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Abstract

By constructing suitable Borcherds forms on Shimura curves and using Schofer's formula for norms of values of Borcherds forms at CM points, we determine all of the equations of hyperelliptic Shimura curves $X_0^D(N)$. As a byproduct, we also address the problem of whether a modular form on Shimura curves $X_0^D(N)/W_{D,N}$ with a divisor supported on CM divisors can be realized as a Borcherds form, where $X_0^D(N)/W_{D,N}$ denotes the quotient of $X_0^D(N)$ by all of the Atkin–Lehner involutions. The construction of Borcherds forms is done by solving certain integer programming problems.

1. Introduction

For an indefinite quaternion algebra B of discriminant D over $\mathbb{Q}$ and a positive integer N with $(D,N)=1$, we let $X_0^D(N)$ be the Shimura curve associated to an Eichler order $\mathcal{O}$ of level N in B. When $D=1$, the Shimura curve $X_0^D(N)$ is simply the classical modular curve $X_0(N)$, which is the coarse moduli space of elliptic curves together with a cyclic subgroup of order N and has been studied extensively in the literature. When $D>1$, the curve $X_0^D(N)$ is the coarse moduli space of principally polarized abelian surfaces with multiplication by $\mathcal{O}$. The arithmetic of such a Shimura curve is similar to those of classical modular curves, but the lack of cusps makes the Diophantine geometry and explicit calculation of such a Shimura curve more interesting and challenging than those of classical modular curves. The primary purpose of the present paper is to address the problem of determining equations of Shimura curves.

In the classical modular case, which has been studied extensively and is well known for admitting Fourier expansions around the cusps, there are many constructions of modular forms and modular functions, such as Eisenstein series, the Dedekind function, theta series, etc., and there are formulas for their Fourier expansions. Thus, it is often easy to determine equations of modular curves. We refer the reader to Galbraith [Gal96], Yang [Yan06], and the references contained therein for more information about equations of modular curves.

On the other hand, when $D \neq 1$, the absence of cusps has been an obstacle for explicit approaches to Shimura curves since modular forms or modular functions on Shimura curves do not have Fourier expansions and, as a result, most of the methods for classical modular curves cannot possibly be extended to the case of general Shimura curves. Up to now, only a few equations of Shimura curves are known. Ihara [Iha79] was perhaps the first to give defining equations of Shimura curves. For example, he found an equation for the curve $X_0^6(1)$ of genus zero. Kurihara [Kur79] extended Ihara's method and determined equations of $X_0^{10}(1)$ and $X_0^{22}(1)$

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of genus zero and $X_0^{14}(1)$, $X_0^{21}(1)$, and $X_0^{46}(1)$ of genus one. Jordan [Jor81] computed equations of two Shimura curves $X_0^{15}(1)$ and $X_0^{33}(1)$ of genus one. Later on, González and Rotger [GR04, GR06] completed the list of equations of Shimura curves $X_0^D(N)$ of genus one and two. For Shimura curves $X_0^D(N)$ of higher genus, Elkies [Elk08] found equations of Shimura curves $X_0^{57}(1)$ and $X_0^{206}(1)$ using the fact that some families of $K3$ surfaces are parameterized by Shimura curves. More recently, Molina [Mol12] found equations of $X_0^{39}(1)$ and $X_0^{55}(1)$ and Atkin–Lehner quotients of some Shimura curves. Also, González and Molina [GM16] determine equations of all Shimura curves $X_0^D(1)$ of genus three. (Note that it happens that all of these curves are hyperelliptic.) We remark that all of the methods in the above-mentioned works other than those in [Elk08] are strongly based on the Cerednik–Drinfeld theory of p-adic uniformization of Shimura curves [BC92], $p|D$, and arithmetic properties of CM points. In addition, other than [Elk08], their methods do not allow us to locate general CM points on the curves.

In this paper, we will adopt a very different approach, using the theory of Borcherds forms and explicit formulas for values of Borcherds forms at CM points to obtain equations of Shimura curves. (See § 2 for a quick introduction to Borcherds forms.) The main result of this paper is a complete list of equations of all hyperelliptic Shimura curves $X_0^D(N)$.

Theorem 1. *The tables in Appendix A give a complete list of equations of hyperelliptic Shimura curves $X_0^D(N)$, $D > 1$.*

The idea of realizing modular forms on Shimura curves as Borcherds forms is not new. For example, as a corollary to his formula for average values of Borcherds forms at CM points, Schofer [Sch09] proved a weak analogue of Gross and Zagier's result [GZ85] on the prime factorization of the norm of the difference of two singular moduli on the classical modular curve $X_0(1)$ for the case of Shimura curves. Later on, Errthum [Err11] applied Schofers formula to compute singular moduli on $X_0^6(1)/W_{6,1}$ and $X_0^{10}(1)/W_{10,1}$, verifying Elkies numerical computation [Elk98], where $W_{D,N}$ denotes the full Atkin–Lehner group on $X_0^D(N)$. However, applications of Borcherds forms to the theory of Shimura curves were not explored any further in the literature. One possible reason is that in order to successfully use Borcherds forms to perform computation on Shimura curves, one needs a systematic method to construct them in the first place, but such a method has not yet been developed in the literature. Thus, our first task here is to develop a systematic method to construct Borcherds forms. We will see that the problem of constructing Borcherds forms reduces to that of solving certain integer programming problems, which we solve by using the AMPL modeling language (http://www.ampl.com) and the Gurobi solver (http://www.gurobi.com).

Note that our method works for any Shimura curve $X_0^D(N)$ such that $X_0^D(N)/W_{D,N}$ has genus zero, but because there are too many of them, here we consider only the hyperelliptic cases. (There are more than 110 non-hyperelliptic Shimura curves $X_0^D(N)$ whose Atkin–Lehner quotient $X_0^D(N)/W_{D,N}$ has genus zero.) In addition, under a certain technical assumption (Assumption 40), it is also possible to determine equations of $X_0^D(N)/W_{D,N}$ even if it is not of genus zero. In § 4.3, we give two such examples. However, the method becomes less systematic and it is not clear whether it will always work in general.

In principle, our list of equations should also be obtainable using Elkies' approach [Elk08], but our approach via Borcherds forms have potential applications to other problems about Shimura curves beyond the scope of the present paper. To illustrate our point, in the arXiv version of the present paper (http://arxiv.org/pdf/1510.06193), we also discuss how our construction of Borcherds forms leads to a method to compute heights of CM points on Shimura curves, again under Assumption 40. Note that both Elkies' and our approaches have an advantage over other methods in that we can determine the coordinates of CM points on Shimura curves, although Elkies' approach sometimes involves exhaustive search and it is not clear whether the method is always guaranteed to work. In the arXiv version of the present paper, we also list the coordinates of the CM points used in determining our equations of Shimura curves.

The rest of the paper is organized as follows. In § 2, we give a quick overview of the theory of Borcherds forms and explain the idea of realizing modular forms on Shimura curves in terms of Borcherds forms. The exposition of this section follows [Yan15]. In § 3, we discuss how to construct Borcherds forms by solving certain integer programming problems. For our purpose, the case of odd D needs special attention. As a byproduct, we find that for (D, N) in Theorem 1 with even D, all meromorphic modular forms with divisors supported on CM divisors (Definition 21) can be realized as Borcherds forms. (We believe that this is also true for odd D, but since it is not the main problem we are concerned with, we will not prove this assertion here.) In § 4, we will give several examples illustrating how to obtain equations of Shimura curves using Borcherds forms we constructed in § 3 and Schofer's formula for values of Borcherds forms at CM points. In § 4.3, we give additional examples where the genus of $X_0^D(N)/W_{D,N}$ is not zero. Specifically, we determine equations of $X_0^{142}(1)/W_{142,1}$ and $X_0^{302}(1)/W_{302,1}$, under Assumption 40.

2. Borcherds forms

2.1 Basic theory

We give a quick introduction to Borcherds forms. For details, see [Bor98, Bor00, Bru02] for the classical setting and [Err11, Kud03, Sch09] for the adelic setting.

Let L be an even lattice with symmetric bilinear form $\langle \cdot\,,\cdot \rangle$ of signature $(n, 2)$ and let $L^\vee$ be the dual lattice of L. We assume that L is nondegenerate and denote by

$$\{e_\eta : \eta \in L^\vee/L\}$$

the standard basis for the group algebra $\mathbb{C}[L^\vee/L]$. Associated to the lattice L, we have a unitary *Weil representation* ρ_L of the metaplectic group

$$\widetilde{\mathrm{SL}}(2,\mathbb{Z}) = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \right\}$$

on the group algebra $\mathbb{C}[L^\vee/L]$ defined by

$$\rho_L(T)e_\eta = e^{-2\pi i\langle \eta,\eta\rangle/2} e_\eta,$$
$$\rho_L(S)e_\eta = \frac{e^{2\pi i(n-2)/8}}{\sqrt{|L^\vee/L|}} \sum_{\delta\in L^\vee/L} e^{2\pi i\langle \eta,\delta\rangle} e_\delta,$$

where

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \quad \text{and} \quad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right),$$

which generate $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$.

Definition 1. A holomorphic function $F : \mathfrak{H} \to \mathbb{C}[L^\vee/L]$ is called a weakly holomorphic vector-valued modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ and type ρ_L on $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ if it satisfies

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) F(\tau)$$

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for all $\tau \in \mathfrak{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ and F is meromorphic at the cusp ∞. The last condition means that the Fourier expansion of F is of the form

$$F(\tau) = \sum_{\eta \in L^\vee/L} \sum_{m \in \mathbb{Z} - \langle \eta, \eta \rangle/2,\ m > -m_0} c_\eta(m) q^m e_\eta, \quad q = e^{2\pi\tau},$$

for some rational number m_0.

For $k = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, let $V(k) = L \otimes k$ and extend the definition of $\langle \cdot, \cdot \rangle$ to $V(k)$ by linearity. Define $O_V(\mathbb{R})$ to be the orthogonal group of the bilinear form $\langle \cdot, \cdot \rangle$ and its subgroup

$$O_V^+(\mathbb{R}) := \{\sigma \in O_V(\mathbb{R}) : \operatorname{spin} \sigma = \operatorname{sgn} \det \sigma\},$$

where if σ is equal to the product of n reflections with respect to the vectors $v_1, \ldots, v_n$, then its spinor norm is defined by $\operatorname{spin} \sigma = (-1)^n \prod_{i=1}^n \operatorname{sgn}\langle v_i, v_i \rangle$. We also define

$$O_L^+ := \{\sigma \in O_V^+(\mathbb{R}) : \sigma(L) = L\}$$

to be the orthogonal group of the lattice L. As the orthogonal group O_L^+ acts on the dual lattice $L^\vee$, there is an induced operation on $\mathbb{C}[L^\vee/L]$ given by

$$\sum_{\eta \in L^\vee/L} c_\eta e_\eta \longmapsto \sum_{\eta \in L^\vee/L} c_\eta e_{\sigma\eta}, \quad \sigma \in O_L^+.$$

Definition 2. Suppose that $F = \sum_{\eta \in L^\vee/L} F_\eta e_\eta$ is a vector-valued modular form. We define the automorphism group $O_{L,F}^+$ of F by

$$O_{L,F}^+ = \{\sigma \in O_L^+ : F_{\sigma\eta} = F_\eta \text{ for all } \eta \text{ in } L^\vee/L\}.$$

Consider the subset

$$K = \{[z] \in \mathbb{P}(V(\mathbb{C})) : \langle z, z \rangle = 0, \langle z, \bar{z} \rangle < 0\}$$

of the projective space $\mathbb{P}(V(\mathbb{C}))$. This set K consists of two connected components and the orthogonal group $O_V^+(\mathbb{R})$ preserves the components. Pick one of them to be K^+. Then it can be checked that $O_V^+(\mathbb{R})$ acts transitively on K^+.

Definition 3. Let $\widetilde{K}^+ = \{z \in V(\mathbb{C}) : [z] \in K^+\}$. For each subgroup of Γ of finite index of O_L^+, we call a meromorphic function $\Psi : \widetilde{K}^+ \to \mathbb{P}(\mathbb{C})$ a modular form of weight k and character χ on Γ if Ψ satisfies:

(i) $\Psi(cz) = c^{-k}\widetilde{\Psi}(z)$ for all $c \in \mathbb{C}^*$ and $z \in \widetilde{K}$;
(ii) $\Psi(hz) = \chi(h)\widetilde{\Psi}(z)$ for all $h \in \Gamma$ and $z \in \widetilde{K}$.

Theorem A [Bor98, Theorem 13.3]. *Let L be an even lattice of signature $(n, 2)$ and $F(\tau)$ be a weakly holomorphic vector-valued modular forms of weigh $1 - n/2$ and type ρ_L with Fourier expansion $F(\tau) = \sum_\eta (\sum_n c_\eta(n) q^n) e_\eta$. Suppose that $c_\eta(n) \in \mathbb{Z}$ for any $\eta \in L^\vee/L$ and $n \leqslant 0$. Then there corresponds a meromorphic function $\Psi_F(z)$, $z \in \widetilde{K}^+$ with the following properties.*

(i) *Here $\Psi_F(z)$ is a meromorphic modular forms of weight $c_0(0)/2$ for the group $O_{L,F}^+$ with respect to some unitary character χ of $O_{L,F}^+$.*

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(ii) *The only zeros or poles of* $\Psi_F(z)$ *lie on the rational quadratic divisor*

$$\lambda^\perp = \{z \in \widetilde{K}^+ : \langle z, \lambda \rangle = 0\}$$

for λ *in* L, $\langle \lambda, \lambda \rangle > 0$, *and are of order*

$$\sum_{0<r\in\mathbb{Q}, r\lambda \in L^\vee} c_{r\lambda}(-r^2\langle \lambda, \lambda \rangle/2).$$

Definition 4. We call the function $\Psi_F(z)$ the *Borcherds form* associated to F.

2.2 Borcherds forms on Shimura curves

We now explain how to realize modular forms on Shimura curves as Borcherds forms. We follow the exposition in [Yan15]. See also [Err11].

Let B be an indefinite quaternion algebra of discriminant D over $\mathbb{Q}$. Consider the vector space

$$V = V(\mathbb{Q}) = \{x \in B : \mathrm{tr}(x) = 0\}$$

over $\mathbb{Q}$ with the natural bilinear form $\langle x, y \rangle = \mathrm{tr}(x\overline{y}) = -\mathrm{tr}(xy)$. Then V has signature $(1, 2)$ and the associated quadratic form is $\mathrm{nrd}(x) = -x^2$, where $\mathrm{nrd}(x)$ denotes the reduced norm of $x \in B$. Given an Eichler order $\mathcal{O}$ of level N in B, we let L be the lattice

$$L = \mathcal{O} \cap V = \{x \in \mathcal{O} : \mathrm{tr}(x) = 0\}.$$

For an invertible element β in $B \otimes \mathbb{R}$, define $\sigma_\beta : V(\mathbb{R}) \to V(\mathbb{R})$ by $\sigma_\beta(\gamma) = \beta\gamma\beta^{-1}$. Then, we can show that

$$O_V^+(\mathbb{R}) = \{\sigma_\beta : \beta \in (B \otimes \mathbb{R})^*/\mathbb{R}^*, \mathrm{nrd}(\beta) > 0\} \times \{\pm 1\}$$

and

$$O_L^+ = \{\sigma_\beta : \beta \in N_B^+(\mathcal{O})/\mathbb{Q}^*\} \times \{\pm 1\}.$$

If we assume that the quaternion algebra is represented by $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ with $a > 0$ and $b > 0$, that is, $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$ with $i^2 = a$, $j^2 = b$, and $ij = -ji$, and fix an embedding $\iota : B \hookrightarrow M(2, \mathbb{R})$ by

$$\iota : i \longmapsto \begin{pmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix},$$

then each class in $K = \{z \in \mathbb{P}(V(\mathbb{C})) : \langle z, z \rangle = 0, \langle z, \bar{z} \rangle < 0\}$ contains a unique representative of the form

$$z(\tau) = \frac{1-\tau^2}{2\sqrt{a}} i + \frac{\tau}{\sqrt{b}} j + \frac{1+\tau^2}{2\sqrt{ab}} ij$$

for some $\tau \in \mathfrak{H}^\pm$, the union of upper and lower half-plane. The mapping $\tau \mapsto z(\tau) \bmod \mathbb{C}^*$ is a bijection of between $\mathfrak{H}^\pm$ and K.

