SOME HOMOLOGICAL PATHOLOGY IN VECTOR LATTICES

To Professor A. D. Wallace on his sixtieth birthday

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1. Introduction. The purpose of this paper is to point out a number of curious phenomena in the category of (real) vector lattices and linear lattice homomorphisms. Birkhoff **(3**, p. 221, Ex. 2 and Problem 96**)** called attention to the question of constructing models of the free objects with more than one generator in this category, a problem recently solved by E. C. Weinberg **(9)**. In §6 we construct a more manageable class of (non-free) projective vector lattices. Here, however, there is a countability restriction which suggests strong connections with free and projective Boolean algebras (in the category of Boolean algebras and their homomorphisms, such algebras must satisfy the countable chain condition **(6)**). That there are no non-trivial injective vector lattices intimates other relations in this direction (injectives are also absent from the category of complete Boolean algebras and complete homomorphisms **(6)**). This latter situation is quite similar to the one encountered, for example, in the category of rings and ring homomorphisms. Of course our category is badly non-abelian and trouble is to be expected.

In §2 we consider an analogue of the Jacobson radical. The radical may be, and often is, arbitrarily bad. The structure of radical vector lattices can be described rather adequately, and this we do in §3. We offer an analogue of the First Principal Wedderburn Structure Theorem. The other two Wedderburn theorems become rather trivial in this context; a simple vector lattice is isomorphic to the real numbers and semi-simple lattices with minimum condition are isomorphic to R^n (real n-space with the usual co-ordinate-wise ordering). In §7 we consider the Frattini sublattice, perforce briefly, since it vanishes, and conclude that the only "covers" in our category are the isomorphisms. On passing to the dual situation, we find that there are too many large "envelopes" (because injectives are missing).

Our terminology is of no permanent significance and merely reflects a homological viewpoint. For general background on vector lattices, we refer to **(3)**, except that we write $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, so that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. An adequate supply of homological machinery is contained in the article of Halmos **(6)**. The letter R will denote the real numbers throughout.

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2. The radical and semi-simplicity. An *ideal* I in a vector lattice A is a linear subspace of A with the property: $|x| \leqslant |y|$ and $y \in I$ imply $x \in I$. The *radical*, denoted $\mathrm{Rad}(A)$, of A is the intersection of all maximal ideals, with the usual convention that $\mathrm{Rad}(A) = A$ if A has no maximal ideals. We recall some of the basic facts.

PROPOSITION 1. *A vector lattice has no non-trivial ideals if and only if it is isomorphic to the real numbers* (**3**, p. 239).

A vector lattice A is termed *semi-simple* if $\mathrm{Rad}(A) = 0$; it is called *radical* if $\mathrm{Rad}(A) = A$.

PROPOSITION 2. *The maximal ideals of A are in a natural* 1–1 *correspondence (up to positive scalar multiples) with the epimorphisms from A to the reals.*

We pause to note that the homomorphisms in our category preserve the absolute value (and hence the other lattice operations) as well as linear structure. Thus A is semi-simple if and only if A has a total (separating) family of real epimorphisms; and A is radical if and only if A admits no non-zero real homomorphisms.

THEOREM 1 (Nakayama **8**). *A vector lattice A is semi-simple if and only if A is isomorphic to a vector lattice of real (finite) valued functions on some set, the operations being the usual pointwise ones.*

Proof. If $\mathrm{Rad}(A) = 0$, there is a family $(f_i)_{i\in I}$ of epimorphisms $f_i : A \to R$ with $\bigcap_{i\in I} \mathrm{Ker} f_i = 0$. Hence the mapping $a \to (f_i(a))_{i \in I}$ is a monomorphic embedding of A into R^I. Conversely, if $A \subset R^I$ and p_i denotes the ith projection, then for $a \neq 0$, there is an index $i \in I$ with $p_i(a) \neq 0$. The restriction $f = p_i|A$ is then an epimorphism with $f(a) \neq 0$.

PROPOSITION 3. *If A and B are vector lattices and $f:A \to B$ is a homomorphism, then* $f(\mathrm{Rad}(A)) \subset \mathrm{Rad}(B)$. *If A is a linear sublattice of B, then* $\mathrm{Rad}(A) \subset \mathrm{Rad}(B)$. *If I is an ideal in A and if* $I \subset \mathrm{Rad}(A)$, *then* $\mathrm{Rad}(A/I) = \mathrm{Rad}(A)/I$.

