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AN INVARIANT DISTRIBUTION FOR THE $G/G/1$ QUEUEING OPERATOR

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Abstract

We consider the $G/G/1$ queue as an operator that maps inter-arrival times to inter-departure times of points, given the service times. For arbitrarily fixed statistics of service times, we are interested in the existence of distributions of inter-arrival times that induce identical distributions on the inter-departure times. In this note we prove, by construction, the existence of one of such distribution.

QUEUEING THEORY

1. Introduction and notation

Consider a single-server first-come-first-served queue with infinite buffer ($G/G/1$). For any $j \in \mathbb{N} := \{0, 1, 2, \cdots\}$, let $t_j \in \mathbb{R}$ be the arrival time of the jth customer ($t_j \leqq t_{j+1}$), $\tau_j = t_{j+1} - t_j \in \mathbb{R}_+$ the inter-arrival time between the jth and the $(j+1)$th customer, and σ_j the service time required by the jth customer. The random sequences $\{\tau_j, j \in \mathbb{N}\}$ and $\{\sigma_j, j \in \mathbb{N}\}$ are defined on some common probability space $(\Omega, \boldsymbol{F}, P)$.

Let $w_j, j \in \mathbb{N}$ be the waiting time of the jth customer between arriving to the queue and starting to be serviced. Assuming that the queue is initially empty (i.e., $\omega_0 = 0$), the sequence $\{w_j, j \in \mathbb{N}\}$ is specified by Lindley's equations: $w_{j+1} = [w_j + \sigma_j - \tau_j]^+$, $j \in \mathbb{N}$, $w_0 = 0([x]^+ := x\mathbf{1}\{x > 0\}$, $x \in \mathbb{R})$. The jth customer leaves the queue at time $t_j^* = t_j + \sigma_j + w_j \in \mathbb{R}$, and $\tau_j^* = t_{j+1}^* - t_j^* \in \mathbb{R}_+$ is the inter-departure time between the jth and the $(j+1)$th customer. The sequence $\{\tau_j^*, j \in \mathbb{N}\}$ is inductively determined by the equations:

$$\tau_j^* = \tau_j + [w_j + \sigma_j - \tau_j]^+ - w_j + \sigma_{j+1} - \sigma_j, \qquad j \in \mathbb{N}. \tag{1}$$

From the above we see that there is a well-defined, Borel-measurable mapping $F : \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \to \mathbb{R}_+^\infty$, such that

$$\{\tau_j^*, j \in \mathbb{N}\} = F(\{\tau_j, j \in \mathbb{N}\}, \{\sigma_j, j \in \mathbb{N}\}). \tag{2}$$

Let P_σ denote the distribution of $\{\sigma_j, j \in \mathbb{N}\}$ and $P_{\tau\sigma}$ its joint distribution with $\{\tau_j, j \in \mathbb{N}\}$. Analogously define P_τ and P_{τ^*}. Observe that P_{τ^*} is induced by $P_{\tau\sigma}$ through (2).

The problem we want to study is the following. Given an arbitrarily fixed distribution of $\{\sigma_j, j \in \mathbb{N}\}$, say $P_\sigma = P^0$, does there exist a distribution P_τ of $\{\tau_j, j \in \mathbb{N}\}$, such that $P_\tau = P_{\tau^*}$? The problem is motivated by the well-known fact that if the σ_j's are independent and identically distributed with exponential distribution, and $\{\tau_j\}$ is independent of $\{\sigma_j\}$ and forms a Poisson point process ($M/M/1$ queue), then, under conditions of stationarity and stability for the queue, $\{\tau_j^*\}$ is also an identical Poisson process (i.e., the input to the queue and the output are identically distributed).

In this brief note we prove, by construction, the existence of one such distribution P_τ, satisfying $P_\tau = P_{\tau^*}$, for an arbitrarily chosen $P_\sigma = P^0$. The constructed distribution is rather

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special, but leads to the formulation, in Section 3, of a general conjecture about the class of distributions with the above property.

2. The construction

Consider a countably infinite number of queues in series, indexed by the integers $k \in \mathbb{Z}$. Customers leaving the kth queue join the $(k+1)$th one. With the kth queue we associate the sequence of service times $\{\sigma_j^k, j \in \mathbb{N}\}$. The random sequences $\{\sigma_j^k, j \in \mathbb{N}\}$, $k \in \mathbb{Z}$ are independent and identically distributed with distribution P_σ, and are defined on the probability space $(\Omega^\infty, \boldsymbol{F}^\infty, P^\infty)$. All the queues are initially empty.

