ABSTRACT THEORY OF PACKINGS AND COVERINGS. I.

by A. M. MACBEATH

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1. Introduction. *The aim of this note is to examine the basic ideas underlying* Minkowski's theorem on lattice points in a symmetrical convex body and related results of Blichfeldt, and to indicate how these can be generalized. Theorems analogous to Minkowski's, on the automorphisms of quadratic forms and other linear groups and on Fuchsian groups of transformations in the complex plane, have been obtained by Siegel [**6**] and Tsuji [**7**]. Generalizations which include these are due to Chabauty [**2**] and Santalo [**5**].

The last two authors both consider groups of transformations of topological spaces. Minkowski's method, however, is really a generalized " box-principle ", and seems to be more a matter of measure theory than topology. In this note, therefore, a topology is not assumed to exist in the space, so that the results are simpler and possibly more general (but see the remarks in § 6). An earlier version of these results was included in the dissertation submitted by the author in 1950 in partial fulfilment of the requirements for the Degree of Doctor of Philosophy at Princeton University.

2. Terminology. We suppose given a measure space $(X, \mathbf{S}, \mu)$ (see [**3**]), consisting of a set of elements X, a σ-ring $\mathbf{S}$ of subsets of X called measurable sets, and a measure μ defined on all sets of $\mathbf{S}$. To take the place of the lattice of number-theory, we assume that there is a countable group G of permutations of X, each of which is measure-preserving. That is, if $E \in \mathbf{S}$, and $g \in G$, then $gE \in \mathbf{S}$ and $\mu(gE) = \mu(E)$. We denote the identical permutation by e.

Definitions. 1. A set $P \in \mathbf{S}$ is called a *G-packing* if $P \cap gP = \phi$ for all $g \neq e$, $g \in G$.

2. A set $C \in \mathbf{S}$ is called a *G-covering* if $GC = X$. Here GC denotes the set of all points gc with $g \in G$, $c \in C$, so that $GC = \bigcup\{gC : g \in G\}$.

3. The *determinant* $\Delta(G)$ is defined to be the greatest lower bound of $\mu(C)$, where C may be any G-covering.

4. A set $F \in \mathbf{S}$ is called a *fundamental domain* for G if F is both a G-covering and a G-packing ; i.e., $F \cap gF = \phi$ and $GF = X$.

Some of the results hold only in a weaker form if no measurable fundamental domain F exists, and necessary and sufficient conditions for F to exist are obtained in the second paper with this title, by S. Świerczkowski. It is worth mentioning here that there are some very simple cases in which no F exists, even when G is without fixed points—for instance if X is the real line and G is the group of all rational translations [**3**, p. 67–70].

3. The generalized Minkowski Theorem.

THEOREM 1. *If C is a G-covering and P a G-packing, then* $\mu(C) \geqslant \mu(P)$.

Proof. If g_1, g_2, are distinct elements of G then $P \cap g_1^{-1}g_2P = \phi$, so that $g_1P \cap g_2P = \phi$, and the sets $C \cap gP$ are disjoint. Thus, if summations are taken over all elements g of G, we have

$$\mu(C) \geqslant \Sigma\mu(C \cap gP) = \Sigma\mu(g^{-1}C \cap P) \geqslant \mu(P \cap (\bigcup g^{-1}C)) = \mu(P \cap X) = \mu(P).$$

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COROLLARY. *If F_1, F_2 are fundamental domains, then $\mu(F_1) = \mu(F_2) = \Delta(G)$.*

From Theorem 1, it is possible to deduce two different generalizations of Minkowski's theorem, but each one requires different extra assumptions about the space X.

First generalization. Suppose given a function $d(x, y)$ defined for all pairs of points $x, y \in X$, and satisfying the conditions given below. In specifying one of these conditions, we require the symbol $S(a\,;\,r)$ to denote the set of all points x such that $d(a, x) < r$. Here $a \in X$, and r is a positive real number.

Conditions :

(i) $d(x, y) = d(y, x) \geqslant 0$.
(ii) $d(x, y) + d(y, z) \geqslant d(x, z)$.
(iii) $d(gx, gy) = d(x, y)$ for all $g \in G$.
(iv) $S(a\,;\,r)$ is measurable for each a, r.

THEOREM 2. *Suppose that the measure of $S(a\,;\,r)$ is greater than $\Delta(G)$. Then there is an element $g \in G$, $g \neq e$, such that*

$$d(a, ga) < 2r.$$

Proof. Since $\mu(S(a\,;\,r))$ is greater than the lower bound of the measure of all coverings, we must have $\mu(S(a\,;\,r)) > \mu(C)$ for some covering C. By Theorem 1, $S(a\,;\,r)$ cannot be a packing. Thus, for some $g \neq e$, the sets $S(a\,;\,r)$ and $gS(a\,;\,r)$ have a common point x, say. Hence $d(a, x) < r$ and, by (iii), $d(ga, x) = d(a, g^{-1}x) < r$.

From (ii) it follows that $d(a, ga) \leqslant d(a, x) + d(ga, x) < 2r$.

Second generalization. Assume that the space X is itself a group, and that G is a subgroup of X acting on it by left translation. (An example which springs to mind at once is the case when X is a locally compact topological group and μ is the left Haar measure. However we do not necessarily require μ to be invariant under *all* left translations, but only under those of G.)

THEOREM 3. *If X is a group and G a subgroup acting by left translation, and if D is a subset of X such that $\mu(D) > \Delta(G)$, then there is a point $g \in G \cap DD^{-1}$ with $g \neq e$.*

Proof. Since, by Theorem 1, D is not a packing, there is a $g \neq e$ such that D and gD intersect; i.e., $d = gd'$, where $d, d' \in D$. Then $g = dd'^{-1} \in DD^{-1}$.