Proof. The second assertion follows trivially from the first, taking f to be the inclusion map. The last statement is a consequence of the fact that the ideals of A containing I are in a natural 1–1 correspondence with the ideals of A/I. Now if $a \in \mathrm{Rad}(A)$ and $g:B \to R$ is a homomorphism, then gf is a real homomorphism on A. Thus $g(f(a)) = 0$ and $f(\mathrm{Rad}(A))$ is annihilated by each real homomorphism on B.

COROLLARY 1. *Every linear sublattice of a semi-simple vector lattice is semi-simple.*

COROLLARY 2. $A/\mathrm{Rad}(A)$ *is semi-simple.*

COROLLARY 3. *The only homomorphism of a radical vector lattice into a semi-simple vector lattice is the zero mapping. Thus an exact sequence*

$$0 \to A \to B \to C \to 0$$

with B semi-simple and C radical never splits.

Examples 1 and 2 below were called to our attention by F. B. Wright.

Example 1. It is generally false that a quotient of a semi-simple lattice is semi-simple. In fact, let $A = \mathfrak{L}^1[0, 1]$ be the vector lattice (clearly semi-simple) of Lebesgue summable functions on the unit interval and let N be the ideal of "null functions," i.e. all $f \in A$ with

$$\int_0^1 |f(x)|dx = 0.$$

Then $A/N = L^1[0, 1]$. But $L^1[0, 1]$ is a radical vector lattice. This can be seen as follows. A linear functional f on a vector lattice is a lattice homomorphism if and only if f is positive and $f(a)f(b) = 0$, whenever $a \wedge b = 0$. Since f is a positive linear functional, it is continuous in the L^1-norm on $L^1[0, 1]$. Composing f with the quotient map $\phi: \mathfrak{L}^1 \to L^1$ we see that $f\phi$ restricted to the characteristic functions of measurable sets is a two-valued measure τ, absolutely continuous with respect to Lebesgue measure (since f is continuous). But then τ is concentrated on some point in $[0, 1]$ by a simple interval splitting argument. Since points have Lebesgue measure zero, $\tau = 0$ and hence $f = 0$.

Example 2. Let A be an abstract (L)-space in the sense of **(3)**. Then A is semi-simple if and only if $A = l^1(S)$, for some index set S (generalized sequence space) **(7)**.

PROPOSITION 4. *The (vector space) direct sum or direct product (ordered co-ordinate-wise) of any family of semi-simple vector lattices is semi-simple.*

The proof is direct and will be omitted.

In ring theory, a ring can be the (Jacobson) radical of another ring if and only if it is a radical ring. The situation with vector lattices is quite different; the radical may be anything.

PROPOSITION 5. *Let A be any vector lattice. Then there is a vector lattice E containing A as its unique maximal ideal so that* $\mathrm{Rad}(E) = A$.

Proof. Let $E = R \oplus A$ be the direct sum of the real line with A, ordered lexicographically: $(\alpha, a) \geqslant 0$ means that either $\alpha > 0$ or $\alpha = 0$ and $a \geqslant 0$. It is easily checked that E is a vector lattice with these elements as positive cone; moreover,

$$|(\alpha, a)| = (\alpha, a) \text{ if } \alpha > 0, \quad |(\alpha, a)| = (-\alpha, -a) \text{ if } \alpha < 0,$$

and

$$|(\alpha, a)| = (0, |a|) \text{ if } \alpha = 0.$$

If we set $e = (1, 0)$ and take $\lambda > |\alpha|$, then $\lambda e \geqslant |(\alpha, a)|$, so e is an *order unit* for E. Clearly the map $a \rightarrow (0, a)$ embeds A monomorphically into E. Equally obvious is the fact that the map $(\alpha, a) \rightarrow \alpha$ is a real epimorphism, so that A appears as a maximal ideal in E. In fact A is the *unique* maximal ideal in E, for if I is any ideal in E which is not contained in A, then we can find $(\alpha, a) \in I$ with $\alpha \neq 0$. We may as well assume that $\alpha > 1$ so that $0 < e < (\alpha, a)$. But then $e \in I$ so that $I = E$, since a proper ideal cannot contain the order unit. Hence the only ideals in E are the ideals in A and A itself. Clearly $A = \text{Rad}(E)$.