Let $\{\tau_j^k, j \in \mathbb{N}\}$ be the sequence of inter-arrival times and $\{w_j^k, j \in \mathbb{N}\}$ the sequence of waiting times at the kth queue ($w_0^k = 0$ for every $k \in \mathbb{Z}$, since all the queues are initially empty). Therefore,

$$w_{j+1}^k = [w_j^k + \sigma_j^k - \tau_j^k]^+, \qquad j \in \mathbb{N}, \qquad k \in \mathbb{Z}. \tag{3}$$

The queues being in series, Equation (1) implies

$$\tau_j^{k+1} = \tau_j^k + [w_j^k + \sigma_j^k - \tau_j^k]^+ - w_j^k + \sigma_{j+1}^k - \sigma_j^k, \qquad j \in \mathbb{N}, \qquad k \in \mathbb{Z}, \tag{4}$$

and Equation (2),

$$\{\tau_j^{k+1}, j \in \mathbb{N}\} = F(\{\tau_j^k, j \in \mathbb{N}\}, (\sigma_j^k, j \in \mathbb{N}\}), \qquad k \in \mathbb{Z}. \tag{5}$$

Define now the random mappings $F^k(\{\tau_j, j \in \mathbb{N}\}) = F(\{\tau_j, j \in \mathbb{N}\}, \{\sigma_j^k, j \in \mathbb{N}\})$, $k \in \mathbb{Z}$, for any $\{\tau_j, j \in \mathbb{N}\} \in \mathbb{R}_+^\infty$ and also, for any $m, n \in \mathbb{Z}$, $m < n$,

$$\Phi_m^n = [F^{n-1} \circ F^{n-2} \circ \cdots \circ F^{m+1} \circ F^m](\{\sigma_{j+1}^{m-1}, j \in \mathbb{N}\}), \tag{6}$$

the symbol $\circ$ denoting composition of mappings. Observe that Φ_m^n is the inter-arrival times sequence $\{\tau_j^n, j \in \mathbb{N}\}$ at the nth queue, if the inter-arrival times sequence at the mth queue is $\{\tau_j^m, j \in \mathbb{N}\} = \{\sigma_{j+1}^{m-1}, j \in \mathbb{N}\}$. This is equivalent to starting the queueing at queue $m-1$ with zero interarrival times, since $\{\sigma_{j+1}^{m-1}, j \in \mathbb{N}\} = F^{m-1}(\{\tau_j^{m-1} = 0, j \in \mathbb{N}\})$.

Theorem. The limit

$$\Phi^n = \{\phi_j^n, j \in \mathbb{N}\} = \lim_{m \to -\infty} \Phi_m^n \tag{7}$$

exists pathwise, for every $n \in \mathbb{Z}$. Moreover, the sequences $\{\phi_j^n, j \in \mathbb{N}\}$, $n \in \mathbb{Z}$ are identically distributed with distribution P_ϕ and, for any $n \in \mathbb{Z}$,

$$\{\phi_j^{n+1}, j \in \mathbb{N}\} = F(\{\phi_j^n, j \in \mathbb{N}\}, \{\sigma_j^n, j \in \mathbb{N}\}). \tag{8}$$

Proof. For any two sequences $\{x_j, j \in \mathbb{N}\}$, $\{y_j, j \in \mathbb{N}\} \in \mathbb{R}_+^\infty$, define $\{x_j, j \in \mathbb{N}\} \geqq \{y_j, j \in \mathbb{N}\}$, iff $x_j \geqq y_j$ for every $j \in \mathbb{N}$.

Observe that, by (4), the function $\tau_j^{k+1}(\tau_j^k, w_j^k, \sigma_j^k, \sigma_{j+1}^k)$ is increasing in τ_j^k, decreasing in w_j^k, and $\tau_j^{k+1} \geqq \sigma_{j+1}^k$ for any $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Also, by (3), the function $w_{j+1}^k(w_j^k, \sigma_j^k, \tau_j^k)$ is increasing in w_j^k and decreasing in τ_j^k for any $k \in \mathbb{Z}$, $j \in \mathbb{N}$. Therefore, by induction on j we can easily prove that, if $\{x_j, j \in \mathbb{N}\} \geqq \{y_j, j \in \mathbb{N}\}$, then $F(\{x_j, j \in \mathbb{N}\}, \{\sigma_j^k, j \in \mathbb{N}\}) \geqq F(\{y_j, j \in \mathbb{N}\}, \{\sigma_j^k, j \in \mathbb{N}\}) \geqq \{\sigma_{j+1}^k, j \in \mathbb{N}\}$, for any $k \in \mathbb{Z}$. In view of the above, it is easily seen that, $\Phi_m^n \leqq \Phi_{m-1}^n$ for any $m, n \in \mathbb{Z}$, $m < n$. So, $\Phi^n = \lim_{m \to -\infty} \Phi_m^n$ exists component-wise.

Since the sequences $\{\sigma_j^k, j \in \mathbb{N}\}$, $k \in \mathbb{Z}$ are independent and identically distributed, Φ_{m+z}^{n+z} and Φ_m^n are identically distributed for any $z \in \mathbb{Z}$. Therefore, taking the limits, we see that Φ^{n+z} and Φ^n are identically distributed for any $n, z \in \mathbb{Z}$. Let P_ϕ be this common distribution.

Finally, by (6) we have $\Phi_m^{n+1} = F(\Phi_m^n, \{\sigma_j^n, j \in \mathbb{N}\})$ for any $m, n \in \mathbb{Z}$, $m < n$. Also, the function F is continuous in its first argument in the usual topology on $\mathbb{R}_+^\infty$. Therefore, taking the limits as $m \to -\infty$, we get

$$\Phi^{n+1} = F(\Phi^n, \{\sigma_j^n, j \in \mathbb{N}\}) \text{ for any } n \in \mathbb{Z}.$$

This completes the proof of the theorem.