Minkowski's theorem follows particularly simply from Theorem 3. Let X be Euclidean space of n dimensions, regarded as a group under vector addition; let G be a lattice, so that $\Delta(G)$ is the determinant of G in the usual sense, and let K be a convex body, symmetrical in the origin, with volume exceeding $2^n\Delta(G)$. Take D in Theorem 1 to be the set $\frac{1}{2}K$. Then $K = D - D$ must contain a lattice point other than the origin.

4. Applications of Theorem 2. We now indicate three different situations in which Theorem 2 can be applied to give known results. The first of these is Minkowski's theorem, but the other two are of more recent origin. We refer the reader to the original papers for the calculations and precise numerical statements of results.

I. Let X be the Euclidean space of n dimensions, G the group of translations of a lattice, and let $d(x, y)$ be the Minkowski distance-function of a convex body K with volume exceeding $2^n\Delta(G)$. Let the point a be the origin. Applying Theorem 2 with $r = \frac{1}{2}$, we get

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Minkowski's theorem.

II. Let X be the interior $|z| < 1$ of the unit circle in the complex plane and let G be a discrete group of transformations of the type

$$w = \frac{az+b}{\bar{b}z+\bar{a}} e^{i\theta}.$$

Such a group is called a Fuchsian group. Define the metric by the differential expression

$$ds = 2\,|dz|\,/(1-|z|^2),$$

and the measure (in polar coordinates) by $4r\,dr\,d\theta/(1-r^2)^2$. We are then led to the results of Tsuji [**7**]. This situation is in fact a particular case ($n = 2$, $B = 0$) of III below, considered by Siegel. However, in the general case the results are less precise because the measure of the "spheres" cannot be computed exactly.

III. Let B be an $n \times n$ matrix which is either symmetric or skew-symmetric. Let X be the space of all positive definite symmetric A such that $A'BA = B$. Let Γ be the group of *all* non-singular T such that $T'BT = B$. Then Γ acts as a group of transformations of X by means of the congruent mapping $A \rightarrow T'AT$. Let G be a discrete subgroup of Γ, and define $d(x, y)$ to be the geodesic distance between x and y, calculated from the differential form

$$ds^2 = \operatorname{tr}\{(dA)A^{-1}(dA)A^{-1}\}.$$

Theorem 2 then implies a result due to Siegel [**6**, p. 714, Theorem 2]. The measure of the set $S(a\,;\; r)$ is not easy to estimate in this case, but Siegel, by an elegant argument, obtains an inequality (*loc. cit.*, Theorem 1) which relates it to the measure of the sphere in Euclidean space with the same radius.

5. The theorems of Blichfeldt and Santalo. The results of this section are generalizations of theorems of Blichfeldt [**1**]. Similar generalizations are due to Santalo [**5**], but Santalo assumes that X admits a transitive group of permutations each of which commutes with G. Without this assumption, the results have to be stated differently, but appear to be essentially the same in the special case which he considers. We state two separate results, the first holding in general, the second and stronger holding only when a fundamental domain F is known to exist.

THEOREM 4. *Let $f(x)$ be a non-negative integrable function and let C and P be a G-covering and a G-packing respectively, with $\mu(P) < \infty$. Then there are points a, $b \in X$ such that*

(i) $\mu(C) \sum_{g \in G} f(ga) \geqslant \int f(x)\, d\mu(x)$,

(ii) $\mu(P) \sum_{g \in G} f(gb) \leqslant \int f(x)\, d\mu(x)$.

Proof. Since the two proofs are similar, we only prove (i). Since $GC = X$ and f is non-negative, we have

$$\int f(x)\, d\mu(x) \leqslant \sum_{g \in G} \int_{gC} f(x)\, d\mu(x) = \sum_{g \in G} \int_{C} f(gx)\, d\mu(x) = \int_{C} \sum_{g \in G} f(gx)\, d\mu(x).$$

The result follows.

On taking $f(x)$ to be the characteristic function of a set D we obtain the following corollary.

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Corollary. *If $\mu(D) > n\Delta(G)$, where n is an integer, then there is a point $a \in X$ such that $ga \in D$ for at least $n+1$ distinct elements g of G.*

If a fundamental domain F exists for G, the following exact formula is valid for all integrable $f(x)$. The interchanges of summation and integration are valid if $f(x)$ is non-negative, and hence the result is true for any integrable $f(x)$, since such a function is the difference of two non-negative integrable functions. We have

$$\int f(x)\,d\mu(x) = \sum_{g \in G} \int_{gF} f(x)\,d\mu(x) = \int_F \sum f(gx)\,d\mu(x).$$

From this we deduce

Theorem 5. *If a fundamental domain exists, if $\Delta(G) < \infty$, and if $f(x)$ is integrable, then there are points $a, b \in X$ such that*

$$\Delta(G) \sum_{g \in G} f(ga) \leqslant \int f(x)\,d\mu(x) \leqslant \Delta(G) \sum_{g \in G} f(gb).$$

6. Comparison with previous results. Though we do not use topological terms, some of our assumptions are almost topological in character. Thus the first generalization of Minkowski's theorem, Theorem 2, requires the function $d(x, y)$, which is almost a metric — only the condition that $d(x, y)$ is zero if and only if $x = y$ is missing. Even without this assumption the spheres $S(a\,;\ r)$ can be taken as a base for a topology, though this topology will not, in general, satisfy a separation axiom.

The conditions in which Theorem 3 holds very nearly imply a topology too, since it is known that a measurable group admits a topological structure which is closely associated with the measure [**3**, p. 275, Theorem *H*].

It does seem, however, that the avoidance of topological assumptions, if it does not result in very much greater generality, certainly does result in improved simplicity and economy of statement.

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Queen's College
Dundee

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