A vector lattice with a unique maximal ideal will be called a *local* vector lattice.

PROPOSITION 6. *Every finite-dimensional totally ordered vector lattice is local.*

Proof. In such a lattice A, there are only a finite number of ideals and they form a chain

$$\{0\} \subset I_1 \subset I_2 \subset \ldots \subset I_n \subset A,$$

where the gaps are one-dimensional. Clearly $I_n = \text{Rad}(A)$.

PROPOSITION 7. *A totally ordered vector lattice is either radical or local.*

Proof. The chain of ideals either has no maximum (radical case) or else does have one (local case).

3. Structure of radical lattices. In this section we examine radical vector lattices in more detail. We first give some simple observations.

PROPOSITION 8. *A totally ordered radical vector lattice has no non-zero positive linear functionals.*

Proof. As remarked in Example 1, a linear functional is a lattice homomorphism if and only if $f \geqslant 0$ and $f(a)f(b) = 0$ if $a \wedge b = 0$. Finally we observe that on a totally ordered vector lattice, a positive linear functional preserves the lattice operations. For if $a \wedge b = 0$, then $0 \leqslant a \leqslant b$ or $0 \leqslant b \leqslant a$, so $a = 0$ or $b = 0$.

PROPOSITION 9. *Every quotient of a radical lattice is radical (or zero).*

Proof. Let $f{:}A \rightarrow B$ be an epimorphism with A radical and $B \neq 0$. If g is a real homomorphism on B, then $gf = 0$. Since f is onto, $g = 0$.

Remark. Every totally ordered radical vector lattice is infinite dimensional (Prop. 7). As we shall presently see, every radical lattice is built up from totally ordered radical lattices. Since every vector lattice has totally ordered quotients (see Prop. 11), any radical lattice is necessarily infinite dimensional.

Following Bonsall **(4)**, we call a vector lattice A *everywhere non-Archimedean* if for any $a \in A$, there is an element $b \in A^+$ with $|a| \leqslant \alpha b$ for all real scalars $\alpha > 0$. It we write $\beta|a| \leqslant b$, where $\beta = 1/\alpha$, then the relation (usually written

$a \ll b$; cf. **(3**, p. 225 and **1**, §8**)** says that the principal ideal $\{x \in A : |x| \leqslant \lambda a,$ for some real $\lambda > 0\}$ generated by a is bounded above in A by b.

PROPOSITION 10. *An everywhere non-Archimedean vector lattice has no non-zero positive linear functionals and is therefore radical.*

Proof. Suppose that A is everywhere non-Archimedean and that f is a non-zero positive linear functional. Then $f(a) \neq 0$, for some $a \in A$. Since $|f(a)| \leqslant f(|a|)$, we may assume that $a > 0$, so that $f(a) > 0$. Then $\alpha a \leqslant b$ for some $b \in A^+$ and all $\alpha > 0$, so that $\alpha f(a) \leqslant f(b)$ for all $\alpha > 0$, which is absurd. Hence $f = 0$.

Example 3. Let T be an infinite totally ordered set having no largest element. Let H be the space of all real (finite) valued functions f on T with anti-well-ordered support (e.g. finite); thus $\{x \in T : f(x) \neq 0\}$ is anti-well-ordered. Call $f > 0$ if f is >0 on the largest element of its support. The set

$$H^+ = \{f : f > 0\} \cup \{0\}$$

is a cone with $H^+ \cap (-H^+) = 0$ and $H = H^+ \cup (-H^+)$; so H^+ totally orders H. This is the *full Hahn group with value set* T **(1)**. Clearly H is everywhere non-Archimedean.