Let K^+ be the image of $\mathfrak{H}^+ = \mathfrak{H}$ under the mapping. Then we obtain compatible actions of $N_B^+(\mathcal{O})/\mathbb{Q}^*$ on K^* and $\mathfrak{H}$ with the action on K^+ by conjugation and the action on $\mathfrak{H}$ by linear fraction transformation. More precisely, this means that for $\alpha \in N_B^+(\mathcal{O})$, if we write $\iota(\alpha) = \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$, then

$$\alpha z(\tau) \alpha^{-1} = \frac{(c_3\tau + c_4)^2}{\mathrm{nrd}(\alpha)} z\left(\frac{c_1\tau + c_2}{c_3\tau + c_4}\right) \equiv z(\iota(\alpha)\tau) \mod \mathbb{C}^*. \tag{1}$$

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Lemma 5 [Yan15, Lemma 6]. *Let $F(\tau)=\sum_\eta(\sum_n c_\eta(n)q^n)e_\eta$ be a weakly holomorphic vector-valued modular form of weight $1/2$ and type ρ_L such that $O^+_{L,F}=O^+_L$ and $c_\eta(n)\in\mathbb{Z}$ whenever $\eta\in L^\vee/L$ and $n\leqslant 0$. Then the function $\psi_F(\tau)$ defined by $\psi_F(\tau)=\Psi_F(z(\tau))$ is a meromorphic modular forms of weight $c_0(0)$ with certain unitary character χ on the Shimura curve $X_0^D(N)/W_{D,N}$.*

Definition 6. With assumptions given as in the lemma, the function $\psi_F(\tau)$ defined by

$$\psi_F(\tau)=\Psi_F(z(\tau))$$

is called the *Borcherds forms* on the Shimura curve $X_0^D(N)/W_{D,N}$ associated to F.

The next lemma gives us the criterion when the character of a Borcherds form $\psi_F(\tau)$ is trivial, under the assumption that the genus of $N_B^+(\mathcal{O})\backslash\mathfrak{H}$ is zero.

Lemma 7 [Yan15, Lemma 8]. *Assume that the genus of $X=N_B^+(\mathcal{O})\backslash\mathfrak{H}$ is zero. Let $\tau_1,\dots,\tau_r$ be the elliptic points of X and assume that their orders are $b_1,\dots,b_r$, respectively. Assume further that, as* CM *points, the discriminant of $\tau_1,\dots,\tau_r$ are $d_1,\dots,d_r$, respectively. Let $F(\tau)=\sum_\eta(\sum_m c_\eta(n)q^m)e_\eta$ be a weakly holomorphic vector-valued modular form of weight $1/2$ and type ρ_L such that $O^+_{L,F}=O^+_L$ and $c_\eta(m)\in\mathbb{Z}$ whenever $\eta\in L^\vee/L$ and $m\leqslant 0$. Assume that $c_0(0)$ is even. Then the Borcherds form $\psi_F(\tau)$ is a modular form with trivial character on X if and only if for j such that $b_j\neq 3$, the order of $\Psi_F(z)$ at $z(\tau_j)$ has the same parity as $c_0(0)/2$.*

We now state Schofer's formula [Sch09, Corollaries 1.2 and 3.5] in the setting of Shimura curves as follows.

Theorem B [Sch09, Corollaries 1.2 and 3.5]. *Let $F(\tau)=\sum_\eta(\sum_m c_\eta(n)q^m)e_\eta$ be a weakly holomorphic vector-valued modular form of weight $1/2$ and type ρ_L for $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ such that $O^+_{L,F}=O^+_L$, $c_0(0)=0$ and $c_\eta(m)\in\mathbb{Z}$ whenever $\eta\in L^\vee/L$ and $n\leqslant 0$. Let $d<0$ be a fundamental discriminant such that the set* $\mathrm{CM}(d)$ *of CM points of discriminant d on $N_B^+(\mathcal{O})\backslash\mathfrak{H}$ is not empty and that the support of* $\operatorname{div}\psi(\tau)$ *does not intersect* $\mathrm{CM}(d)$. *Then we have*

$$\sum_{\tau\in\mathrm{CM}(d)}\log|\psi_F(\tau)|=-\frac{|\mathrm{CM}(d)|}{4}\sum_{\gamma\in L^\vee/L}\sum_{m\geqslant 0}c_\gamma(-m)\kappa_\gamma(m),$$

where $\kappa_\gamma(m)$ are certain sums involving derivatives of Fourier coefficients of some incoherent Eisenstein series.

We refer the reader to [Err11, Yan15] for strategies to compute $\kappa_\gamma(m)$ explicitly.

3. Construction of Borcherds forms

3.1 Errthum's method

In this section, we will review Errthum's method [Err11] for constructing vector-valued modular forms out of scalar-valued modular forms. Here the notation D, N, $\mathcal{O}$, L, etc. has the same meaning as in § 2.2. The level N is always assumed to be squarefree.

Let us first describe the structure of the lattice L.

Lemma 8. *Assume that N is squarefree. Let q be a prime number such that $q \equiv 1 \bmod 4$ and*

$$\left(\frac{q}{p}\right) = \begin{cases} -1 & \text{if } p|D, \\ 1 & \text{if } p|N. \end{cases} \tag{2}$$

Then $B = \left(\frac{DN,q}{\mathbb{Q}}\right)$ is a quaternion algebra of discriminant D over $\mathbb{Q}$. Moreover, let a be an integer such that $a^2DN \equiv 1 \bmod q$. Then the $\mathbb{Z}$-module $\mathcal{O}$ generated by

$$e_1 = 1, \quad e_2 = \frac{1+j}{2}, \quad e_3 = \frac{i+ij}{2}, \quad e_4 = \frac{aDNj+ij}{q} \tag{3}$$

is an Eichler order of level N in B. Also, let L be the set of elements of trace zero in $\mathcal{O}$ and let

$$\ell_1 = j, \quad \ell_2 = \frac{i+ij}{2}, \quad \ell_3 = \frac{aDNj+ij}{q}. \tag{4}$$

Then

$$L = \mathbb{Z}\ell_1 + \mathbb{Z}\ell_2 + \mathbb{Z}\ell_3, \quad L^\vee = \mathbb{Z}\frac{\ell_1}{2} + \mathbb{Z}\frac{\ell_2}{DN} + \mathbb{Z}\frac{\ell_3}{DN}.$$

Proof. The conditions in (2) imply that B is ramified at prime divisors of D and unramified at prime divisors of N. Also, by the quadratic reciprocity law, we have $\left(\frac{DN}{q}\right) = 1$. Thus, the discriminant of B is D.

We check that

$$\begin{aligned}
e_2^2 &= \frac{q-1}{4}e_1 + e_2, \\
e_2e_3 &= \frac{aDN(q-1)}{4}e_1 + \frac{aDN(1-q)}{2}e_2 + \frac{1-q}{2}e_3 + \frac{q(q-1)}{4}e_4, \\
e_2e_4 &= aDNe_1 - aDNe_2 - e_3 + \frac{q+1}{2}e_4, \\
e_3^2 &= \frac{DN(1-q)}{4}e_1, \\
e_3e_4 &= -\frac{DN(a^2DN(q-1)+q+1)}{2q}e_1 + \frac{DN(a^2DN(q-1)+1)}{q}e_2 \\
&\quad + aDNe_3 + \frac{aDN(1-q)}{2}e_4,
\end{aligned}$$

so that $\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4$ is an order in B. Also, the Gram matrix

$$(\operatorname{tr}(e_i\bar{e}_j)) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & (q-1)/2 & 0 & -aDN \\ 0 & 0 & DN(q-1)/2 & DN \\ 0 & -aDN & DN & 2DN(1-a^2DN)/q \end{pmatrix}$$

has determinant $(DN)^2$. Thus, it is an Eichler order of level N.

Moreover, it is clear that ℓ_1, ℓ_2 and ℓ_3 span L. Also, the Gram matrix of L with respect to this basis is

$$\begin{pmatrix} -2q & 0 & -2aDN \\ 0 & DN(q-1)/2 & DN \\ -2aDN & DN & 2DN(1-a^2DN)/q \end{pmatrix}, \tag{5}$$

and its determinant is $2D^2N^2$. From the Gram matrix of L, it is easy to check that $L^\vee$ is spanned by $\ell_1/2$, ℓ_2/DN and ℓ_3/DN. This proves the lemma. $\square$

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Corollary 9. *Assume that N is squarefree. The discriminant of the lattice L is*

$$|L^\vee/L| = 2(DN)^2$$

and the level of L is

$$\begin{cases} 4DN & \text{if } DN \text{ is odd,} \\ 2DN & \text{if } DN \text{ is even.} \end{cases}$$

Proof. The result follows directly from the proof of the previous lemma since the determinant of the Gram matrix in (5) is $2(DN)^2$ and $L^\vee/L \simeq (\mathbb{Z}/2) \times (\mathbb{Z}/DN)^2$. □

We now recall Errthum's method [Err11] for constructing weakly holomorphic vector-valued modular forms. Let χ_θ denote the character associated to the Jacobi theta function $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$. That is, χ_θ is defined by

$$\theta(\gamma\tau) = \chi_\theta(\gamma)(c\tau + d)^{1/2}\theta(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$ and all $\tau \in \mathfrak{H}$.

Lemma 10 [Bar03, Theorem 4.2.9]. *Let M be the level of the lattice L. Suppose that $f(\tau)$ is a weakly holomorphic scalar-valued modular form of weight $1/2$ such that*

$$f(\gamma\tau) = \chi_\theta(\gamma)(c\tau + d)^{1/2} f(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$. Then the function $F_f(\tau)$ defined by

$$F_f(\tau) = \sum_{\gamma \in \widetilde{\Gamma}_0(M)\backslash \widetilde{\mathrm{SL}}(2,\mathbb{Z})} f|_\gamma(\tau)\rho_L(\gamma^{-1})e_0 \tag{6}$$

is a weakly holomorphic vector-valued modular form of weight $1/2$ and type ρ_L.

Lemma 11 [Err11, Theorem 5.8]. *Let $f(\tau)$ and $F_f(\tau)$ be given as in the previous lemma. Then for η and $\eta' \in L^\vee/L$ with $\langle \eta, \eta\rangle = \langle \eta', \eta'\rangle$, the e_η component and $e_{\eta'}$ component of $F_f(\tau)$ are equal. Consequently, we have $O^+_{L,F_f} = O^+_L$.*

Lemma 12 [Bor00, Theorem 6.2]. *Let M be the level of the lattice L. Suppose that r_d, $d|M$, are integers satisfying the conditions:*

(i) $\sum_{d|M} r_d = 1$;
(ii) $|L^\vee/L| \prod_{d|M} d^{r_d}$ *is a square in* $\mathbb{Q}^*$;
(iii) $\sum_{d|M} dr_d \equiv 0 \bmod 24$; *and*
(iv) $\sum_{d|M} (M/d)r_d \equiv 0 \bmod 24$.

Then $\prod_{d|M} \eta(d\tau)^{r_d}$ is a weakly holomorphic scalar-valued modular form satisfying the condition for $f(\tau)$ in Lemma 10.

Definition 13. If an eta product satisfies the conditions in Lemma 12, then we say it is *admissible.*

To have a better control over the divisors of Borcherds forms constructed, we will use certain special admissible eta products.

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Definition 14. Let M be the level of the lattice L and let S be a subset of the cusps of $\Gamma_0(M)$. If f is a weakly holomorphic modular form of weight $1/2$ on $\Gamma_0(M)$ whose only poles are at the cusps in S, then we say f is a S-weakly holomorphic scalar-valued modular form of weight $1/2$ on $\Gamma_0(M)$.

Later on, we will use $\{\infty\}$-weakly holomorphic modular forms to construct Borcherds forms for the case of even D and $\{\infty, 0\}$-weakly holomorphic modular forms for the case of odd D. Therefore, let us introduce the following definitions.

Definition 15. Let D_0 be the odd part of DN. We let $M^!(4D_0)$ denote the space of all $\{\infty\}$-weakly holomorphic modular forms of weight $1/2$ on $\Gamma_0(4D_0)$. Also, for a nonnegative integer n, let $M^!_n(4D_0)$ be the subspace of $M^!(4D_0)$ consisting of modular forms with a pole of order at most n at ∞. If j is a positive integer such that there does not exist a modular form in $M^!(4D_0)$ with a pole of order j at ∞, then we say j is a *gap* of $M^!(4D_0)$.

Similarly, we let $M^{!,!}(4D_0)$ be the space of all $\{\infty, 0\}$-weakly holomorphic modular forms of weight $1/2$ on $\Gamma_0(4D_0)$. For nonnegative integers m and n, let $M^{!,!}_{m,n}(4D_0)$ be the subspace of $M^{!,!}(4D_0)$ consisting of modular forms with a pole of order at most m at ∞ and a pole of order at most n at zero.

Remark 16. Note that the space $M^!_0(4D_0)$ is simply the space of holomorphic modular forms of weight $1/2$ on $\Gamma_0(4D_0)$. Since D_0 is assumed to be squarefree, by [SS77, Theorem A], the space $M^!_0(4D_0)$ is one-dimensional and spanned by $\theta(\tau)$.

3.2 Case of even D

In this section, we assume that D is even and N is squarefree. In Proposition 18, we will see how the problem of constructing Borcherds forms becomes the problem of solving certain integer programming problem. Ultimately, in Proposition 23, we will show that for (D, N) in Theorem 1 with $2|D$, every meromorphic modular form of even weight on $X_0^D(N)/W_{D,N}$ with divisor supported on CM divisors (see Definition 21) can be realized as a Borcherds form. Note that Bruinier [Bru14] and Heim and Murase [HM15] studied when a modular form on an orthogonal group $O(n, 2)$ can be realized as a Borcherds form, but as the integer n is assumed to be at least two, their results do not apply to the case of Shimura curves. In fact, it is pointed out in [Bru14, § 1] that counterexamples exist in the case $n = 1$ (see also [BO10, § 8.3]). It will be a very interesting problem to characterize those modular forms on Shimura curves $X_0^D(N)/W_{D,N}$ that can be realized as Borcherds forms.

Let D_0 be the odd part of DN. Then according to Corollary 9, the level of the lattice under consideration is $4D_0$. Let us first determine the dimensions of $M^!_n(4D_0)$.

Lemma 17. *Let D_0 be the odd part of DN and g be the genus of the modular curve $X_0(4D_0)$. Then for a nonnegative integer n with*

$$n \geqslant 2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor,$$

we have

$$\dim_{\mathbb{C}} M^!_n(4D_0) = n + \sum_{d|D_0} \lfloor d/4 \rfloor + 1 - g.$$

Moreover, the number of gaps of $M^!(4D_0)$ is $g - \sum_{d|D_0} \lfloor d/4 \rfloor$.

Proof. Let $\theta(\tau)=\sum_n q^{n^2}$ be the Jacobi theta function. For a divisor d of $4D_0$, let C_d represent the cusp $1/d$. As a modular form on $\Gamma_0(4D_0)$, we have

$$\operatorname{div}\theta=\sum_{d|D_0}\frac{d}{4}(C_{2d}).$$

A modular form f is contained in $M_n^!(4D_0)$ if and only if the modular function $g=f/\theta$ on $\Gamma_0(4D_0)$ satisfies

$$\operatorname{div} g\geqslant -n(\infty)-\sum_{d|D_0}\lfloor d/4\rfloor(C_{2d}).$$

Then by the Riemann–Roch theorem, when n is a nonnegative integer such that $n\geqslant 2g-2-\sum_{d|D_0}\lfloor d/4\rfloor$, the dimension of the space $M_n^!(4D_0)$ is

$$n+\sum_{d|D_0}\lfloor d/4\rfloor+1-g.$$

Now since D_0 is squarefree, by [SS77, Theorem A], the space $M_0^!(4D_0)$ is one-dimensional and spanned by θ, which implies that there is no modular form in $M^!(4D_0)$ having a zero at ∞. Therefore, from the dimension formula for $M_n^!(4D_0)$, we see that the number of gaps is $n+1-\dim M_n^!(4D_0)=g-\sum_{d|D_0}\lfloor d/4\rfloor$. □

PROPOSITION 18. *For (D,N) in Theorem 1 with even D, the space $M^!(4D_0)$ is spanned by admissible eta products. Moreover, there exists a positive integer m such that, for each positive integer $j\geqslant m$, there exists a modular form f_j in $M^!(4D_0)\cap\mathbb{Z}((q))$ whose order of pole at ∞ is j and whose leading coefficient is one.*

Proof. Let g be the genus of the modular curve $X_0(4D_0)$ and set

$$n_0=\max\biggl(2g-2-\sum_{d|D_0}\lfloor d/4\rfloor,0\biggr).$$

According to Lemma 17, if n is an integer such that $n\geqslant n_0$, then there exists a modular form in $M^!(4D_0)$ with a pole of order n at ∞. Now suppose that we can find an eta product $t(\tau)$ such that t is a modular function on $\Gamma_0(4D_0)$ with a unique pole at ∞. Let k be the order of the pole of t at ∞. Now Lemma 17 implies that for each integer $j\geqslant n_0$, there is a modular form in $M^!(4D_0)$ with a pole of order j at ∞. Thus, for all $n\geqslant n_0$, we have

$$M_{n+k}^!(4D_0)=M_n^!(4D_0)+tM_n^!(4D_0).$$

Therefore, to prove the assertion about $M^!(4D_0)$, it suffices to find such a modular function t and show that the space $M_{n_0+k}^!(4D_0)$ can be spanned by eta products and that there exists a positive integer $m\geqslant n_0$ such that for each integer j with $m\leqslant j\leqslant m+k-1$, there exists a modular form in $M_j^!(4D_0)\cap\mathbb{Z}((q))$ whose order of pole at ∞ is j and whose leading coefficient is one.