According to the theorem, $\Phi^1 = F(\Phi^0, \{\sigma_j^0, j \in \mathbb{N}\})$. The distribution of $\{\sigma_j^0, j \in \mathbb{N}\}$ is P_σ and the distribution of both Φ^0 and Φ^1 is P_Φ, providing a solution to the problem introduced in Section 1.

An interesting observation is that, since the sequences $\{\sigma_j^k, j \in \mathbb{N}\}$, $k \in \mathbb{Z}$ are independent and identically distributed, we have Φ_{-n}^0 and Φ_0^n identically distributed. Thus, taking the limits $n \to \infty$, we see that the distribution of Φ_0^n converges also to P_Φ as $n \to \infty$.

It is also interesting to note, that the previous construction would work analogously, if the sequences $\sigma^k = \{\sigma_j^k, j \in \mathbb{N}\}$, $k \in \mathbb{Z}$ were not independent, but the sequence $\{\sigma^k, k \in \mathbb{Z}\}$ was just stationary with respect to the transformation $\theta\{\sigma^k\} = \{\sigma^{k+1}\}$.

3. Conclusions and further research

In this note we have constructed, for an arbitrarily fixed distribution of the service times P_σ, a distribution of the inter-arrival times that induces the same distribution of the inter-departure times. The construction is special indeed. In Equation (6) the interarrival times at queue m have a rate equal to the service rate, rendering the queue m 'critical' (see Loynes [2]), and the result about stable $M/M/1$ queues is not covered. In particular, the important question of finiteness of the limit (7) has not been addressed. This could potentially be done under more restrictive conditions on the service times. However, this is one of the very few results on this problem, important in view of its connection to the analysis of tandem networks with a large number of queues. Indeed, consider the queueing network of an infinite number of queues in tandem, indexed by $\{1, 2, 3, \cdots\}$. Customers leaving the kth queue join the $(k+1)$th one. The sequence $\{\sigma^k = \{\sigma_j^k, j \in \mathbb{N}\}, k \in \{1, 2, 3, \cdots\}\}$ is stationary with respect to the transformation $\theta\{\sigma^k\} = \{\sigma^{k+1}\}$. The previous sequence can be extended to the corresponding one $\{\sigma^k, k \in \mathbb{Z}\}$, which is again θ-stationary and agrees in distribution with the original one on the positive integers (see Doob [1]). This corresponds to appending fictitious queues to the system, indexed by negative integers. Initially $(t = 0)$, there is an infinite number of customers in the infinite-capacity buffer of the first queue and all the other queues are empty. We let the system evolve according to the tandem queueing discipline, the customers being dispersed to the following queues. This is equivalent to feeding customers to the first queue from an infinite-capacity pool, without ever letting that queue become idle. Following the notation used in the previous section, but working on the θ-stationary sequence $\{\sigma^k, k \in \mathbb{Z}\}$ now, we note that the sequence of interarrival times in the nth queue is Φ_2^n and is identically distributed to Φ_n^{-2}, due to the stationarity of $\{\sigma^k\}$. Therefore, we see that passing through a large number of consecutive queues $(n \to \infty)$, the sequence of interarrival times $\{\tau_j \in n, j \in \mathbb{N}\}$ eventually converges in distribution to the θ-invariant distribution P_Φ.

Concerning the problem posed in this note, the following general conjecture is plausible, supported by the result proven here. Consider a $G/G/1$ queue with sequence of inter-arrival and service times $\{(\tau_j, \sigma_j), j \in \mathbb{Z}\}$ being stationary and ergodic under the transformation $\theta^\#\{(\tau_j, \sigma_j), j \in \mathbb{Z}\} = \{(\tau_{j+1}, \sigma_{j+1}), j \in \mathbb{Z}\}$. As has been proven by Loynes [2], if $E[\sigma_0] < E[\tau_0]$, then there exists a unique finite, stationary sequence of waiting times $\{w_j, j \in \mathbb{Z}\}$ corresponding to $\{(\tau_j, \sigma_j), j \in \mathbb{Z}\}$ and thus a uniquely induced finite, stationary sequence of inter-departure times $\{\tau_j^*, j \in \mathbb{Z}\}$. Also, $E[\tau_0] = E[\tau_0^*]$. We conjecture that, given an arbitrarily fixed stationary and ergodic sequence of service times $\{\sigma_j, j \in \mathbb{Z}\}$ and a number $d > E[\sigma_0]$, there exists a stationary and ergodic sequence of inter-arrival times $\{\tau_j, j \in \mathbb{Z}\}$ with $E[\tau_0] = d$, such that the sequence of inter-departure times $\{\tau_j^*, j \in \mathbb{Z}\}$ is distributed as the sequence $\{\tau_j, j \in \mathbb{Z}\}$. This would be a direct generalization of the standard result about stable $M/M/1$ queues.

References

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