Conversely, we have

THEOREM 2. *Every totally ordered everywhere non-Archimedean vector lattice is isomorphic to a linear sublattice of a full Hahn group whose totally ordered value set T has no largest element.*

This is essentially the classical Hahn embedding theorem for totally ordered abelian groups. The set T is taken to be the collection of all local ideals (i.e. ideals, which, when regarded as linear sublattices of the entire lattice, are local vector lattices). T is also isomorphic to the "orders of magnitude" of elements in the lattice; see **(1)** for details and a proof of the Hahn theorem. Note that the restriction that T has no largest element is essential, for otherwise the lattice would be local.

PROPOSITION 11. *Let A be a radical vector lattice. Then for $0 \neq a \in A$, there is an ideal $P \not\ni a$ such that A/P is totally ordered (P* is *prime) and everywhere non-Archimedean*; cf. **(4**, §4, Lemma 5 and Theorem 10**)**, but note the difference in the definition of ideals.

Proof. From the inductive collection of all ideals not containing a, choose a maximal one P. We may assume $a > 0$, for if not, we replace a by $0 < |a| \notin P$. Let $\bar{a}$ denote the residue class of a modulo P. Then $\bar{a} > 0$ lies in every ideal of A/P. Now for $x \in A/P$, $0 < \bar{a} \leqslant \alpha x^+, \beta x^-$, for suitable scalars α and β, since the ideals

$$(z) = \{y \in A/P : |y| \leqslant \lambda z\}$$

generated by $z = x^+, x^-$ contain a. Since $(\alpha x^+) \wedge (\beta x^-) = 0$ we must have $x^+ = 0$ or $x^- = 0$ and hence A/P is totally ordered. If $B = A/P$ is everywhere non-Archimedean, we are finished. If not, set

$$M = \{y \in B : |y| \leqslant \alpha b, \text{ some real } \alpha > 0, b \in B^+\}.$$

Clearly M is an ideal properly contained in B and B/M is totally ordered. But it is also Archimedean, for if $|\bar{x}| \leqslant \alpha \bar{u}$ for all $\alpha > 0$ with $x, u \in B$, then $|x| \leqslant \alpha u$, for all $\alpha > 0$, since B is totally ordered. Thus $x \in M$ and $\bar{x} = 0$. Hence $B/M \cong R$ and M induces a maximal ideal in A, contradicting the fact that $A = \text{Rad}(A)$. Finally, then, B is everywhere non-Archimedean.

As an immediate consequence, we have

THEOREM 3. *Every radical vector lattice is a subdirect sum of totally ordered everywhere non-Archimedean vector lattices.*

We note that, in Example 1, A/N is an *Archimedean* radical lattice, since it possesses a norm with the property: $||x|| \leqslant ||y||$, if $|x| \leqslant |y|$.

Example 4. Let X be any infinite set and let $F = R^X$ be the (semi-simple) vector lattice of all real finite-valued functions on X, ordered pointwise. Let B be the ideal of all bounded functions and set $A = F/B$. Then A is everywhere non-Archimedean, for if $a > 0$ in A, there is an unbounded function $f > 0$ with $a = \bar{f}$. Define a sequence b_n of constant functions $(\in B)$ by the formula: $b_n(x) = n^2/4$. Then for each positive integer n, $nf \leqslant f^2 + b_n$, so $nf \leqslant f^2 \pmod{B}$, i.e. $na \leqslant b$, where $b = \bar{f}^2$. Moreover, A is not totally ordered; for if we identify the integers with a subset of X, the function $f(n) = n$ $(n = 0, \pm 1, \pm 2, \ldots)$ and $f(x) = 0$ if $x \neq n$ has unbounded positive and negative parts f^+ and f^- which pass to non-zero disjoint elements in the quotient.

This example also shows that a prime ideal need not be contained in any maximal ideal. For let P be a prime ideal in A (as in Prop. 11, there are always "enough" of these). Then P induces a prime ideal Q in F which is contained in *no* maximal ideal. Being semi-simple, however, F has an abundance of maximal ideals.

Finally, we have

THEOREM 4. *Let A be a totally ordered vector lattice. Then $A = \text{Rad}(A)$ if and only if A is everywhere non-Archimedean.*

Proof. A is radical if A is everywhere non-Archimedean by Prop. 10. If A is radical, then its "value set" T (cf. Theorem 2) has no largest element or what amounts to the same thing, the "orders of magnitude" are unbounded. But this is precisely the statement that A is everywhere non-Archimedean.