Consider the case of a maximal order first. Assume that $D=2p$ for some odd prime p. By Lemma 12, for an eta product $\prod_{d|4p}\eta(d\tau)^{r_d}$ to be admissible, the integers r_d must satisfy

$$\begin{array}{ccccccccccccc}
r_1 & + & r_2 & + & r_4 & + & r_p & + & r_{2p} & + & r_{4p} & = & 1\\
 & & r_2 & & & & & + & r_{2p} & & & = & 1+2\delta_2\\
 & & & & & & r_p & + & r_{2p} & + & r_{4p} & = & 2\delta_p\\
r_1 & + & 2r_2 & + & 4r_4 & + & pr_p & + & 2pr_{2p} & + & 4pr_{4p} & = & 24\epsilon_1\\
4pr_1 & + & 2pr_2 & + & pr_4 & + & 4r_p & + & 2r_{2p} & + & r_{4p} & = & 24\epsilon_2
\end{array}\qquad(7)$$

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for some integers δ_2, δ_p, ϵ_1 and ϵ_2. Moreover, the congruence subgroup $\Gamma_0(4p)$ has six cusps, represented by $1/c$ with $c|4p$. The orders of the eta function $\eta(d\tau)$ at these cusps, multiplied by 24, are given by the following.

	1	$1/2$	$1/4$	$1/p$	$1/2p$	$1/4p$
$\eta(\tau)$	$4p$	p	p	4	1	1
$\eta(2\tau)$	$2p$	$2p$	$2p$	2	2	2
$\eta(4\tau)$	p	p	$4p$	1	1	4
$\eta(p\tau)$	4	1	1	$4p$	p	p
$\eta(2p\tau)$	2	2	2	$2p$	$2p$	$2p$
$\eta(4p\tau)$	1	1	4	p	p	$4p$

Thus, in order for an eta product to be in $M_n^!(4p)$, the exponents r_d should satisfy

$$\begin{array}{rcrcrcrcrcrcl} r_1 & + & 2r_2 & + & 4r_4 & + & pr_p & + & 2pr_{2p} & + & 4pr_{4p} & \geqslant & -24n \\ r_1 & + & 2r_2 & + & r_4 & + & pr_p & + & 2pr_{2p} & + & pr_{4p} & \geqslant & 0 \\ 4r_1 & + & 2r_2 & + & r_4 & + & 4pr_p & + & 2pr_{2p} & + & pr_{4p} & \geqslant & 0 \\ pr_1 & + & 2pr_2 & + & 4pr_4 & + & r_p & + & 2r_{2p} & + & 4r_{4p} & \geqslant & 0 \\ pr_1 & + & 2pr_2 & + & pr_4 & + & r_p & + & 2r_{2p} & + & r_{4p} & \geqslant & 0 \\ 4pr_1 & + & 2pr_2 & + & pr_4 & + & 4r_p & + & 2r_{2p} & + & r_{4p} & \geqslant & 0 \end{array} \tag{8}$$

In literature, problems of solving a set of equalities and inequalities in integers are called *integer programming problems*. Solving (7) and (8) using the AMPL modeling language (http://www.ampl.com) and the gurobi solver (http://www.gurobi.com), we can produce many admissible eta products.

To find t, we replace the first two equations in (7) by $r_1 + r_2 + r_4 + r_p + r_{2p} + r_{4p} = 0$ and $r_2 + r_{2p} = 2\delta_2$ and solve the integer programming problem. We find that we can choose

$$t(\tau) = \frac{\eta(4\tau)^4\eta(2p\tau)^2}{\eta(2\tau)^2\eta(4p\tau)^4}$$

with $k = (p-1)/2$.

In the other cases when $N > 1$, N is always a prime. The modular curve $X_0(4D_0)$ has 12 cusps and there are more inequalities and equalities in the integer programming problem. Nevertheless, we can easily find t and many admissible eta products by solving the integer programming problem.

Having found $t(\tau)$ and many admissible eta products, we check case by case that eta products do span $M_{n_0+k}^!(4p)$ and that there exists positive integer $m \geqslant n_0$ such that for each integer j with $m \leqslant j \leqslant m+k-1$, there exists a modular form $f_j \in M_{n_0+k}^!(4D_0) \cap \mathbb{Z}((q))$ whose order of pole at ∞ is j and whose leading coefficient is one. (Sometimes, f_j will be a linear combination of eta products with rational coefficients. To show that all Fourier coefficients are integers, we use Sturm's theorem.) Here we omit the details, providing only one example as below. □

Example 19. Consider the case $D = 26$ and $N = 1$. The modular curve $X_0(52)$ has genus five. Thus, by Lemma 17, the number of gaps of $M^!(52)$ is $5 - \sum_{d|13} \lfloor d/4 \rfloor = 2$. The modular function

$$t(\tau) = \frac{\eta(4\tau)^4\eta(26\tau)^2}{\eta(2\tau)^2\eta(52\tau)^4}$$

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has a unique pole of order six at ∞. According to the proof of Proposition 18, we need to show that the space $M^!_{11}(52)$ can be spanned by eta products. Using the gurobi solver, we find the following solutions $(r_1, r_2, r_4, r_{13}, r_{26}, r_{52})$ to the integer programming problem in (7) and (8) with $n = 11$

$$\begin{gathered}(-3,6,0,-3,11,-10),\ (-1,3,1,3,2,-7),\ (3,-3,3,-1,8,-9),\ (1,1,1,1,4,-7),\\ (-1,1,1,3,4,-7),\ (0,-3,6,-2,8,-8),\ (3,-1,1,-1,6,-7),\ (-5,12,-4,-1,5,-6),\\ (-1,2,0,-5,15,-10),\ (1,-1,1,1,6,-7),\ (1,3,-1,1,2,-5),\ (-1,3,-1,3,2,-5),\\ (3,1,-1,-1,4,-5),\ (-2,3,2,0,2,-4),\ (0,-1,2,-2,6,-4),\ (-2,5,-2,0,0,0).\end{gathered}$$

Suitable linear combinations of these eta products $\prod_{d|52} \eta(d\tau)^{r_d}$ yield a basis consisting of

$$\begin{aligned} f_0 &= 1 + 2q + 2q^4 + 2q^9 + \cdots, & f_3 &= q^{-3} + q^{-1} + q^3 + q^9 + \cdots, \\ f_4 &= q^{-4} - q^{-1} - q + q^3 + \cdots, & f_5 &= q^{-5} + q^{-2} - 2q + q^2 + \cdots, \\ f_6 &= q^{-6} + q^{-2} - 2q + 2q^2 + \cdots, & f_7 &= q^{-7} - q^{-2} + 2q - q^2 + \cdots, \\ f_8 &= q^{-8} + q^{-2} + q^2 + 2q^5 + \cdots, & f_9 &= q^{-9} + 2q^{-1} + 3q + 2q^3 + \cdots, \\ f_{10} &= q^{-10} + 3q^{-1} + q - q^3 + \cdots, & f_{11} &= q^{-11} + 2q^2 + q^5 + 4q^7 + \cdots, \end{aligned}$$

for the space $M^!_{11}(52)$. In fact, since all of these modular forms have integral coefficients, multiplying these f_j by powers of t, we find that for each non-gap positive integer j, there exists a modular form f_j in $M^!(52) \cap \mathbb{Z}((q))$ with a pole of order j at ∞ and a leading coefficient of one.

Remark 20. Quite curiously, our computation shows that whenever $N = 1$, i.e. whenever $D_0 = p$ is an odd prime, the space $M^!(4D_0)$ has the property that for each non-gap positive integer j, there exists a modular form f in $M^!(4D_0) \cap \mathbb{Z}((q))$ such that f has a pole of order j at ∞ with leading coefficient one.

The smallest D_0 such that $M^!(4D_0)$ does not have this property is $D_0 = 51$. We can show that the gaps of $M^!(204)$ are $1, \ldots, 14$, and 20 and there exists a modular form f in $M^!(204) \cap \mathbb{Z}((q))$ with a Fourier expansion $2q^{-22} - q^{-20} - 2q^{-14} + 2q^{-12} + \cdots$. As 20 is a gap, there cannot exist $g \in M^!(204) \cap \mathbb{Z}((q))$ with a Fourier expansion $q^{-22} + \cdots$.

We now show that for (D, N) in Theorem 1 with even D, all meromorphic modular forms of even weights on $X_0^D(N)/W_{D,N}$ with divisors supported on CM divisors, which we define below, can be realized as Borcherds forms.

Definition 21. For a negative discriminant d, we let $\mathrm{CM}(d)$ denote the set of CM points of discriminant d on $X_0^D(N)/W_{D,N}$, $h_d = |\mathrm{CM}(d)|$, and P_d be the divisor

$$P_d = \sum_{\tau \in \mathrm{CM}(d)} \tau.$$

(If $h_d = 0$, then P_d simply means zero.) We call P_d the CM *divisor* of discriminant d. Note that sometimes we wish to keep track the degree of the divisor P_d. In such as a case, we will write $P_d^{\times h_d}$ instead of P_d.

Lemma 22. *Let f be an element in $M^!(4D_0) \cap \mathbb{Z}((q))$, F_f be the vector-valued modular form constructed using f as given by (6), and $\psi_{F_f}(\tau)$ be the Borcherds form on $X_0^D(N)/W_{D,N}$*

corresponding to F_f as defined in Definition 6. Suppose that the Fourier expansion of f is $\sum_m c_m q^m$. Then

$$\operatorname{div}\psi_{F_f} = \sum_{m<0} c_m \sum_{r\in\mathbb{Z}^+, 4m/r^2 \text{ is a discriminant}} \frac{1}{e_{4m/r^2}} P_{4m/r^2},$$

where e_d is the cardinality of the stabilizer subgroup of $\tau \in \mathrm{CM}(d)$ in $N_B^+(\mathcal{O})/\mathbb{Q}^*$.

Proof. This follows from [Err11, Proposition 5.4] and [Yan15, Lemma 7]. □

Proposition 23. *For (D, N) in Theorem 1 with $2|D$, all meromorphic modular forms of even weights on $X_0^D(N)/W_{D,N}$ with a divisor supported on CM divisors can be realized as Borcherds forms.*

Proof. We will prove only the case $(D, N) = (26, 1)$. The proof of the other cases is similar.

We claim that:

(i) there is a Borcherds form ψ of weight two with a trivial character; and

(ii) every modular function on $X_0^{26}(1)/W_{26,1}$ with divisor supported on CM divisors can be realized as a Borcherds form.

Then observe that if ϕ is a modular form of even weight k, then $\phi/\psi^{k/2}$ has weight zero. The two claims imply that ϕ can be realized as a Borcherds form.

The Shimura curve $X_0^{26}(1)/W_{26,1}$ has genus zero and precisely five elliptic points of order two. Among the five elliptic points, one is a CM point of discriminant -8, one is a CM point of discriminant -52 and the remaining three are CM points of discriminant -104. Also, if ψ is a meromorphic modular form of even weight k on $X_0^{26}(1)/W_{26,1}$, then the degree of $\operatorname{div}\psi$ is $k/4$. Thus, by Lemmas 7 and 22, for $f = \sum_m c_m q^m \in M^!(52) \cap \mathbb{Z}((q))$, the Borcherds form ψ_{F_f} has even weight k and a trivial character if and only if

$$\sum_{m<0} c_m \sum_{r\in\mathbb{Z}^+, 4m/r^2 \text{ is a discriminant}} \frac{1}{e_{4m/r^2}} |\mathrm{CM}(4m/r^2)| = k/4 \tag{9}$$

and

$$\sum_{m=-2n^2} c_m \equiv \sum_{m=-13n^2} c_m \equiv \sum_{m=-26n^2} c_m \equiv k/2 \mod 2. \tag{10}$$

Now from Example 19, we know that for each $j \geqslant 3$, there exists a unique element f_j in $M^!(52) \cap \mathbb{Z}((q))$ such that its Fourier expansion is of the form $f_j = q^{-j} + c_{-2}q^{-2} + c_{-1}q^{-1} + \cdots$. In particular, we find

$$\begin{aligned} f_7 &= q^{-7} - q^{-2} + 2q + \cdots, \\ f_{13} &= q^{-13} - q^{-2} - 2q^{-1} + q + \cdots, \\ f_{26} &= q^{-26} + q^{-1} - q + \cdots. \end{aligned}$$

The modular form

$$f = f_{26} - f_{13} + 2f_7 = q^{-26} - q^{-13} + 2q^{-7} - q^{-2} + 3q^{-1} + 2q + \cdots,$$

satisfies the conditions in (9) and (10) with $k = 2$. (Note that no CM points of discriminants -4 and -7 exist on the Shimura curve $X_0^{26}(1)$, so the presence of the terms q^{-7} and q^{-1} will not contribute anything to the divisor of the Borcherds form.) This proves claim (i).

To prove claim (ii), it suffices to show that for each discriminant $d<0$, there exists a modular form f in $M^!(52)\cap\mathbb{Z}((q))$ satisfying (9) and (10) with $k=0$ such that $\operatorname{div}\psi_{F_f}=P_d^{\times h_d}-h_dP_{-8}$. For the special cases $d=-52$ and $d=-104$, we may choose f to be $2f_{13}$ and $2f_{26}+6f_7$, respectively. If $d\neq -52,-104$ and d is a fundamental discriminant, we choose f to be $f_{|d|}+af_7$ with a proper integer a such that the coefficient of q^{-2} is $-2h_d$. (If $4|d$, we may choose $f_{|d|/4}+bf_7$ instead.) Now assume that d is not a fundamental discriminant, say, $d=d_0n^2$ for some fundamental discriminant d_0. We let a be the integer such that the coefficient of q^{-2} in $f=\sum_{r|n}\mu(r)f_{|d|/r^2}+af_7$ is $-2h_d$, where $\mu(r)$ is the Möbius function. Then $\operatorname{div}\psi_{F_f}=P_d^{\times h_d}-h_dP_{-8}$. (The case $d_0=-8$ needs a special treatment, but it is completely analogous.) This proves claim (ii) and, hence, the proposition for the case $(D,N)=(26,1)$. □

3.3 Case of odd D

The construction of Borcherds forms in the case of odd D is a little more complicated than the case of even D. The idea of using $\{\infty\}$-weakly holomorphic modular forms to construct Borcherds forms is no longer sufficient for our purpose. The reason is that if the divisor of a Borcherds form arising from a $\{\infty\}$-weakly holomorphic modular form is supported at a CM point of discriminant d, $d\equiv 1 \bmod 4$, then it also is supported at CM points of discriminant $4d$. However, in practice, we are often required to construct Borcherds forms whose divisors are supported at CM points of discriminant d, but not at CM points of discriminant $4d$. Thus, in the case of odd D, we will need to use $\{\infty,0\}$-weakly holomorphic modular forms to construct desired Borcherds forms.

