}$.

It is shown in [5] that if $(X, \mathfrak{T})$ is a topological space, then there exists a compatible quasi-uniformity $\mathfrak{U}$ on X.

3. Finite topological spaces. It is clear that every finite topological space has a finite compatible quasi-uniformity. Let $(X, \mathfrak{T})$ be a topological

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space with a finite compatible quasi-uniformity $\mathcal{U}'$ and let $\mathcal{U}$ be the quasi-uniform structure generated by $\mathcal{U}'$. Let $U = \bigcap \{V : V \in \mathcal{U}\}$. Then $U \in \mathcal{U}$. Thus there exists $W \in \mathcal{U}$ such that $W \circ W \subset U$. But $U \subset U \circ U \subset W \circ W \subset U$. Thus U is a reflexive and transitive relation, and since U is exactly the collection of all supersets of U, $\{U\}$ is a compatible quasi-uniformity.

DEFINITION. Let $(X, \mathfrak{T})$ be a topological space with a finite compatible quasi-uniformity $\mathcal{U}$. Let $U = \bigcap \{V : V \in \mathcal{U}\}$. Then $\{U\}$ is called the _fundamental quasi-uniformity of_ $(X, \mathfrak{T})$ _with respect to_ $\mathcal{U}$.

THEOREM 3.1. _Let_ $(X, \mathfrak{T})$ _be a topological space with a finite compatible quasi-uniformity and let_ $\mathcal{U} = \{U\}$ _be a compatible fundamental quasi-uniformity. Then for each_ $x \in X$, $U(x)$ _is the smallest open set which contains_ x.

Proof. Let $x \in X$. For any quasi-uniform space it is true that $x \in [U(x)]^{o}$. Hence $U(x) \subset [U(x)]^{o}$. Moreover, if $x \in A \in \mathfrak{T} = \mathfrak{T}_{\mathcal{U}}$, then $U(x) \in A$ by definition.

THEOREM 3.2. _Let_ $(X, \mathfrak{T})$ _be a topological space with a finite compatible quasi-uniformity and let_ $\mathcal{U} = \{U\}$ _be a compatible fundamental quasi-uniformity. Let_ $x, y \in X$ _with_ $x \neq y$. _Then_ $(x, y) \in U$ _if and only if_ $U(y) \subset U(x)$.

Proof. Suppose $(x, y) \in U$. Then $y \in U(x)$ so that $U(y) \subset U \circ U(x) = U(x)$. The reverse implication follows from the fact that $y \in U(y)$.

COROLLARY. _Let_ $x, y \in X$ _with_ $x \neq y$. _Then_ $U(x) = U(y)$ _if and only if_ $y \in U(x)$ _and_ $x \in U(y)$.

THEOREM 3.3. _Let_ $(X, \mathfrak{T})$ _be a topological space and suppose there exists a finite compatible quasi-uniformity_ $\mathcal{U}'$. _Then the quasi-uniform structure_ $\mathcal{U}$ _generated by_ $\mathcal{U}'$ _is the largest compatible quasi-uniform structure on_ X.

Proof. Let $\mathcal{V}$ be the universal quasi-uniform structure for $(X, \mathfrak{T})$. By definition $\mathcal{U} \subset \mathcal{V}$, and $\mathcal{V}$ is the largest compatible quasi-uniform structure on X. Let $V \in \mathcal{V}$ and let $U = \bigcap \{W : W \in \mathcal{U}'\}$. Let $x \in X$. Since $U(x)$ is a subset of any open set which contains x, $U(x) \subset [V(x)]^{o} \subset V(x)$. Thus $U \subset V$ so that $V \in \mathcal{U}$. Hence $\mathcal{U} = \mathcal{V}$.

COROLLARY. _Let_ $(X, \mathfrak{T})$ _be a finite topological space. Then there exists exactly one quasi-uniform structure which is compatible with_ $(X, \mathfrak{T})$.

COROLLARY. _Each topological space with a finite compatible quasi-uniformity has exactly one compatible fundamental quasi-uniformity._

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COROLLARY [6]. Let X be a finite set. There is a one-to-one correspondence between the collection of all topologies on X and the collection of all reflexive, transitive relations on X.

COROLLARY [3]. The number of topologies on a finite set X with exactly n elements is less than or equal to $2^{n(n-1)}$.

Proof. There are $2^{n(n-1)}$ subsets of $X \times X - \Delta$. Hence there are at most $2^{n(n-1)}$ fundamental quasi-uniformities on X.