Example 5. Some *Archimedean* vector lattices which have no non-zero positive linear functionals (and are therefore radical) are: (1) $L^p[0, 1]$ for $0 < p < 1$ and Lebesgue measure on the unit interval; (2) the space (S) of Banach, of all measurable functions on $[0, 1]$ modulo null functions; (3) the space of all Baire functions on $[0, 1]$ modulo null functions. Since all of these vector lattices are boundedly complete (see the remarks preceding Theorem 7), they are Archimedean. None of them can be "compatibly" normed (note after Theorem 3) or even supplied with a compatible locally convex topology (see Goffman **(5)** for a discussion of these matters).

4. Injectives. The sole purpose of this section is to prove

THEOREM 5. *In the category of vector lattices and linear lattice homomorphisms, there are no non-trivial injective objects.*

Proof. Suppose $A \neq 0$ is injective. Let $E = R \oplus A$ be the extension of A constructed in Proposition 5 and let $i:A \to E$ be the natural embedding $i(a) = (0, a)$ of A as a maximal ideal in E. As before, let $e = (1, 0)$ denote the order unit of E. Let $u:A \to A$ be the identity map. Then by injectivity u can be extended over E to a homomorphism $f:E \to A$ satisfying $fi = u$. Now the kernel K of f is an ideal. But we know the ideal structure of E from Proposition 5. We list the possible cases:

(1) $K = i(A)$. Then $fi = 0 = u$, a contradiction.

(2) $K = E$. Then $f = 0$, contradicting $u \neq 0$.

(3) K is a proper ideal (non-zero) in $i(A)$. If $0 \neq x \in K$, then $i(y) = x$ for some $y \in A$, $y \neq 0$. But $fi(y) = f(x) = 0$ and $u(y) = y \neq 0$, contradicting $fi = u$.

(4) $K = 0$. Then f is a monomorphism. Since $fi = u$, f is an epimorphism; hence f is an isomorphism. But consider the order unit $e = (1, 0)$ in E. If $a = f(e)$ we have $fi(a) = u(a) = uf(e) = f(e)$, so $i(a) = e$, which is clearly absurd. Thus our original assumption $A \neq 0$ is untenable.

Certain categories of algebras are also lacking non-trivial injectives. For example, let K be any integral domain and consider (1) the category of all K-algebras and K-algebra homomorphisms; (2) the category of all K-algebras with unit element and unit-preserving K-algebra homomorphisms. The first of these categories has no non-zero injectives and the second has none (but both have enough free and projective objects). For suppose $A \neq 0$ is injective in the first category. Since A can be embedded in a K-algebra B with unit, the identity map on A can be extended (by injectivity) to a K-algebra homomorphism of B onto A. Thus A must have a unit element. It follows that A must be injective in the second category as well. Now take any field F (a K-algebra) containing the field of quotients of K and consider the canonical homomorphism $k \to k.1$ of K into the centre of A. Injectivity of A then requires that A contain an isomorphic copy of F, which is clearly absurd. Thus $A = 0$.

5. Free vector lattices. E. C. Weinberg has recently given a general construction which in the present context explicitly describes the free objects. We shall merely outline his construction, the details of which can be found in **(9)**.

Let $\mathfrak{M}$ be any cardinal number and let S be a set of cardinality $\mathfrak{M}$. We construct the real vector space A of dimension $\mathfrak{M}$ having S as basis and consider the family $\{P_i\}_{i \in I}$ of *all* cones which totally order A. Thus each cone P in this family has the properties:

(1) $a + b \in P$ if $a, b \in P$,

(2) $\alpha a \in P$ if $a \in P$ and $\alpha \geqslant 0$ is real,

(3) $P \cap (-P) = 0$,

(4) $A = P \cup (-P)$.

Using Zorn's lemma, one can easily show that for any $0 \neq a \in A$, there is such a cone P not containing a.

Now let A_i denote the real vector space A totally ordered by the cone P_i. We form the direct product

$$V = \prod\{A_i : i \in I\}$$

and identify A with the diagonal in V. Under the co-ordinate-wise ordering, V is a vector lattice, and, in particular, a distributive lattice. We let $\mathrm{FVL}(\mathfrak{M})$ denote the distributive sublattice of V generated by the diagonal A. Thus a typical element $x \in \mathrm{FVL}(\mathfrak{M})$ has the form

$$x = \sup_i \inf_k a_{ik},$$

where $\{a_{ik}\}$ is a finite subset of A.