Assume that D is odd and N is squarefree. As usual, we let $\mathcal{O}$ be an Eichler order of level N in the quaternion algebra B of discriminant D, and L be the lattice formed by elements of trace 0 in $\mathcal{O}$. For convenience, for a modular form f, we let $P(f)$ denote the principal part of f at ∞, i.e. the sums of the terms with negative exponents in the Fourier expansion of f. Similarly, for a vector-valued modular form $F=\sum_{\eta\in L^\vee/L}F_\eta e_\eta$, we let

$$P(F)=\sum_\eta P(F_\eta)e_\eta.$$

Lemma 24. *Let M be the level of L. Suppose that f is a $\{\infty,0\}$-weakly holomorphic scalar-valued modular of weight $1/2$ on $\Gamma_0(M)$ and F_f was given in Lemma 10. Assume that $P(f|_{1/2}S)=\sum_{n>0}b_nq^{-n/M}$. Then*

$$P(F_f)=P(f)e_0+\frac{Me^{2\pi i/8}}{\sqrt{|L^\vee/L|}}\sum_{n>0}b_nq^{-n/M}\sum_{\eta\in L^\vee/L,\operatorname{nrd}(\eta)\in n/M+\mathbb{Z}}e_\eta.$$

Proof. Since f is of $\{\infty,0\}$-weakly holomorphic, if γ is an element of $\mathrm{SL}(2,\mathbb{Z})$ such that $\gamma\infty$ is not equivalent to the cusp ∞ or zero, then we have $P(f|_{1/2}\gamma)=0$. Now $\gamma=I$ is the only right coset representative of $\Gamma_0(M)$ in $\mathrm{SL}(2,\mathbb{Z})$ with $\gamma\infty\sim\infty$ and $\gamma=ST^j$, $j=0,\ldots,M-1$, are the only right coset representatives with $\gamma\infty\sim 0$. Thus,

$$P(F_f)=P(f)e_0+\sum_{j=0}^{M-1}P(f|_{1/2}ST^j)\rho_L(T^{-j}S^{-1})e_0.$$

Since

$$\rho_L(S^{-1})e_\delta=\frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}}\sum_{\eta\in L^\vee/L}e^{-\langle\eta,\delta\rangle}e_\eta,$$

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we find

$$\begin{aligned}
P(F_f) &= P(f)e_0 + \frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}} \sum_{j=0}^{M-1} P(f|_{1/2}ST^j)\rho_L(T^{-j}) \sum_{\eta\in L^\vee/L} e_\eta \\
&= P(f)e_0 + \frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}} \sum_{n>0} b_n q^{-n/M} \sum_{j=0}^{M-1} \sum_{\eta\in L^\vee/L} e^{2\pi i j(-n/M+\mathrm{nrd}(\eta))} e_\eta \\
&= P(f)e_0 + \frac{Me^{2\pi i/8}}{\sqrt{|L^\vee/L|}} \sum_{n>0} b_n q^{-n/M} \sum_{\eta\in L^\vee/L,\mathrm{nrd}(\eta)\in n/M+\mathbb{Z}} e_\eta.
\end{aligned}$$

This proves the lemma. □

In general, the principal part $e^{2\pi i/8}|L^\vee/L|^{-1/2}P(f|_{1/2}S)$ in the lemma lie in $\mathbb{C}[q^{-1/M}]$. For our purpose, we will only consider those f such that

$$P(f)\in\mathbb{Z}[q^{-1}], \quad \frac{Me^{2\pi i/8}}{\sqrt{|L^\vee/L|}}P(f|_{1/2}S)\in\mathbb{Z}[q^{-1/4}].$$

LEMMA 25. *Let f be as in the lemma above. Suppose that $P(f)$ and $P(f|_{1/2}S)$ are of the form*

$$P(f)=\sum_{n>0,n\in\mathbb{Z}} a_n q^{-n}, \quad \frac{Me^{2\pi i/8}}{\sqrt{|L^\vee/L|}}P(f|_{1/2}S)=\sum_{n>0,n\in\mathbb{Z}} b_n q^{-n/4}$$

for some integers a_n and b_n. Then

$$\begin{aligned}
\operatorname{div}\psi_{F_f} &= \sum_n a_n \sum_{r\in\mathbb{Z}^+,-4n/r^2 \text{ is a discriminant}} \frac{1}{e_{-4n/r^2}} P_{-4n/r^2} \\
&\quad + \sum_n b_n \sum_{r\in\mathbb{Z}^+,-N^2n/r^2 \text{ is a discriminant}} \frac{1}{e_{-N^2n/r^2}} P_{-N^2n/r^2},
\end{aligned}$$

where e_d is the cardinality of the stabilizer subgroup of a CM point of discriminant d in $N_B^+(\mathcal{O})/\mathbb{Q}^*$.

Proof. Let q be a prime satisfying the condition in Lemma 8 so that $B=\left(\frac{DN,q}{\mathbb{Q}}\right)$ is a quaternion algebra of discriminant D. Let $\mathcal{O}$ be the Eichler order of level N spanned by $e_1,\dots,e_4$ given in (3) and $\{\ell_1,\ell_2,\ell_3\}$ be given as in (4). The contribution from $P(f)e_0$ to the divisor of ψ_{F_f} is described in Lemma 22. Here we are mainly concerned with the contribution from $P(f|S)$.

Consider the case of odd N first. Let λ be an element in $L^\vee=\mathbb{Z}\ell_1/2+\mathbb{Z}\ell_2/DN+\mathbb{Z}\ell_3/DN$ satisfying $\mathrm{nrd}(\lambda)=n/4$ for some positive integer n. We need to determine the discriminant of the optimal embedding $\phi:\mathbb{Q}(\sqrt{-n})\hookrightarrow B$ that maps $\sqrt{-n}$ to 2λ.

Observe that $2DN\lambda\in\mathcal{O}$ and $\mathrm{nrd}(2DN\lambda)=-D^2N^2n$. By [AB04, Proposition 1.53], we must have $2N\lambda\in\mathcal{O}$, i.e. $\lambda=c_1\ell_1/2+c_2\ell_2/N+c_3\ell_3/N$ for some integers c_1, c_2 and c_3, and the discriminant of the optimal embedding ϕ is $-4N^2n/r^2$ for some integer r.

From the Gram matrix in (5), we have

$$\mathrm{nrd}(N\lambda)=-\frac{qN^2c_1^2}{4}+\frac{q-1}{4}DNc_2^2+\frac{1-a^2DN}{q}DNc_3^2-aDN^2c_1c_3+DNc_2c_3.$$

As q is congruent to 1 modulo 4, this shows that $\mathrm{nrd}(2N\lambda) \equiv 0, 3 \bmod 4$. Therefore, if $n \equiv 1$, $2 \bmod 4$, then there does not exist $\lambda \in L^\vee$ such that $\mathrm{nrd}(\lambda) = n/4$. Also, if $n \equiv 3 \bmod 4$, then c_1 must be odd and

$$\frac{1+N\lambda}{2} = \frac{1-Nc_1}{2}e_1 + Nc_1e_2 + c_2e_3 + c_3e_4 \in \mathcal{O}.$$

In this case, the discriminant of the optimal embedding is $-N^2n/r^2$ for some r. If $n \equiv 0 \bmod 4$, then c_1 is even. It follows that $N\lambda \in \mathcal{O}$ and the optimal embedding has discriminant $-N^2n/r^2$ for some r.

Conversely, given a CM point τ of discriminant $-N^2n/r^2$, there exists an element $\lambda = d_1\ell_1 + d_2\ell_2 + d_3\ell_3 \in L$ fixing τ and having norm

$$\mathrm{nrd}(\lambda) = \begin{cases} -N^2n/4 & \text{if } n \equiv 0 \bmod 4, \\ -N^2n & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

Note that if n is odd, then we must have $(1+\lambda)/2 \in \mathcal{O}$. In other words, d_2 and d_3 are even and d_1 is odd. On the other hand,

$$\mathrm{nrd}(\lambda) = -qd_1^2 + \frac{q-1}{4}DNd_2^2 + \frac{1-a^2DN}{q}DNd_3^2 - 2aDNd_1d_3 + DNd_2d_3.$$

Since N is squarefree, this implies that $N|d_1$. Setting

$$\lambda' = \begin{cases} \lambda/N & \text{if } n \equiv 0 \bmod 4, \\ \lambda/(2N) & \text{if } n \equiv 3 \bmod 4, \end{cases}$$

we find $\lambda' \in L^\vee$ with $\mathrm{nrd}(\lambda') = n/4$. This proves the lemma for the case of odd N. The proof of the case of even N is similar and is omitted. □

Lemma 26. *Let M be the level of the lattice L and let $f(\tau) = \prod_{d|M} \eta(d\tau)^{r_d}$ be an admissible eta product. (See Definition 13.) Then we have*

$$\frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}}(f|_{1/2}S)(\tau) = \frac{1}{\sqrt{|L^\vee/L|}} \prod_{d|M} \frac{1}{d^{r_d/2}} \eta(\tau/d)^{r_d} \in \mathbb{Q}((q^{1/M})).$$

Proof. The lemma follows immediately from the formula $\eta(-1/\tau) = e^{-2\pi i/8}\sqrt{\tau}\eta(\tau)$ and the assumptions that $\sum r_d = 1$ and that $|L^\vee/L| \prod_{d|M} d^{r_d}$ is a square in $\mathbb{Q}^*$. □

Lemma 27. *Let D_0 be the odd part of DN and g be the genus of the modular curve $X_0(4D_0)$.*

(1) *For nonnegative integers m and n with*

$$m + n \geqslant 2g - 2 - \sum_{d|D_0} \lfloor d/4 \rfloor,$$

we have

$$\dim_{\mathbb{C}} M^{!,!}_{m,n}(4D_0) = m + n + \sum_{d|D_0} \lfloor d/4 \rfloor + 1 - g.$$

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(2) *Let m be a nonnegative integers such that $m \geqslant 2g-2-\sum_{d|D_0}\lfloor d/4\rfloor$. Then for each positive integer n, there exists a modular form f_n in $M^{!,!}_{m,n}(4D_0)$ with a pole of order n at 0. Furthermore, the space $M^{!,!}(4D_0)$ is spanned by $M^!(4D_0)$ and $f_1, f_2, \ldots$.*

Proof. The proof of part (1) is similar to that of Lemma 17 and is omitted. To prove part (2), we note that part (1) implies that when $m \geqslant 2g-2-\sum_{d|D_0}\lfloor d/4\rfloor$, the space $M^{!,!}_{m,0}(4D_0)$ has co-dimension n in $M^{!,!}_{m,n}(4D_0)$. It follows that for each integer k with $1 \leqslant k \leqslant n$, there exists a modular form f_k in $M^{!,!}_{m,n}(4D_0)$ with a pole of order k at zero. Now if f is a modular form in $M^{!,!}(4D_0)$, then for some linear combination $\sum c_n f_n$, we have $f - \sum c_n f_n \in M^!(4D_0)$. This proves part (2). $\square$

PROPOSITION 28. *For (D, N) in Theorem 1 with odd D and squarefree N, the space $M^{!,!}(4D_0)$ is spanned by admissible eta products. Moreover, if $f(\tau) \in M^{!,!}(4D_0) \cap \mathbb{Q}((q))$, then*

$$\frac{e^{2\pi i/8}}{\sqrt{|L^\vee/L|}}(f|_{1/2}S)(\tau) \in \mathbb{Q}((q^{1/(4D_0)})).$$

Proof. Suppose that we can find an eta product $t(\tau)$ such that $t(\tau)$ is a modular function on $X_0(4D_0)$ with a unique pole at ∞. Let k be the order of pole of $t(\tau)$ at ∞. Then $t(-1/(4D_0\tau))$ is a modular function on $X_0(4D_0)$ with a unique pole of order k at zero. Let g be the genus of $X_0(4D_0)$ and m be an integer with $m \geqslant 2g-2-\sum_{d|D_0}\lfloor d/4\rfloor$. By Lemma 27, for each positive integer j, there exist a modular form in $M^{!,!}_{m,j}(4D_0)$ with a pole of order j at zero. It follows that

$$M^{!,!}_{m,n+k}(4D_0) = M^{!,!}_{m,n}(4D_0) + t(-1/4D_0\tau)M^{!,!}_{m,n}(4D_0).$$

Thus, to prove the proposition, it suffices to show that:

(i) there exists an eta product $t(\tau)$ such that $t(\tau)$ is a modular function on $\Gamma_0(4D_0)$ with a unique pole of at ∞;
(ii) admissible eta products span $M^!(4D_0)$; and
(iii) admissible eta products span $M^{!,!}_{m,k}(4D_0)$, where k is the order of pole of $t(\tau)$ at ∞.

For conditions (i) and (ii), the integer programming problem involved in the construction of $t(\tau)$ and admissible eta products is the same as that in Proposition 18. For condition (iii), the integer programming problem is slightly different. For the case $D_0 = p$ is a prime, instead of (8), we have

$$\begin{array}{rcrcrcrcrcrcl}
r_1 & + & 2r_2 & + & 4r_4 & + & pr_p & + & 2pr_{2p} & + & 4pr_{4p} & \geqslant & -24m \\
r_1 & + & 2r_2 & + & r_4 & + & pr_p & + & 2pr_{2p} & + & pr_{4p} & \geqslant & 0 \\
4r_1 & + & 2r_2 & + & r_4 & + & 4pr_p & + & 2pr_{2p} & + & pr_{4p} & \geqslant & 0 \\
pr_1 & + & 2pr_2 & + & 4pr_4 & + & r_p & + & 2r_{2p} & + & 4r_{4p} & \geqslant & 0 \\
pr_1 & + & 2pr_2 & + & pr_4 & + & r_p & + & 2r_{2p} & + & r_{4p} & \geqslant & 0 \\
4pr_1 & + & 2pr_2 & + & pr_4 & + & 4r_p & + & 2r_{2p} & + & r_{4p} & \geqslant & -24k
\end{array}$$

where the last inequality corresponds to the condition that the order of the pole at zero is at most k. After setting up the integer programming problems, we check case by case that admissible eta products do expand $M^{!,!}(4D_0)$.

Since every modular form $f(\tau)$ in $M^{!,!}(4D_0) \cap \mathbb{Q}((q))$ is a $\mathbb{Q}$-linear combination of admissible eta products, the assertion about rationality of Fourier coefficients of $f|_{1/2}S$ follows from Lemma 26. $\square$

Example 29. Consider the Shimura curve $X_0^{15}(1)/W_{15,1}$. We have $|L^\vee/L| = 450$ and the level of the lattice L is 60. By solving the relevant integer programming problem, we find that

$$t(\tau) = \frac{\eta(2\tau)\eta(12\tau)^6\eta(20\tau)^2\eta(30\tau)^3}{\eta(4\tau)^2\eta(6\tau)^3\eta(10\tau)\eta(60\tau)^6} = q^{-8} - q^{-6} + q^{-4} + q^{-2} + q^4 + \cdots .$$

is a modular function on $\Gamma_0(60)$ with a unique pole of order eight at ∞. Also, the genus of $X_0(60)$ is seven. By Lemma 17, the number of gaps of $M^!(60)$ is three, and for $n \geqslant 8$, we have $\dim M_n^!(60) = n-2$. According to the proof of Proposition 18, we should find an integer n_0 such that $M_{n_0+8}^!(60)$ is spanned by eta products and for each integer j with $n_0 < j \leqslant n_0 + 8$, there exists a modular form in $M_{n_0+8}^!(60)$ with a pole of order j at ∞. It turns out that we can choose $n_0 = 3$. (In other words, we will see that the gaps are $1, 2, 3$.)

For convenience, we let $(r_1, r_2, r_3, r_4, r_5, r_6, r_{10}, r_{12}, r_{15}, r_{20}, r_{30}, r_{60})$ represents the eta product $\prod_{d|60} \eta(d\tau)^{r_d}$. By solving the integer programming program, we find that there are at least 96 eta products in $M_{11}^!(60)$. Among them, we choose

$$\begin{aligned}
f_{11} &= (0,1,0,-1,-1,-1,0,2,5,2,1,-7), & f_{10} &= (0,0,-1,0,2,1,0,1,1,1,2,-6),\\
f_9 &= (0,0,-1,0,-1,2,-1,0,2,3,4,-7), & f_8 &= (0,0,-1,1,-2,0,1,1,5,1,0,-5),\\
f_7 &= (0,1,1,-1,2,-1,-2,1,-1,3,3,-5), & f_6 &= (0,1,0,-1,0,-1,0,2,2,1,1,-4),\\
f_5 &= (0,0,-1,0,-1,1,2,1,2,0,0,-3), & f_4 &= (0,0,-1,0,0,2,-1,0,-1,2,4,-4),\\
f_0 &= (-2,5,0,-2,0,0,0,0,0,0,0,0).
\end{aligned}$$

They form a basis for $M_{11}^!(60)$. (The subscripts are the orders of poles at ∞.) Then multiplying those modular forms by suitable powers of $t(\tau)$, we get, for each a non-gap integer $j > 0$, a modular form in $M^!(60) \cap \mathbb{Z}((q))$ with a unique pole of order j at ∞ and a leading coefficient of one.