4. Topological properties of finite topological spaces.

THEOREM 4.1. Let $(X, \mathfrak{T})$ be a finite topological space and let $\mathcal{U} = \{U\}$ be the compatible fundamental quasi-uniformity. Then $(X, \mathfrak{T}_{\mathcal{U}})$ is connected if and only if $\{X, \mathfrak{T}_{\mathcal{U}^{-1}}\}$ is connected.

Proof. Let $a \in X$. Then by [4, Theorem 1.15], $\overline{\{a\}} = \bigcap \{U(a) : U^{-1} \in \mathcal{U}\} = U^{-1}(a)$. Thus every $\mathfrak{T}_{\mathcal{U}^{-1}}$ - open set is $\mathfrak{T}_{\mathcal{U}}$ - closed and the theorem follows.

COROLLARY [6]. If X is a finite set there are an even number of non-trivial connected topologies on X.

The following theorem originally proved in [1] illustrates the force of quasi-uniform theory on finite topological spaces. All of the equivalences of this theorem are immediate consequences of basic theorems about quasi-uniform spaces.

THEOREM 4.2. [1]. Let $(X, \mathfrak{T})$ be a finite topological space and let $\mathcal{U} = \{U\}$ be the compatible fundamental quasi-uniformity. The following are equivalent.

(a) U is symmetric;

(b) $(X, \mathfrak{T})$ is regular;

(c) $(X, \mathfrak{T})$ is completely regular;

(d) $(X, \mathfrak{T})$ is 0-dimensional;

(e) $(X, \mathfrak{T})$ is $\mathfrak{R}_1$

(f) $(X, \mathfrak{T})$ is $\mathfrak{R}_0$.

Proof. It is well known that $a \Rightarrow c$, $c \Rightarrow d$, and $e \Rightarrow f$.

b ⇒ a: [4, Theorem 3.17 ii].

a ⇒ e: $U(a) = U^{-1}(a) = \bigcap\{U^{-1}(a) : U \in \mathcal{U}\} = \overline{\{a\}}$.

f ⇒ a: [4, Theorem 3.8].

d ⇒ b: If $U(a)$ is closed, then by [4, Theorem 1.15], $U(a) = U^{-1} \circ U(a)$. By [4, Theorem 3.17 iii], $(X, \mathfrak{T})$ is regular.

Perhaps the most important separation axiom in finite topological spaces is the T_o separation axiom. By [4, Theorem 3.1], a finite topological space $(X, \mathfrak{T})$ is T_o if and only if the member of its fundamental quasi-uniformity is anti-symmetric.

5. A quasi-uniform property of finite topological spaces.

Finite topological spaces have a noteworthy quasi-uniform property: namely, if $(X, \mathfrak{T})$ is a finite topological space then $(X, \mathfrak{T})$ has a unique compatible quasi-uniform structure. Uniform spaces having a unique compatible uniform structure have been characterized by R. Doss [2], and it is well known that the unique compatible uniform structure of a compact Hausdorff space is the collection of all neighborhoods of the diagonal in $X \times X$. On the other hand, it is not difficult to find compact Hausdorff spaces which have two distinct compatible quasi-uniform structures. Such an example is given in [5]. I conjecture that a topological space $(X, \mathfrak{T})$ admits exactly one compatible quasi-uniform structure if and only if $\mathfrak{T}$ is finite. The following theorem supports this conjecture.

THEOREM 5.1. *If $(X, \mathfrak{T})$ is a Tychonoff space with a unique compatible quasi-uniform structure, then $(X, \mathfrak{T})$ is a finite topological space.*

Proof. The unique quasi-uniform structure $\mathcal{U}$ is the Pervin quasi-uniform structure. Since $\mathfrak{T} = \mathfrak{T}_{\mathcal{U}^{-1}}$ is T_1, $\mathfrak{T}_{\mathcal{U}^{-1}}$ is the discrete topology and since $(X, \mathfrak{T})$ is uniformizable $\mathfrak{T}_{\mathcal{U}} = \mathfrak{T}_{\mathcal{U}^{-1}}$. Thus $(X, \mathfrak{T}_{\mathcal{U}})$ is discrete and hence metrizable. It follows that $\mathcal{U}$ is complete and since $\mathcal{U}$ is the Pervin quasi-uniform structure, $\mathcal{U}$ is totally bounded. Then $(X, \mathfrak{T}_{\mathcal{U}})$ is compact and discrete. Hence X is finite.

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