Because of the translation invariance of $\vee$ and $\wedge$, we see immediately that $\mathrm{FVL}(\mathfrak{M})$ is a linear subspace of V and is therefore a vector lattice. If B is any vector lattice and $f:S \to B$ is any function, then f can be (uniquely) extended to a linear map $f:A \to B$. One then extends f to $\mathrm{FVL}(\mathfrak{M})$ by defining

$$f(x) = \sup_i \inf_k f(a_{ik}).$$

After checking that this is well-defined (the only non-trivial point), we have

THEOREM 6. $\mathrm{FVL}(\mathfrak{M})$ *is the free vector lattice on* $\mathfrak{M}$ *generators. The real linear dimension d of* $\mathrm{FVL}(\mathfrak{M})$ *is determined by* $\mathfrak{M}$ *as follows:*

(1) $d = 2$ *if* $\mathfrak{M} = 1$,

(2) $d = \aleph_1$ *if* $\mathfrak{M} > 1$ *is finite or* $\mathfrak{M} = \aleph_0$,

(3) $d = \mathfrak{M}$ *if* $\mathfrak{M} \geqslant \aleph_1$.

(Here we assume the Generalized Continuum Hypothesis.)

Next we show that $\mathrm{FVL}(\mathfrak{M})$ is semi-simple. This will provide an affirmative answer to Birkhoff's Problem 107. The arguments in the remainder of this section were supplied by W. C. Holland.

LEMMA 1. *Let A be a real vector space of dimension at least two and let X be any countable subset of A not containing the origin. Then there is a hyperplane (through the origin) which misses X.*

Proof. From the dimensionality assumption and a simple counting argument, there is evidently a line in A through the origin which misses X.

By Zorn's lemma, we may choose a maximal linear subspace H of A missing X. If H is not a hyperplane, consider $B = A/H$ and the image Y of X in B. Again one can find a line through the origin in B missing Y, since B is at least two-dimensional by assumption and Y is at most countable. Moreover, this line induces a proper linear subspace in A containing H properly and missing X, contrary to the maximality of H. Thus H is a hyperplane.

THEOREM 7. FVL$(\mathfrak{M})$ *is semi-simple. In particular, every vector lattice is a quotient of a semi-simple one*; cf. (**3**, p. 242, Problem 107).

Proof. Given $0 \neq x \in \mathrm{FVL}(\mathfrak{M})$, it is enough to exhibit a maximal ideal in FVL$(\mathfrak{M})$ which does not contain x.

Now $x = \sup \inf a_{ik}$ (assume each $a_{ik} \neq 0$). Let $X = \{a_{ik}\}$. Since X is finite, Lemma 1 tells us that there is a hyperplane H missing X. By an easy application of Zorn's lemma, we can find a maximal cone P_0 which totally orders A and lies on one side of H. Let A_0 denote the component of the direct product $V = \Pi A_i$ which is ordered by the cone P_0. Clearly the hyperplane H is a maximal ideal in A_0. We define M to be the set of all elements in FVL$(\mathfrak{M})$ whose co-ordinates are arbitrary except in the A_0 position—and there we require the co-ordinate to lie in H. It is manifest that M is a maximal ideal and that $x \notin M$.

E. C. Weinberg has also obtained this result, but the above proof is somewhat shorter and more direct. Theorem 7 was observed independently by the author in conversation with N. Alling, but the means employed (ultrapowers and η_α-fields) did not seem to justify the end.

6. Projectives. We first recall some standard homological devices and pause to check that they apply to the concrete situation at hand. If $f:A \to B$ is an epimorphism, then by a *cross-section* to f we mean a homomorphism $g:B \to A$ such that fg is the identity on B. Clearly g must be a monomorphism in this case.

PROPOSITION 12. *A vector lattice B is projective if and only if every epimorphism $f:A \to B$ admits a cross-section.*

Proof. First suppose that B is projective and let $f:A \to B$ be an epimorphism. By projectivity, the identity map i on B lifts through f into A to give a cross-section $g:B \to A$ with $fg = i$.