Furthermore, we find that there are at least 102 eta products in $M_{3,8}^{!,!}(60)$. Among them, we choose

$$\begin{aligned}
g_1(\tau) &= \frac{\eta(2\tau)\eta(3\tau)\eta(4\tau)\eta(5\tau)\eta(12\tau)\eta(30\tau)}{\eta(\tau)^2\eta(6\tau)\eta(60\tau)^2} = q^{-3} + 2q^{-2} + 4q^{-1} + \cdots ,\\
g_2(\tau) &= \frac{\eta(2\tau)^4\eta(3\tau)^2\eta(10\tau)^3\eta(12\tau)}{\eta(\tau)^3\eta(4\tau)\eta(5\tau)\eta(6\tau)^2\eta(20\tau)\eta(60\tau)} = q^{-2} + 3q^{-1} + 5 + 8q + \cdots ,\\
g_3(\tau) &= \frac{\eta(4\tau)^2\eta(6\tau)\eta(10\tau)^2}{\eta(\tau)^2\eta(20\tau)\eta(60\tau)} = q^{-2} + 2q^{-1} + 5 + 10q + 18q^2 + \cdots ,\\
g_4(\tau) &= \frac{\eta(2\tau)^3\eta(3\tau)^4\eta(5\tau)\eta(12\tau)^2\eta(30\tau)}{\eta(\tau)^4\eta(4\tau)\eta(6\tau)^3\eta(15\tau)\eta(60\tau)} = q^{-1} + 4 + 11q + 24q^2 + \cdots ,\\
g_5(\tau) &= \frac{\eta(2\tau)^5\eta(3\tau)\eta(6\tau)\eta(10\tau)}{\eta(\tau)^5\eta(12\tau)\eta(60\tau)} = q^{-2} + 5q^{-1} + 15 + 39q + 90q^2 + \cdots ,\\
g_6(\tau) &= \frac{\eta(2\tau)^3\eta(3\tau)^2\eta(5\tau)\eta(6\tau)^2}{\eta(\tau)^5\eta(12\tau)\eta(60\tau)} = q^{-2} + 5q^{-1} + 17 + 48q + \cdots ,\\
g_7(\tau) &= \frac{\eta(2\tau)^2\eta(3\tau)\eta(4\tau)\eta(5\tau)^3\eta(6\tau)}{\eta(\tau)^5\eta(12\tau)\eta(15\tau)} = 1 + 5q + 18q^2 + 54q^3 + \cdots ,\\
g_8(\tau) &= \frac{\eta(2\tau)^4\eta(3\tau)^2\eta(5\tau)^3\eta(12\tau)^2\eta(15\tau)}{\eta(\tau)^6\eta(4\tau)\eta(6\tau)^2\eta(10\tau)\eta(60\tau)} = q^{-1} + 6 + 23q + 72q^2 + \cdots
\end{aligned}$$

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of weight 1/2 on $\Gamma_0(60)$. By Lemma 26,

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_1|S)(\tau) = \frac{2}{3}(q^{-2/60} + 2q^{-1/60} + 4 + 8q^{1/60} + 14q^{2/60} + \cdots),$$
$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_2|S)(\tau) = q^{-2/60} + q^{-1/60} + 2 + 4q^{1/60} + 6q^{2/60} + 8q^{3/60} + \cdots,$$
$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_3|S)(\tau) = 2(q^{-3/60} + q^{-2/60} + 2q^{-1/60} + 4 + 6q^{1/60} + \cdots),$$
$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_4|S)(\tau) = 2(q^{-4/60} + q^{-3/60} + q^{-2/60} + 2q^{-1/60} + 3 + \cdots),$$
$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_5|S)(\tau) = q^{-5/60} + q^{-4/60} + 2q^{-3/60} + 3q^{-2/60} + 5q^{-1/60} + \cdots,$$
$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_6|S)(\tau) = \frac{2}{3}(q^{-6/60} + q^{-5/60} + 2q^{-4/60} + 3q^{-3/60} + \cdots),$$
$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_7|S)(\tau) = \frac{1}{5}(q^{-7/60} + q^{-3/60} + q^{-2/60} + 2q^{1/60} + \cdots),$$
$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(g_8|S)(\tau) = \frac{2}{15}(q^{-8/60} + q^{-7/60} + 2q^{-6/60} + 3q^{-5/60} + \cdots).$$

Thus, letting

$$h_1 = 3g_1/2 - g_2, \quad h_2 = g_2, \quad h_3 = g_3/2, \quad h_4 = g_4/2,$$

and

$$h_5 = g_5, \quad h_6 = 3g_6/2, \quad h_7 = 5g_7, \quad h_8 = 15g_8/2,$$

we get a sequence h_j, $j = 1, \ldots, 8$, of modular forms such that

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(h_j|S)(\tau) = q^{-j/60} + \cdots.$$

Now we have

$$t(-1/60\tau) = 5\frac{\eta(2\tau)^3\eta(3\tau)^2\eta(5\tau)^6\eta(30\tau)}{\eta(\tau)^6\eta(6\tau)\eta(10\tau)^3\eta(15\tau)^2} = 5 + 30q + 120q^2 + 390q^3 + \cdots,$$

which is a modular function on $\Gamma_0(60)$ having a unique pole of order eight at the cusp zero. Thus, by multiplying h_j with suitable powers of $t(-1/60\tau)$, we get, for each positive integer m, an $\{\infty, 0\}$-weakly holomorphic modular form h_m whose order of pole at ∞ is bounded by three, while

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(h_m|S)(\tau) = q^{-m/60} + \cdots.$$

Remark 30. We expect that, as in the case of even D, for (D, N) in Theorem 1 with odd D and squarefree N, all meromorphic modular forms of even weights on $X_0^D(N)/W_{D,N}$ with a divisor supported on CM divisors can be realized as a Borcherds form. However, a proof along the line of that of Proposition 23 will be a little complicated because the Fourier expansions at zero of a modular form in $M^{!,!}(4D_0) \cap \mathbb{Z}((q))$ may not be integral.

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Example 31. Here we give an example showing how to construct a Borcherds form with a desired divisor on $X_0^{15}(1)/W_{15,1}$ using modular forms in $M^{!,!}(60)$.

Suppose that we wish to construct a Borcherds form with a divisor $P_{-12}-P_{-3}$. For a positive integer j, we let h_j be the modular form in $M^{!,!}(60)$ constructed in Example 29 with the properties that its order of pole at ∞ is bounded by three and

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(h_j|S)(\tau) = q^{-j/60} + \cdots .$$

A suitable linear combination of these h_m will yield a function h with

$$h(\tau) = q^{-2} + 11 + \cdots , \quad \frac{60e^{2\pi i/8}}{15\sqrt{2}}(h|S)(\tau) = 2q^{-3/4} + 4q^{1/60} + 4q^{2/60} + \cdots . \tag{11}$$

By Lemma 25,

$$\operatorname{div} \psi_{F_h} = \tfrac{1}{3}P_{-3}.$$

Let

$$f = f_8 - f_5 + f_4 = q^{-8} + 2q^{-3} + q^{-2} + 2q^2 + \cdots$$

where f_j are as given in Example 29. By Lemma 25,

$$\operatorname{div} \psi_{F_f} = P_{-12} + \tfrac{1}{3}P_{-3}.$$

Therefore, we find that $\psi_{F_{f-4h}}$ is a Borcherds form with a divisor $P_{-12} - P_{-3}$.

4. Equations of hyperelliptic Shimura curves

Recall that a compact Riemann surface X of genus at least two is hyperelliptic if and only if there exists a double covering $\pi : X \to \mathbb{P}(\mathbb{C})$ or, equivalently, if there exists an involution $w : X \to X$ such that X/w has genus zero. The involution w is unique and is called the hyperelliptic involution.

Theorem C [Ogg83, Theorems 7 and 8]. *Let $g(D,N)$ denote the genus of $X_0^D(N)$. Table 1 gives the full list of hyperelliptic Shimura curves, $D > 1$, and their hyperelliptic involutions.*

4.1 Method

Let us briefly explain our method to compute equations of these hyperelliptic Shimura curves. Before doing that, we remark that in addition to Borcherds forms and Schofer's formula, arithmetic properties of CM points are also crucial in our computation. We refer the reader to [GR06, § 5] for an explicit description of the Shimura reciprocity law.

Let $X_0^D(N)$ be one of the curves in Ogg's list. Since the hyperelliptic involution of $X_0^D(N)$ is an Atkin–Lehner involution, the genus of $X_0^D(N)/W_{D,N}$ is necessarily zero. Moreover, it turns out that any of these $X_0^D(N)/W_{D,N}$ has at least three rational CM points τ_1, τ_2 and τ_3 of discriminants d_1, d_2 and d_3, respectively. Thus, there is a Hauptmodul $s(\tau)$ on $X_0^D(N)/W_{D,N}$ with $s(\tau_1) = \infty$, $s(\tau_2) = 0$ and $s(\tau_3) \in \mathbb{Q}$.

Let W be a subgroup of index two of $W_{D,N}$. Suppose that w_m is an element of $W_{D,N}$ not in W. Then $X_0^D(N)/W \to X_0^D(N)/W_{D,N}$ is a double cover ramified at certain CM points that are fixed points of the Atkin–Lehner involutions $w_{mn/\gcd(m,n)^2}$, $w_n \in W$. Thus, an equation of $X_0^D(N)/W$ is

$$y^2 = a \prod_{\tau \text{ ramified}, s(\tau)\neq\infty} (s - s(\tau)), \tag{12}$$

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TABLE 1. List of hyperelliptic Shimura curves and their hyperelliptic involutions.

D	N	$g(D,N)$	w
26	1	2	w_{26}
35	1	3	w_{35}
38	1	2	w_{38}
39	1	3	w_{39}
51	1	3	w_{51}
55	1	3	w_{55}
57	1	3	w_{19}
58	1	2	w_{29}
62	1	3	w_{62}
69	1	3	w_{69}
74	1	4	w_{74}
82	1	3	w_{41}
86	1	4	w_{86}
87	1	5	w_{87}
93	1	5	w_{31}
94	1	3	w_{94}
95	1	7	w_{95}
111	1	7	w_{111}
119	1	9	w_{119}
134	1	6	w_{134}
146	1	7	w_{146}
159	1	9	w_{159}
194	1	9	w_{194}
206	1	9	w_{206}

D	N	$g(D,N)$	w
6	11	3	w_{66}
6	17	3	w_{34}
6	19	3	w_{114}
6	29	5	w_{174}
6	31	5	w_{186}
6	37	5	w_{222}
10	11	5	w_{110}
10	13	3	w_{65}
10	19	5	w_{38}
10	23	9	w_{230}
14	3	3	w_{14}
14	5	3	w_{14}
15	2	3	w_{15}
15	4	5	w_{15}
21	2	3	w_{7}
22	3	3	w_{66}
22	5	5	w_{110}
26	3	5	w_{26}
39	2	7	w_{39}

where a is a rational number depending on the arithmetic of $X_0^D(N)/W$. Specifically, a must be a rational number such that $(a\prod_{\tau \text{ ramified}}(-s(\tau)))^{1/2}$ is in the field of definition of a CM point of discriminant d_2 on $X_0^D(N)/W$. As an additional check, note that when τ_1 is not a ramified point, the right-hand side of (12) is a polynomial of even degree and a must be a rational number such that $\sqrt{a}$ is in the field of definition of a CM point of discriminant d_1 on $X_0^D(N)/W$.

To determine the coefficients of the polynomial on the right-hand side of (12), we simply have to know the values of s and y^2 at sufficiently many points. For this purpose, we observe that s and y^2 are both modular functions on $X_0^D(N)/W_{D,N}$ with divisors supported on CM divisors. Thus, they are both realizable as Borcherds forms. (This is proved in Proposition 23 for the case of even D. We do not try to give a proof for the case of odd D, but, in practice, we are always able to realize modular forms encountered as Borcherds forms.) Then Schofer's formula gives us the absolute values of norms of values of s and y^2 at CM points.

In order to obtain the actual values of s, not just the absolute values, we let $\widetilde{s}$ be another Hauptmodul with $\widetilde{s}(\tau_1)=\infty$, $\widetilde{s}(\tau_3)=0$ and $\widetilde{s}(\tau_2)\in\mathbb{Q}$. We may also realize $\widetilde{s}$ as a Borcherds form. Then the absolute values of $s(\tau_3)$ and $\widetilde{s}(\tau_2)$ obtained using Schofer's formula determine the relation $\widetilde{s}=bs+c$ between s and $\widetilde{s}$. If d is a discriminant such that there is only one CM point τ_d of discriminant d, then knowing the values of $|s(\tau_d)|$ and $|\widetilde{s}(\tau_d)|=|bs(\tau_d)+c|$ from Schofer's formula is enough to determine the value of $s(\tau_d)$. If there are two CM points τ_d and

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τ_d' of discriminant d, then from the values of $|s(\tau_d)s(\tau_d')|$ and $|(bs(\tau_d)+c)(bs(\tau_d')+c)|$ we get four possible candidates for the minimal polynomial of $s(\tau_d)$. In almost all cases we consider, there is precisely one of the four candidates that have roots in the correct field. This gives us the values of $s(\tau_d)$ and $s(\tau_d')$. In practice, we do not need information from discriminants with more than two CM points.

The determination of values of y^2 from absolute values is easier. For example, when d is a discriminant such that there is only one CM point of discriminant d on $X_0^D(N)/W_{D,N}$, $y(\tau_d)$ is either $\sqrt{|y(\tau_d)^2|}$ or $\sqrt{-|y(\tau_d)^2|}$, but only one of them is in the correct field.

Having determined values of s and y^2 at sufficiently many CM points, it is straightforward to determine the equation of $X_0^D(N)/W$. Then we will either work out equations of $X_0^D(N)/W'$ for various other subgroups W' of $W_{D,N}$ of index two or use arithmetic properties of $X_0^D(N)$ to determine equations of $X_0^D(N)$. We will give several examples in the next section.

4.2 Examples

Example 32. Consider $X_0^{15}(1)$. In [Jor81, Proposition 3.2.1], it is shown that an equation of $X_0^{15}(1)$ is

$$3y^2 + (x^2+3)(x^2+243) = 0.$$

In this example, we will use Borcherds forms and Schofer's formula to obtain this result.

The curve $X = X_0^{15}(1)$ and its various Atkin–Lehner quotients have the following geometric information.

Curve	Genus	Elliptic points
X	1	$\mathrm{CM}(-3)^{\times 2}$
X/w_3	0	$\mathrm{CM}(-3)^{\times 2}, \mathrm{CM}(-12)^{\times 2}$
X/w_5	1	$\mathrm{CM}(-3)$
X/w_{15}	0	$\mathrm{CM}(-3), \mathrm{CM}(-15)^{\times 2}, \mathrm{CM}(-60)^{\times 2}$
$X/W_{15,1}$	0	$\mathrm{CM}(-3), \mathrm{CM}(-12), \mathrm{CM}(-15), \mathrm{CM}(-60)$

According to the method described in the previous section, we should first determine the equation of X/W for some subgroup W of $W_{15,1}$ of index two. Here we choose $W = \langle w_3 \rangle$. The double cover $X/w_3 \to X/W_{15,1}$ is ramified at the CM points τ_{-15} and τ_{-60} of discriminants -15 and -60. Let $s(\tau)$ be a Hauptmodul on $X/W_{15,1}$ taking values zero and ∞ at CM points τ_{-12} and τ_{-3} of discriminants -12 and -3, respectively, and satisfying $s(\tau_{-40}) \in \mathbb{Q}$, where τ_{-40} is the unique CM point of discriminant -40 on $X/W_{15,1}$. Then an equation of X/w_3 is

$$y^2 = a(s - s(\tau_{-15}))(s - s(\tau_{-60})),$$

where $a = -3r^2$ for some $r \in \mathbb{Q}$ since a CM point of discriminant -3 on X/w_3 is defined over $\mathbb{Q}(\sqrt{-3})$. The divisor of y^2, as a function on $X/W_{15,1}$, is $P_{-15} + P_{-60} - 2P_{-3}$. Let also $\widetilde{s}$ be a Hauptmodul with $\widetilde{s}(\tau_{-15}) = \infty$, $\widetilde{s}(\tau_{40}) = 0$, and $\widetilde{s}(\tau_{-60}) \in \mathbb{Q}$. According to our method, we should construct Borcherds forms with divisors $P_{-12} - P_{-3}$, $P_{-40} - P_{-3}$ and $P_{-15} + P_{-60} - 2P_{-3}$. A Borcherds form $P_{-12} - P_{-3}$ is constructed in Example 31. Denote this Borcherds form by ψ_1. Here let us construct the other two Borcherds forms.

Using the notation in Example 29 and letting h be the modular form in (11), we find that

$$f_{10} - f_7 + f_5 - 2f_4 - 3h = q^{-10} - 3q^{-2} + q^{-1} - 35 + \cdots$$

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and

$$\frac{60e^{2\pi i/8}}{15\sqrt{2}}(f_{10}-f_7+f_5-2f_4-3h)|S = 6q^{-3/4}+c_0+c_1q^{1/60}+\cdots$$

for some c_j. Thus, by Lemma 25, the Borcherds form ψ_2 associated to this modular form has a divisor $P_{-40}-P_{-3}$. Also, we have

$$\begin{aligned}&2f_{15}+4f_{13}+2f_{12}-2f_{10}-4f_9-7f_8-10f_7+10f_6+3f_5-23f_4-6h\\&\quad=2q^{-15}-q^{-8}-5q^{-2}-2q^{-1}-78+\cdots\end{aligned}$$

and

$$\begin{aligned}&\frac{60e^{2\pi i/8}}{15\sqrt{2}}(2f_{15}+4f_{13}+2f_{12}-2f_{10}-4f_9-7f_8-10f_7+10f_6+3f_5-23f_4-6h)|S\\&\quad=12q^{-3/4}+c_0'+c_1'q^{1/60}+\cdots\end{aligned}$$

for some c_j'. Therefore, the Borcherds form ψ_3 associated to this modular form has a divisor $P_{-15}+P_{-60}-2P_{-3}$. An application of Schofer's formula yields the following values of Borcherds forms at CM points.

	-3	-7	-12	-15	-40	-43	-60
$\lvert\psi_1\rvert$	∞	1	0	3	$1/2$	$1/16$	$1/27$
$5^{-3/2}\lvert\psi_2\rvert$	∞	$1/9$	$1/27$	$5/27$	0	$1/24$	$25/3^6$
$\lvert\psi_3\rvert$	∞	$35/3^6$	$5/2^43^5$	0	$5^4/2^63^6$	$43^15^17^2/2^{12}3^6$	0

Observe that multiplying ψ_j by a scalar of absolute value 1 does not change the absolute value of its value at a CM point. Thus, we may as well assume that $\psi_1(\tau_{-15})=-3$, $5^{-3/2}\psi_2(\tau_{-15})=5/27$ and $\psi_3(\tau_{-7})=-35/3^6$. Also, we choose s, $\widetilde{s}$ and y such that $s(\tau_{-15})=-243$, $\widetilde{s}(\tau_{-15})=5$ and $y(\tau_{-7})^2=-2^43^47$. Therefore, we have

$$s=81\psi_1,\quad \widetilde{s}=27\cdot 5^{-3/2}\psi_2,\quad y^2=\frac{2^43^{10}}{5}\psi_3.$$

Then from the table above, we obtain

$$|s(\tau_{-12})|=0,\quad |\widetilde{s}(\tau_{-12})|=1,\quad |s(\tau_{-40})|=81/2,\quad |\widetilde{s}(\tau_{-40})|=0,$$

which implies that $\widetilde{s}$ is equal to one of $\pm 2s/81\pm 1$. As $s(\tau_{-15})=-243$ and $\widetilde{s}(\tau_{-15})=5$, we find that $\widetilde{s}=-2s/81-1$. Then the table above and the requirement that $y(\tau_d)$ must lie in the correct field yield the following.

	-3	-7	-12	-15	-40	-43	-60
s	∞	81	0	-243	$-81/2$	$81/16$	-3
$\widetilde{s}$	∞	-3	-1	5	0	$-9/8$	$-25/27$
y^2	∞	-2^43^47	-3^5	0	$-3^45^3/4$	$-3^47^243/2^8$	0

It follows that an equation of X/w_3 is $3y^2+(s+243)(s+3)=0$.

Furthermore, the double cover $X/w_{15} \to X/W_{15,1}$ is ramified at CM points of discriminants -3 and -12. Thus, an equation of X/w_{15} is $x^2 = bs$ for some b. As CM points of discriminant -7 are rational points on X/w_{15}, we find that b must be a square, which we may assume to be one. That is, we have $s = x^2$. Therefore, we have $3y^2 + (x^2 + 243)(x^2 + 3) = 0$, which can be taken to be an equation of X, agreeing with Jordan's result.

We remark that Elkies [Elk98] has used Schwarzian differential equations to compute numerically the values of s at many CM points. (His modular function differs from our s by a factor of -3.) Using Borcherds forms, we verify that all of the entries in [Elk98, Table 6] are correct.

Example 33. Consider the Shimura curve $X = X_0^{26}(1)$. In [GR04], González and Rotger proved that an equation of X is

$$y^2 = -2x^6 + 19x^4 - 24x^2 - 169.$$

In this example, we will obtain this result using Borcherds forms.

We have the following information about X and its Atkin–Lehner quotients.

Curve	Genus	Elliptic points
X	2	None
X/w_2	1	$\mathrm{CM}(-8)^{\times 2}$
X/w_{13}	1	$\mathrm{CM}(-52)^{\times 2}$
X/w_{26}	0	$\mathrm{CM}(-104)^{\times 6}$
$X/W_{26,1}$	0	$\mathrm{CM}(-8), \mathrm{CM}(-52), \mathrm{CM}(-104)^{\times 3}$

The double cover $X/w_{13} \to X/W_{26,1}$ is ramified at the CM point of discriminant -8 and the three CM points of discriminant -104. Let s be a Hauptmodul on $X/W_{26,1}$ with $s(\tau_{-8}) = \infty$, $s(\tau_{-52}) = 0$, and $s(\tau_{-11}) \in \mathbb{Q}$. Then an equation of X/w_{13} is

$$y^2 = a \prod_{\tau:\text{CM points of discriminant} -104} (s - s(\tau))$$

for some nonzero rational number a. As a modular function on $X/W_{26,1}$, we have $\operatorname{div} y^2 = P_{-104} - 3P_{-8}$. Let $\widetilde{s}$ be another Hauptmodul on $X/W_{26,1}$ with $\widetilde{s}(\tau_{-8}) = \infty$, $\widetilde{s}(\tau_{-11}) = 0$, and $\widetilde{s}(\tau_{-52}) \in \mathbb{Q}$. We now realize s, $\widetilde{s}$, and y^2 as Borcherds forms.

Let f_j be modular forms in $M^!(52) \cap \mathbb{Z}((q))$ with a pole of order j at ∞ and a leading coefficient of one constructed in Example 19. Using these f_j, we find three modular forms

$$\begin{aligned} g_1 &= 2q^{-13} - 2q^{-2} - 4q^{-1} + 2q - 2q^2 - 2q^3 + \cdots, \\ g_2 &= q^{-11} + 2q^{-7} - 2q^{-2} + 4q + 4q^4 + \cdots, \\ g_3 &= 2q^{-26} + 6q^{-7} - 6q^{-2} + 2q^{-1} + 10q - 8q^2 + \cdots \end{aligned}$$

in $M^!(52)$. Let ψ_j, $j = 1, 2, 3$, be the Borcherds forms associated to g_j. By Lemma 22,

$$\operatorname{div} \psi_1 = P_{-52} - P_{-8}, \quad \operatorname{div} \psi_2 = P_{-11} - P_{-8}, \quad \operatorname{div} \psi_3 = P_{-104} - 3P_{-8}.$$

Thus, ψ_j are scalar multiples of s, $\widetilde{s}$ and y^2, respectively. Applying Schofer's formula, we obtain the following result.

	-8	-11	-19	-20	-24	-52	-67
$\lvert\psi_1\rvert$	∞	1	9	5	3	0	$81/25$
$\lvert\psi_2\rvert$	∞	0	64	32	32	8	$2^6 7/5^2$
$13^{-3}\lvert\psi_3\rvert$	∞	$2^{10}11$	$2^{10}19$	2^{12}	2^{13}	$2^6 13^5$	$2^{10}41^2 67/5^6$

Since multiplying ψ_j by a suitable factor of absolute value one does not change the absolute value of its value at a CM point, we may as well assume that $\psi_1(\tau_{-11}) = 1$, $\psi_2(\tau_{-52}) = 8$ and $\psi_3(\tau_{-11}) = -2^{10}11^1 13^3$. Also, we choose s, $\widetilde{s}$ and y in a way such that $s(\tau_{-11}) = 1$, $\widetilde{s}(\tau_{-52}) = 1$ and $y(\tau_{-11})^2 = -2^4 11$, i.e. $s = \psi_1$, $\widetilde{s} = \psi_2/8$ and $y^2 = \psi_3/2^6 13^3$. Then we have $\widetilde{s} = 1 - s$ and from the table above we obtain the following result.

	-8	-11	-19	-20	-24	-52	-67
s	∞	1	9	5	-3	0	$81/25$
$\widetilde{s} = 1 - s$	∞	0	-8	-4	4	1	$-56/25$
y^2	∞	$-2^4 11$	$-2^4 19$	-2^6	2^7	-13^2	$-2^4 41^2 67/5^6$

(The signs of $y(\tau_d)^2$ are determined by the Shimura reciprocity law.) From the data, we easily deduce that the relation between y and s is

$$y^2 = -2s^3 + 19s^2 - 24s - 169,$$

which is an equation for $X_0^{26}(1)/w_{13}$.

On the other hand, the cover $X_0^{26}(1)/w_{26} \to X_0^{26}(1)/W_{26,1}$ is ramified at the CM points of discriminants -8 and -52. Thus, there is a modular function x on $X_0^{26}(1)/w_{26}$ with $x^2 = cs$ for some rational number c. Since CM points of discriminant -11 are rational points on $X_0^{26}(1)/w_{26}$, we conclude that c can be chosen to be 1. Hence, $y^2 = -2x^6 + 19x^4 - 24x^2 - 169$ is an equation for $X_0^{26}(1)$ and the Atkin–Lehner involutions are given by

$$w_2 : (x, y) \mapsto (-x, -y), \quad w_{26} : (x, y) \mapsto (x, -y).$$

Example 34. Consider $X = X_0^{111}(1)$. We have the following information.

Curve	Genus	Elliptic points
X	7	None
X/w_3	4	None
X/w_{37}	3	$\mathrm{CM}(-148)^{\times 4}$
X/w_{111}	0	$\mathrm{CM}(-111)^{\times 8}, \mathrm{CM}(-444)^{\times 8}$
$X/W_{111,1}$	0	$\mathrm{CM}(-148)^{\times 2}, \mathrm{CM}(-111)^{\times 4}, \mathrm{CM}(-444)^{\times 4}$

Let s and $\widetilde{s}$ be modular functions on $X/W_{111,1}$ such that $s(\tau_{-15}) = \widetilde{s}(\tau_{-15}) = \infty$, $s(\tau_{-60}) = 0$, $\widetilde{s}(\tau_{-24}) = 0$, $s(\tau_{-24}) = 1$ and $\widetilde{s}(\tau_{-60}) = 1$, so that $\widetilde{s} = 1 - s$. Then an equation for X/w_{37} is

$$y^2 = a \prod_{\tau \in \mathrm{CM}(-111), \mathrm{CM}(-444)} (s - s(\tau)). \tag{13}$$

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As CM points of discriminant -60 on X/w_{37} lie in $\mathbb{Q}(\sqrt{-3})$, we choose y such that $y(\tau_{-60})^2 = -27$. Then realizing s, $\widetilde{s}$ and y^2 as Borcherds forms and using Schofer's formula, we deduce the following values of these modular functions at rational CM points.

	-15	-19	-24	-43	-51	-60	-163	-267	-555
s	∞	3	1	-3	-1	0	$3/5$	$1/3$	5
y^2	∞	$-2^8 3^2 19$	$-2^8 3$	$-2^8 3^2 43$	$-2^8 3$	-27	$-2^8 3^2 13^2 163/5^8$	$-2^8 13^2/3^7$	$-2^8 3^1 37^2$

As the right-hand side of (13) is a polynomial of degree eight, these CM values are not sufficient to determine the equation and we will need values of s and y^2 at some degree-two CM points.

Let τ_{-39} and τ'_{-39} be the two CM points of discriminant -39 on $X/W_{111,1}$. Schofer's formula yields

$$|s(\tau_{-39})s(\tau'_{-39})| = 3, \quad |(1 - s(\tau_{-39}))(1 - s(\tau'_{-39}))| = 4.$$

From the Shimura reciprocity law, we know that $s(\tau_{-39}) \in \mathbb{Q}(\sqrt{-3})$. Thus,

$$s(\tau_{-39})s(\tau'_{-39}) = 3, \quad (1 - s(\tau_{-39}))(1 - s(\tau'_{-39})) = 4.$$

From these, we deduce that $s(\tau_{-39}) = \pm\sqrt{-3}$. Likewise, we find that the values of s at the two CM points τ_{-52}, τ'_{-52} of discriminants -52 are $1 \pm 2\sqrt{-1}$. Also, we have

$$y(\tau_{-39})^2 y(\tau'_{-39})^2 = 2^{16}3^2 13, \quad y(\tau_{-52})^2 y(\tau'_{-52})^2 = 2^{16}13^2.$$

These data are enough to determine the equation of X/w_{37}. We find that it is

$$y^2 = -(3s^4 - 6s^3 + 28s^2 - 10s + 1)(s^4 - 2s^3 + 4s^2 + 18s + 27). \tag{14}$$

Similarly, we can compute an equation for X/w_{111} by observing that $X/w_{111} \to X/W_{111,1}$ is ramified at the two CM points of discriminant -148, constructing a Borcherds form with divisor $P_{-148} - 2P_{-15}$, and evaluating at various CM points and obtain

$$t^2 = 5s^2 - 18s + 45.$$

The conic has rational points $(s, t) = (3, \pm 6)$ corresponding the two CM points of discriminant -19 on X/w_{111}, so it admits a rational parameterization. Specifically, let x be a Hauptmodul on X/w_{111} that has a pole and a zero at the two CM points of discriminant -19, respectively, and takes rational values at CM points of discriminant -43. (In terms of (s, t), the coordinates are $(-3, \pm 12)$.) Then

$$x = \frac{c(s-3)}{s-t+3}$$

for some rational number c. Choose $c = 2$ so that it takes values ± 1 at the CM points of discriminant -43. We have

$$(s, t) = \left(\frac{3x^2 - 3x - 3}{x^2 + x - 1}, \frac{6x^2 + 6}{x^2 + x - 1}\right).$$

Plugging in $s = (3x^2 - 3x - 3)/(x^2 + x - 1)$ in (14) and making a slight change of variables, we find that an equation of $X_0^{111}(1)$ is

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$$\begin{aligned} z^2 &= -(x^8 - 3x^5 - x^4 + 3x^3 + 1) \\ &\quad \times (19x^8 - 44x^7 - 16x^6 + 55x^5 + 37x^4 - 55x^3 - 16x^2 + 44x + 19) \end{aligned}$$

with the actions of the Atkin–Lehner involutions given by

$$w_{37} : (x, z) \mapsto \left(-\frac{1}{x}, \frac{z}{x^8}\right), \quad w_{111} : (x, z) \mapsto (x, -z).$$

Example 35. Consider $X = X_0^{146}(1)$. Let s be the Hauptmodul of $X/W_{146,1}$ such that $s(\tau_{-43}) = 0$, $s(\tau_{-11}) = \infty$ and $s(\tau_{-20}) = 1$. Let y be a modular function on X/w_{73} such that y^2 is a modular function on $X/W_{146,1}$ with $\operatorname{div} y^2 = P_{-584} - 8P_{-11}$. Realizing s and y^2 as Borcherds forms and suitably scaling y^2, we find that an equation for X/w_{73} is

$$y^2 = -11s^8 + 82s^7 - 309s^6 + 788s^5 - 1413s^4 + 1858s^3 - 1803s^2 + 1240s - 688. \tag{15}$$

Similarly, we find that an equation for X/w_{146} is $t^2 = s^2 + 4$, where the roots of $s^2 + 4$ correspond the to CM points of discriminant -292. We choose a rational parameterization of the conic to be

$$(s, t) = \left(\frac{x^2 - 1}{x}, \frac{x^2 + 1}{x}\right),$$

where x is actually a modular function on X/w_{146} that has a pole and a zero at the two CM points of discriminant -11 and is equal to ± 1 at the two CM points of discriminant -43 on X/w_{146}. Substituting $s = (x^2 - 1)/x$ in (15) and making a change of variables, we find that an equation for X is

$$\begin{aligned} z^2 = -11x^{16} &+ 82x^{15} - 221x^{14} + 214x^{13} + 133x^{12} - 360x^{11} - 170x^{10} + 676x^9 \\ &- 150x^8 - 676x^7 - 170x^6 + 360x^5 + 133x^4 - 214x^3 - 221x^2 - 82x - 11, \end{aligned}$$

where the Atkin–Lehner involutions are given by

$$w_{73} : (x, y) \mapsto \left(-\frac{1}{x}, \frac{y}{x^8}\right), \quad w_{146} : (x, y) \mapsto (x, -y).$$

Example 36. Let $X = X_0^{14}(5)$. Let s be the Hauptmodul of $X/W_{14,5}$ such that $s(\tau_{-4}) = \infty$, $s(\tau_{-11}) = 1$ and $s(\tau_{-35}) = 0$. We find that an equation for $X/\langle w_5, w_7\rangle$ is

$$y^2 = -16s^3 - 347s^2 + 222s - 35,$$

which is isomorphic to the elliptic curve E_{14A5} in Cremona's table [Cre97]. (In fact, we can use the Čerednik–Drinfeld theory of p-adic uniformization of Shimura curves [BC92] to determine the singular fibers of $X/\langle w_5, w_7\rangle$ and conclude that it is isomorphic to E_{14A5}.) The double cover $X/\langle w_5, w_{14}\rangle \to X/W_{14,5}$ is ramified at the CM point of discriminant -4 and the CM point of discriminant -35, so that there is a Hauptmodul t of $X/\langle w_5, w_{14}\rangle$ such that $t^2 = cs$ for some rational number c. In addition, the CM points of discriminant -11 on $X/\langle w_5, w_{14}\rangle$ are rational points. Thus, we may choose $c = 1$ and find that an equation for X/w_5 is

$$y^2 = -16t^6 - 347t^4 + 222t^2 - 35. \tag{16}$$

We next determine an equation of X/w_{14}. The double cover $X/w_{14} \to X/\langle w_5, w_{14}\rangle$ is ramified at the two CM points of discriminant -280. Using Schofer's formula, we find $s(\tau_{-280}) = 5/16$ and, thus, an equation for X/w_{14} is $u^2 = d(16t^2 - 5)$ for some rational number. The point such that $t=0$ is the CM point of discriminant -35. Therefore, we may choose $d=-1$ and find that an equation for X/w_{14} is

$$u^2+16t^2=5.$$

This is a conic with rational points and a rational parameterization is

$$(t,u)=\left(\frac{x^2-x-1}{2x^2+2},\frac{x^2+4x-1}{x^2+1}\right).$$

Substituting $t=(x^2-x-1)/(2x^2+2)$ into (16) and making a change of variables, we conclude that an equation for $X_0^{14}(5)$ is

$$z^2=-23x^8-180x^7-358x^6-168x^5-677x^4+168x^3-358x^2+180x-23,$$

on which the actions of the Atkin–Lehner operators are given by

$$w_2:(x,z)\mapsto\left(-\frac{1}{x},\frac{z}{x^4}\right),\quad w_{14}:(x,z)\mapsto(x,-z),$$

and

$$w_{35}:(x,z)\mapsto\left(\frac{x+2}{2x-1},\frac{25z}{(2x-1)^4}\right).$$

Note that $X_0^{14}(5)/w_{14}$ is an example of Shimura curves of genus zero that is isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$ but none of the rational points is a CM point.

Example 37. Let $X=X_0^{10}(19)$. Let s be the Hauptmodul of $X/W_{10,19}$ such that $s(\tau_{-8})=0$, $s(\tau_{-40})=\infty$, and $s(\tau_{-3})=1$. We find that an equation for $X/\langle w_2,w_{95}\rangle$ is

$$y^2=-8s^3+57s^2-40s+16,$$

which is isomorphic to the elliptic curve $E_{190\mathrm{A}1}$ in Cremona's table [Cre97]. Also, the double cover $X/\langle w_5,w_{38}\rangle\to X/W_{10,19}$ is ramified at the CM point of discriminant -8 and the CM point of discriminant -40. The CM points of discriminant -3 are rational points on $X/\langle w_5,w_{38}\rangle$. Thus, arguing as before, we deduce that an equation for X/w_{190} is $y^2=-8x^6+57x^4-40x^2+16$. Moreover, the double cover $X/w_{38}\to X/\langle w_5,w_{38}\rangle$ is ramified at the two CM points of discriminant -760. Since $s(\tau_{760})=32/5$ and the point with $s=0$ is a CM point of discriminant -8, we see that an equation for X/w_{38} is $z^2=5x^2-32$. We conclude that an equation for X is

$$\begin{cases}y^2=-8x^6+57x^4-40x^2+16,\\ z^2=5x^2-32,\end{cases}$$

with the actions of the Atkin–Lehner involutions given by

$$\begin{aligned}w_2&:(x,y,z)\mapsto(-x,y,z),\\ w_5&:(x,y,z)\mapsto(x,-y,-z),\\ w_{19}&:(x,y,z)\mapsto(-x,-y,z).\end{aligned}$$

Note that as the conic $z^2=5x^2-32$ has only real points, but no rational points, the Shimura curve X is hyperelliptic over $\mathbb{R}$, but not over $\mathbb{Q}$.

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Remark 38. In [Ogg83], Ogg mentioned that $X_0^{10}(19)$ and $X_0^{14}(5)$ are the only two hyperelliptic curves that he could not determine whether they are hyperelliptic over $\mathbb{Q}$. Our computation shows that $X_0^{14}(5)$ is hyperelliptic over $\mathbb{Q}$ because the curve $X_0^{14}(5)/w_{14}$ has rational points, but $X_0^{10}(19)$ is not hyperelliptic over $\mathbb{Q}$.

Remark 39. Note that there is a curve, namely, $X = X_0^{15}(4)$, whose equation is not obtained using our method. This is because the normalizer of the Eichler order in this case is larger than the Atkin–Lehner group. For this special curve, we use the result of Tu [Tu14]. In [Tu14, Lemma 13], it is shown that there is a Hauptmodul t_4 on $X/\langle w_3, w_5\rangle$ that takes values $\pm 1/\sqrt{-3}$, $\pm\sqrt{-15}/5$ and $(\pm 1 \pm \sqrt{-15})/8$ at CM points of discriminants -12, -15 and -60, respectively. Since the double cover $X/w_3 \to X/\langle w_3, w_5\rangle$ ramifies at CM points of discriminants -15 and -60, while the cover $X/w_{15} \to X/\langle w_3, w_5\rangle$ ramifies at CM points of discriminant -12, we find that there are rational numbers a and b such that the equations of X/w_3 and X/w_{15} are

$$y^2 = a(4t_4^2 - t_4 + 1)(4t_4^2 + t_4 + 1)(5t_4^2 + 3), \quad z^2 = b(3t_4^2 + 1),$$

respectively. To determine the constants a and b, we further recall that [Tu14, Lemma 13] shows that there is a Hauptmodul t_2 on $X_0^{15}(2)/\langle w_3, w_5\rangle$ with

$$t_2 = \frac{5t_4^2 + 2t_4 + 1}{7t_4^2 - 2t_4 + 3}.$$

From this, the CM values of t_2 obtained using Schofer's formula, and arithmetic properties of CM points, we see that we can choose $a = b = -1$. Note that $X_0^{15}(4)$ is one of the hyperelliptic Shimura curves that are not hyperelliptic over $\mathbb{R}$ (see [Ogg83]).

4.3 Additional examples

In the previous section, we determine the equations of hyperelliptic Shimura curves $X_0^D(N)$ whose Atkin–Lehner involutions act as hyperelliptic involutions. In particular, the curves $X_0^D(N)/W_{D,N}$ are of genus zero, so that Lemma 7 applies and we have a simple criterion for a Borcherds form to have a trivial character. Throughout this section, we make the following assumption.

Assumption 40. The criterion for a Borcherds form to have a trivial character is also valid for the case when $N_B^+(\mathcal{O})\backslash\mathfrak{H}$ has a positive genus.

Remark 41. Recall that a Fuchsian group of the first kind is generated by some elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n$ with defining relations

$$[\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g]\gamma_1 \ldots \gamma_n = 1, \quad \gamma_i^{k_i} = 1,\ i = 1, \ldots, n,$$

where α_j, β_j are hyperbolic elements, $[\alpha_j, \beta_j]$ denotes the commutator, g is the genus and k_i is an integer at least two or ∞. (See, for instance, [Kat92].) Let χ be the character of a Borcherds form on $N_B^+(\mathcal{O})\backslash\mathfrak{H}$. The proof of Lemma 7 given in [Yan15] shows that $\chi(\gamma_i) = 1$ for all i if and only if the condition in Lemma 7 holds. Thus, what we really assume in Assumption 40 is that for all hyperbolic elements α, we have $\chi(\alpha) = 1$.

It turns out that sometimes our methods can also be used to determine equations of $X_0^D(N)/W_{D,N}$ even when they have positive genera, under Assumption 40. However, the method becomes less systematic and it is not clear whether our methods will always work in general, so we will only give two examples in this section.

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Example 42. Let $X = X_0^{142}(1)/W_{142,1}$. It is of genus one and has rational points (for instance, the CM point of discriminant -3). Thus, X is a rational elliptic curve. From the Jacquet–Langlands correspondence, we know that it must lie in the isogeny class 142A in Cremona's table [Cre97], whose corresponding cusp form on $\Gamma_0(142)$ has eigenvalues -1 for the Atkin–Lehner involutions w_2 and w_{71}. Since the isogeny class contains only one curve, we immediately conclude that the equation for X is $E_{142\mathrm{A}1} : y^2+xy+y = x^3-x^2-12x+15$. Here we will use our method to obtain the same conclusion. An advantage of our method is that we can determine the coordinates of all CM points on the curve. In the arXiv version of this paper, we discuss the heights of these CM points and verify Zhang's formula [Zha01] for heights of CM points in this particular case.

By finding many suitable eta products, we construct four modular forms f_1, f_2, f_3, f_4 in $M^!(284)$ with Fourier expansions

$$\begin{aligned}
f_1 &= -2q^{-87} - 2q^{-71} - 2q^{-48} - 2q^{-36} + 2q^{-16} - 2q^{-15} - 2q^{-12} + 2q^{-9} \\
&\quad - 2q^{-7} - 2q^{-3} + 2q^{-2} + 2q^{-1} - 4q \cdots, \\
f_2 &= 2q^{-116} - q^{-87} - q^{-79} + 2q^{-71} + 2q^{-60} - 2q^{-48} + q^{-43} - 2q^{-29} \\
&\quad + q^{-19} - 4q^{-15} - 2q^{-12} + 2q^{-7} - 2q^{-3} - 4q + \cdots, \\
f_3 &= q^{-87} - 2q^{-79} + q^{-76} - 2q^{-71} + 2q^{-48} - q^{-40} + 3q^{-32} - 2q^{-20} \\
&\quad - q^{-19} - 2q^{-12} + 2q^{-10} + q^{-8} - 2q^{-7} - 2q^{-2} + 4q + \cdots, \\
f_4 &= -q^{-79} + q^{-76} + q^{-48} - q^{-40} + q^{-32} - q^{-20} - q^{-12} + q^{-10} + q^{-6} \\
&\quad - q^{-5} - q^{-2} - q^{7} + \cdots.
\end{aligned}$$

Let ψ_j, $j = 1, \ldots, 4$, be the Borcherds form associated to f_j. Under Assumption 40, these Borcherds forms have trivial characters. We have

$$\begin{aligned}
\operatorname{div}\psi_1 &= P_{-4} + P_{-8} - 2P_{-3}, \quad & \operatorname{div}\psi_2 &= P_{-19} + P_{-43} - 2P_{-3}, \\
\operatorname{div}\psi_3 &= P_{-8} + P_{-40} - 2P_{-20}, \quad & \operatorname{div}\psi_4 &= P_{-19} + P_{-24} - 2P_{-20}.
\end{aligned}$$

It is easy to show that ψ_2 is a polynomial of degree one in ψ_1 and ψ_4 is a polynomial of degree one in ψ_3. Thus, there are modular functions x and y on X such that x has a double pole at τ_{-3} with $x(\tau_{-4}) = x(\tau_{-8}) = 0$ and $x(\tau_{-19}) = x(\tau_{-43}) = 1$ and y has a double pole at τ_{-20} with $y(\tau_{-8}) = y(\tau_{-20}) = 0$ and $y(\tau_{-19}) = y(\tau_{-24}) = 1$. Computing singular moduli using Schofer's formula and choosing proper scalars of modulus one for ψ_j, we find

$$x = 2^{-10}\psi_1, \quad 1 - x = \psi_2, \quad y = \psi_3/2, \quad 1 - y = \psi_4/2,$$

and the values of x and y at various CM points are given in the following table.

	-3	-4	-8	-19	-20	-24	-40	-43	-148	-232
x	∞	0	0	1	-1	$1/2$	$-1/2$	1	-1	$-1/2$
y	2	$1/2$	0	1	∞	1	0	$3/2$	-2	-5

Since $y(\tau_{-4}) \neq y(\tau_{-8})$, y cannot lie in $\mathbb{C}(x)$. Therefore, x and y generate the field of modular functions on X. From the table above, we determine that the relation between x and y is

$$2(x+1)^2y^2 - (8x^2 + 11x + 1)y + 4x(2x+1) = 0.$$

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Set

$$x_1 = -\frac{2(x+1)^2y - 5x^2 - 3x - 1}{x^2},$$
$$y_1 = -\frac{(4x^3+6x^2-2)y - 5x^3 - 6x^2 + x + 1}{x^3}.$$

We find $y_1^2 + x_1y_1 + y_1 = x_1^3 - x_1^2 - 12x_1 + 15$, which is indeed the elliptic curve $E_{142\mathrm{A}1}$. The coordinates of the CM points above on this model are given in the following table.

-3	-4	-8	-19	-20	-24	-40	-43	-148	-232
$-Q$	$-2Q$	O	Q	$2Q$	$3Q$	$4Q$	$-3Q$	$-4Q$	$-6Q$

Here $Q = (1, 1)$ generates the group of rational points on $E_{142\mathrm{A}1}$.

Example 43. We next consider $X = X_0^{302}(1)/W_{302,1}$, which has genus two. We can construct four modular forms $f_1, \ldots, f_4$ in $M^!(604)$ whose associated Borcherds forms $\psi_1, \ldots, \psi_4$ have divisors

$$\begin{aligned}
\operatorname{div}\psi_1 &= P_{-43} + P_{-72} - P_{-19} - P_{-88},\\
\operatorname{div}\psi_2 &= P_{-20} + P_{-36} - P_{-19} - P_{-88},\\
\operatorname{div}\psi_3 &= P_{-8} + 2P_{-40} - P_{-4} - 2P_{-88},\\
\operatorname{div}\psi_4 &= P_{-11} + P_{-19} + P_{-43} - P_{-4} - 2P_{-88},
\end{aligned}$$

respectively. In addition, under Assumption 40, they have trivial characters. Thus, ψ_1 generates the unique genus-zero subfield of degree two of the hyperelliptic function field, and ψ_2 is a polynomial of degree one in ψ_1. Also, ψ_4 must be a polynomial of degree one in ψ_3. To see this, we observe that there exists a suitable linear combination $a\psi_3 + b\psi_4$ such that it is a function of degree at most two on X and, hence, is contained in $\mathbb{C}(\psi_1)$. If this linear combination is not a constant function, then it must have a pole at τ_{-88}; otherwise it will have only a pole of order one at τ_{-4}, which is impossible. It follows that τ_{-19} is also a pole of this linear combination. However, τ_{-19} can never be a pole of this function. Therefore, we conclude that this linear combination is a constant function.

Let x be the unique function on X with $\operatorname{div} x = \operatorname{div}\psi_1$ and $x(\tau_{-20}) = 2$ and y be the unique function with $\operatorname{div} y = \operatorname{div}\psi_3$ and $y(\tau_{-11}) = 1$. Computing using Schofer's formula, we find the following result.

d	-4	-8	-11	-19	-20	-40	-43	-88	-148	-232
x	-1	$3/2$	1	∞	2	1	0	∞	$5/3$	$5/3$
y	∞	0	1	1	-1	0	1	∞	$-1/9$	$-1/2$

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From the coordinates at τ_{-4}, τ_{-8}, τ_{-19}, τ_{-40} and τ_{-88}, we see that the relation between x and y is

$$a(x+1)y^2 + (-2x^3 + bx^2 + cx + d)y + (2x-3)(x-1)^2 = 0$$

for some rational numbers a, b, c, and d. Then the information at the other CM points yields

$$a = 1, \quad b = 11, \quad c = -13, \quad d = 2.$$

Setting

$$x_0 = \frac{3-x}{1-x}, \quad y_0 = \frac{4(2x^3 - 11x^2 + 13x - 2 - 2xy - 2y)}{(1-x)^3},$$

we obtain a Weierstrass model

$$y_0^2 = x_0^6 - 18x_0^4 + 113x_0^2 - 32$$

for X. Then letting

$$x_1 = x_0^2, \quad y_1 = y_0, \quad x_2 = -32/x_0^2, \quad y_2 = 32y_0/x_0^3,$$

we obtain modular parameterization of two elliptic curves

$$y_1^2 = x_1^3 - 18x_1^2 + 113x_1 - 32, \quad y_2^2 = x_2^3 + 113x_2^2 + 576x_2 + 1024.$$

The minimal models of these two elliptic curves are $E_{302\mathrm{C}1} : Y^2 + XY + Y = X^3 - X^2 + 3$ and $E_{302\mathrm{A}1} : Y^2 + XY + Y = X^3 + X^2 - 230X + 1251$, respectively, in Cremona's table. The coordinates of the CM points on the two curves are as follows.

	-4	-8	-11	-19	-20	-40	-43	-88	-148	-232
$E_{302\mathrm{A}1}$	$2P-Q$	$3P-Q$	$2P$	$4P$	P	$3P$	$3P-Q$	P	$3P+Q$	$2P-Q$
$E_{302\mathrm{C}1}$	$5R$	R	O	$2R$	$2R$	O	$-R$	$-2R$	$5R$	$-5R$

Here $P = (-32, 256)$ generates the torsion subgroup of order five and $Q = (-96, 320)$ generates the free part of $E_{302\mathrm{A}1}(\mathbb{Q})$, and $R = (9, 16)$ generates the group of rational points on $E_{302\mathrm{C}1}$. In the arXiv version of the present paper, we also address the issue of heights of CM points on the Jacobians of these elliptic curves.

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Appendix A. Tables for equations of hyperelliptic Shimura curves

We list defining equations of hyperelliptic Shimura curves in Tables A.1 and A.2.

Table A.1. Equations of level one.

$X_0^{26}(1)$	$y^2 = -2x^6 + 19x^4 - 24x^2 - 169$
	$w_2(x,y) = (-x,-y)$, $w_{26}(x,y) = (x,-y)$
$X_0^{35}(1)$	$y^2 = -(x^2+7)(7x^6 + 51x^4 + 197x^2 + 1)$
	$w_5(x,y) = (-x,-y)$, $w_{35}(x,y) = (x,-y)$
$X_0^{38}(1)$	$y^2 = -16x^6 - 59x^4 - 82x^2 - 19$
	$w_2(x,y) = (-x,-y)$, $w_{38}(x,y) = (x,-y)$
$X_0^{39}(1)$	$y^2 = -(x^4 - x^3 - x^2 + x + 1)(7x^4 - 23x^3 + 5x^2 + 23x + 7)$
	$w_{13}(x,y) = \left(-\frac{1}{x}, \frac{y}{x^4}\right)$, $w_{39}(x,y) = (x,-y)$
$X_0^{51}(1)$	$y^2 = -(x^2+3)(243x^6 + 235x^4 - 31x^2 + 1)$
	$w_3(x,y) = (-x,y)$, $w_{51}(x,y) = (x,-y)$
$X_0^{55}(1)$	$y^2 = -(x^4 - x^3 + x^2 + x + 1)(3x^4 + x^3 - 5x^2 - x + 3)$
	$w_5(x,y) = \left(-\frac{1}{x}, \frac{y}{x^4}\right)$, $w_{55}(x,y) = (x,-y)$
$X_0^{57}(1)$	$y^2 = (3s+1)(3s^3 + 11s^2 + 17s + 1)$, $x^2 = -4s^2 + 2s - 1$
	$w_{19}(s,x,y) = (s,x,-y)$, $w_{57}(s,x,y) = (s,-x,y)$
$X_0^{58}(1)$	$y^2 = -2x^6 - 78x^4 - 862x^2 - 1682$
	$w_2(x,y) = (-x,-y)$, $w_{29}(x,y) = (x,-y)$

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TABLE A.1. Equations of level one (continued).

$X_0^{62}(1)$	$y^2 = -64x^8 - 99x^6 - 90x^4 - 43x^2 - 8$	$w_2(x,y) = (-x,y)$, $w_{62}(x,y) = (x,-y)$
$X_0^{69}(1)$	$y^2 = -243x^8 + 1268x^6 - 666x^4 - 2268x^2 - 2187$	$w_3(x,y) = (-x,y)$, $w_{69}(x,y) = (x,-y)$
$X_0^{74}(1)$	$y^2 = -2x^{10} + 47x^8 - 328x^6 + 946x^4 - 4158x^2 - 1369$	$w_2(x,y) = (-x,-y)$, $w_{74}(x,y) = (x,-y)$
$X_0^{82}(1)$	$y^2 = 4s^4 + 4s^3 + s^2 - 2s + 1$ $x^2 = -19s^2 + 18s - 11$	$w_2(x,y) = (-x,-y)$, $w_{41}(x,y) = (x,-y)$
$X_0^{86}(1)$	$y^2 = -16x^{10} + 245x^8 - 756x^6 - 1506x^4 - 740x^2 - 43$	$w_2(x,y) = (-x,-y)$, $w_{86}(x,y) = (x,-y)$
$X_0^{87}(1)$	$y^2 = -(x^6 - 7x^4 + 43x^2 + 27)(243x^6 + 523x^4 + 369x^2 + 81)$	$w_3(x,y) = (-x,y)$, $w_{87}(x,y) = (x,-y)$
$X_0^{93}(1)$	$y^2 = (3s^3 - 7s^2 - 3s - 1)(3s^3 + s^2 - 3s - 9)$, $x^2 = -4s^2 - 6s - 9$	$w_3(s,x,y) = (s,-x,-y)$, $w_{31}(s,x,y) = (s,x,-y)$
$X_0^{94}(1)$	$y^2 = -8x^8 + 69x^6 - 234x^4 + 381x^2 - 256$	$w_2(x,y) = (-x,y)$, $w_{94}(x,y) = (x,-y)$

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TABLE A.1. Equations of level one (continued).

$X_0^{95}(1)$	$y^2 = -(x^8 + x^7 - x^6 - 4x^5 + x^4 + 4x^3 - x^2 - x + 1) \times (7x^8 + 19x^7 + 21x^6 - 13x^4 + 21x^2 - 19x + 7)$
	$w_5(x,y) = \left(-\frac{1}{x}, \frac{y}{x^8}\right)$, $w_{95}(x,y) = (x,-y)$
$X_0^{111}(1)$	$y^2 = -(19x^8 - 44x^7 - 16x^6 + 55x^5 + 37x^4 - 55x^3 - 16x^2 + 44x + 19) \times (x^8 - 3x^5 - x^4 + 3x^3 + 1)$
	$w_{37}(x,y) = \left(-\frac{1}{x}, \frac{y}{x^8}\right)$, $w_{111}(x,y) = (x,-y)$
$X_0^{119}(1)$	$y^2 = -(7x^{10} - 171x^8 + 758x^6 + 3418x^4 + 4851x^2 + 2401) \times (x^{10} + 3x^8 + 26x^6 + 278x^4 + 373x^2 + 343)$
	$w_7(x,y) = (-x,y)$, $w_{119}(x,y) = (x,-y)$
$X_0^{134}(1)$	$y^2 = -16x^{14} - 347x^{12} - 2518x^{10} - 13341x^8 - 91876x^6 + 32859x^4 - 2518x^2 - 67$
	$w_2(x,y) = (-x,-y)$, $w_{134}(x,y) = (x,-y)$
$X_0^{146}(1)$	$y^2 = -11x^{16} + 82x^{15} - 221x^{14} + 214x^{13} + 133x^{12} - 360x^{11} - 170x^{10} + 676x^9 - 150x^8 - 676x^7 - 170x^6 + 360x^5 + 133x^4 - 214x^3 - 221x^2 - 82x - 11$
	$w_{73}(x,y) = \left(-\frac{1}{x}, \frac{y}{x^8}\right)$, $w_{146}(x,y) = (x,-y)$
$X_0^{159}(1)$	$y^2 = -(81x^{10} + 207x^8 + 874x^6 - 130x^4 - 11x^2 + 3) \times (2187x^{10} + 8389x^8 + 8878x^6 + 42x^4 - 41x^2 + 1)$
	$w_3(x,y) = (-x,y)$, $w_{159}(x,y) = (x,-y)$
$X_0^{194}(1)$	$y^2 = -19x^{20} - 92x^{19} - 286x^{18} - 592x^{17} - 921x^{16} - 1016x^{15} - 872x^{14} + 460x^{13} + 1545x^{12} + 1752x^{11} + 34x^{10} - 1752x^9 + 1545x^8 - 460x^7 - 872x^6 + 1016x^5 - 921x^4 + 592x^3 - 286x^2 + 92x - 19$
	$w_{97}(x,y) = \left(-\frac{1}{x}, -\frac{y}{x^{10}}\right)$, $w_{194}(x,y) = (x,-y)$
$X_0^{206}(1)$	$y^2 = -8x^{20} + 13x^{18} + 42x^{16} + 331x^{14} + 220x^{12} - 733x^{10} - 6646x^8 - 19883x^6 - 28840x^4 - 18224x^2 - 4096$
	$w_2(x,y) = (-x,y)$, $w_{206}(x,y) = (x,-y)$

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TABLE A.2. Equations of level greater than one.

$X_0^6(11)$	$y^2 = -19x^8 - 166x^7 - 439x^6 - 166x^5 + 612x^4 + 166x^3 - 439x^2 + 166x - 19$
	$w_2(x,y) = \left(\frac{x+1}{x-1}, -\frac{4y}{(x-1)^4}\right)$, $w_3(x,y) = \left(-\frac{1}{x}, -\frac{y}{x^4}\right)$, $w_{66}(x,y) = (x,-y)$
$X_0^6(17)$	$z^2 = -3x^2 - 16$, $y^2 = 17x^4 - 10x^2 + 9$
	$w_2(x,y,z) = (-x,y,z)$, $w_3(x,y,z) = (x,-y,-z)$, $w_{34}(x,y,z) = (x,-y,z)$
$X_0^6(19)$	$y^2 = -19x^8 + 210x^6 - 625x^4 + 210x^2 - 19$
	$w_2(x,y) = \left(-\frac{1}{x}, -\frac{y}{x^4}\right)$, $w_3(x,y) = \left(\frac{1}{x}, \frac{y}{x^4}\right)$, $w_{114}(x,y) = (x,-y)$
$X_0^6(29)$	$y^2 = -64x^{12} + 813x^{10} - 3066x^8 + 4597x^6 - 12264x^4 + 13008x^2 - 4096$
	$w_2(x,y) = (-x,y)$, $w_3(x,y) = \left(-\frac{2}{x}, \frac{8y}{x^6}\right)$, $w_{174}(x,y) = (x,-y)$
$X_0^6(31)$	$y^2 = -243x^{12} + 11882x^{10} - 177701x^8 + 803948x^6 - 1599309x^4 + 962442x^2 - 177147$
	$w_2(x,y) = \left(\frac{3}{x}, -\frac{27y}{x^6}\right)$, $w_3(x,y) = (-x,y)$, $w_{186}(x,y) = (x,-y)$
$X_0^6(37)$	$y^2 = -4096x^{12} - 18480x^{10} - 40200x^8 - 51595x^6 - 40200x^4 - 18480x^2 - 4096$
	$w_2(x,y) = (-x,y)$, $w_3(x,y) = \left(\frac{1}{x}, \frac{y}{x^6}\right)$, $w_{222}(x,y) = (x,-y)$

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TABLE A.2. Equations of level greater than one (continued).

$X_0^{10}(11)$	$y^2 = -8x^{12} - 35x^{10} + 30x^8 + 277x^6 + 120x^4 - 560x^2 - 512$
	$w_{10}(x,y) = \left(-\frac{2}{x}, -\frac{8y}{x^6}\right)$, $w_{22}(x,y) = \left(\frac{2}{x}, \frac{8y}{x^6}\right)$, $w_{110}(x,y) = (x,-y)$
$X_0^{10}(13)$	$z^2 = -2x^2 - 25$, $y^2 = 5x^4 - 74x^2 + 325$
	$w_2(x,y,z) = (x,-y,-z)$, $w_5(x,y,z) = (-x,-y,-z)$, $w_{65}(x,y,z) = (x,-y,z)$
$X_0^{10}(19)$	$z^2 = 5x^2 - 32$, $y^2 = -8x^6 + 57x^4 - 40x^2 + 16$
	$w_2(x,y,z) = (-x,y,z)$, $w_5(x,y,z) = (x,-y,-z)$, $w_{38}(x,y,z) = (x,-y,z)$
$X_0^{10}(23)$	$y^2 = -43x^{20} + 318x^{19} - 1071x^{18} + 3014x^{17} - 10540x^{16} + 28266x^{15} - 72217x^{14} + 81478x^{13} - 62765x^{12} - 68732x^{11} + 18840x^{10} + 68732x^9 - 62765x^8 - 81478x^7 - 72217x^6 - 28266x^5 - 10540x^4 - 3014x^3 - 1071x^2 - 318x - 43$
	$w_2(x,y) = \left(\frac{2x+1}{x-2}, -\frac{5^5 y}{(x-2)^{10}}\right)$, $w_5(x,y) = \left(-\frac{1}{x}, -\frac{y}{x^{10}}\right)$, $w_{230}(x,y) = (x,-y)$
$X_0^{14}(3)$	$z^2 = -9x^2 - 2$ $y^2 = -7x^4 + 22x^2 + 1$
	$w_2(x,y,z) = (-x,y,z)$, $w_3(x,y,z) = (x,-y,-z)$, $w_{14}(x,y,z) = (x,-y,z)$
$X_0^{14}(5)$	$y^2 = -23x^8 - 180x^7 - 358x^6 - 168x^5 - 677x^4 + 168x^3 - 358x^2 + 180x - 23$
	$w_2(x,y) = \left(-\frac{1}{x}, \frac{y}{x^4}\right)$, $w_{14}(x,y) = (x,-y)$, $w_{35}(x,y) = \left(\frac{x+2}{2x-1}, -\frac{25y}{(2x-1)^4}\right)$

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Table A.2. Equations of level greater than one (continued).

$X_0^{15}(2)$	$y^2 = -(x^2+3)(3x^2+4)(x^4-x^2+4)$	$w_2(x,y) = \left(\frac{2}{x}, -\frac{4y}{x^4}\right)$, $w_3(x,y) = (-x,y)$, $w_5(x,y) = (-x,-y)$
$X_0^{15}(4)$	$z^2 = -3x^2-1$, $y^2 = -(4x^2-x+1)(4x^2+x+1)(5x^2+3)$	$w_4(x,y,z) = (-x,-y,-z)$, $w_3(x,y,z) = (x,y,-z)$, $w_5(x,y) = (x,-y,-z)$
$X_0^{21}(2)$	$z^2 = -x^2-3$, $y^2 = -(3x-1)(3x+1)(x^2+7)(x^2+3)$	$w_2(x,y,z) = (-x,-y,-z)$, $w_3(x,y,z) = (x,y,-z)$, $w_7(x,y) = (x,-y,z)$
$X_0^{22}(3)$	$y^2 = -27x^8-308x^6-2146x^4-308x^2-27$	$w_2(x,y) = \left(-\frac{1}{x}, -\frac{y}{x^4}\right)$, $w_3(x,y) = (-x,y)$, $w_{66}(x,y) = (x,-y)$
$X_0^{22}(5)$	$y^2 = -11x^{12}-80x^{10}-240x^8-362x^6-240x^4-80x^2-11$	$w_2(x,y) = \left(\frac{1}{x}, \frac{y}{x^6}\right)$, $w_5(x,y) = \left(-\frac{1}{x}, -\frac{y}{x^6}\right)$, $w_{110}(x,y) = (x,-y)$
$X_0^{26}(3)$	$z^2 = -8x^2-3$, $y^2 = x^6-2x^4+9x^2+8$	$w_2(x,y,z) = (-x,-y,-z)$, $w_3(x,y,z) = (x,-y,-z)$, $w_{26}(x,y,z) = (x,-y,z)$
$X_0^{39}(2)$	$y^2 = -(x^8+11x^7+52x^6+140x^5+243x^4+280x^3+208x^2+88x+16)$ $\times(7x^4+24x^3+32x^2+24x+16)(x^4+3x^3+8x^2+12x+7)$	$w_2(x,y,z) = \left(\frac{2}{x}, -\frac{16y}{x^8}\right)$, $w_3(x,y,z) = \left(-\frac{x+2}{x+1}, -\frac{y}{(x+1)^8}\right)$, $w_{39}(x,y) = (x,-y)$

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