For the converse, suppose that every epimorphism onto B admits a cross-section. Let $f':A' \to B'$ be an epimorphism and let $h:B \to B'$ be a homomorphism. Our task is to show that h can be lifted through f' into A'. We accomplish this by constructing the "inverse image" of the fibering $f':A' \to B'$ as follows. Let

$$A = \{(b, a') \in B \times A' : h(b) = f'(a')\}.$$

Then it is a trivial matter to check that A is a vector lattice with the definition $|(b, a')| = (|b|, |a'|)$. Moreover, the maps f and k defined by $f(b, a') = b$ and $k(b, a') = a'$ are homomorphisms and f is an epimorphism. By assumption, f has a cross-section $g:B \to A$. Defining $\phi = kg$, we have

$$f'\phi = f'kg = hfg = h,$$

as required, so B is indeed projective.

If an epimorphism $f:A \to B$ admits a cross-section, we say that f is a *retraction* ("B is a *retract* of A").

COROLLARY 4. *Every retract of a projective vector lattice is projective.*

COROLLARY 5. *Every projective quotient of a vector lattice is a retract. Hence a vector lattice is projective if and only if it is a retract of a free vector lattice. Every projective vector lattice is semi-simple.*

The proofs are simple exercises in "diagram chasing" and are left to the reader.

One of our objects, of course, is to describe as many projective vector lattices in as reasonable a fashion as possible. It is to be expected that some of the non-free projectives will be relatively easy to characterize in familiar terms. We shall produce one such class; in the process, a number of negative results arise. A few preliminaries are in order.

LEMMA 2. *If $a_1, \ldots, a_n$ is a finite set of elements in a vector lattice with $a_i > 0$ and $a_i \wedge a_k = 0$ for $i \neq k$, and if $\alpha_1, \ldots, \alpha_n, \beta_{n+1}, \ldots, \beta_m$ are non-negative reals, then*

$$\left(\sum_{i=1}^{n} \alpha_i a_i\right) \wedge \left(\sum_{k=n+1}^{m} \beta_k a_k\right) = 0.$$

The proof uses the fact that if $a, b, c \geqslant 0$, then

$$(a + b) \wedge c \leqslant a \wedge c + b \wedge c.$$

The details are left to the reader.

LEMMA 3. *Let $\{a_i\}_{i \in I}$ be an indexed family (not necessarily finite) of elements of a vector lattice satisfying $a_i > 0$ and $a_i \wedge a_k = 0$ for $i \neq k$. Then $\{a_i\}$ is linearly independent and for $\alpha_1, \ldots, \alpha_n$ real, we have*

$$\left|\sum_{i=1}^{n} \alpha_i a_i\right| = \sum_{i=1}^{n} |\alpha_i| a_i.$$

Proof. Suppose that

$$a = \sum_{i=1}^{n} \alpha_i a_i = 0$$

with each $\alpha_i \neq 0$. Then $a^+ = a^- = 0$. Now

$$u = \sum_{\alpha_i > 0} \alpha_i a_i \quad \text{and} \quad v = \sum_{\alpha_k < 0} (-\alpha_k) a_k$$

satisfy $u \wedge v = 0$ by Lemma 2, and by the uniqueness of the Jordan decomposition, $u = a^+$ and $v = a^-$. Thus it is enough to show that

$$\sum_{i=1}^{n} \alpha_i a_i = 0$$

with $\alpha_i \geqslant 0$ implies $\alpha_i = 0$, for each i. But $\alpha_1 a_1 \geqslant 0$ and

$$\alpha_1 a_1 = - \sum_{i=2}^{n} \alpha_i a_i \leqslant 0,$$

so $\alpha_1 a_1 = 0$ and finally $\alpha_1 = 0$ since $a_1 > 0$. Similarly $\alpha_i = 0$ for each i, contradicting the original assumption and proving linear independence.

Now

$$a^+ = \sum_{\alpha_i > 0} \alpha_i a_i \quad \text{and} \quad a^- = \sum_{\alpha_k < 0} (-\alpha_k) a_k$$

for $a \neq 0$. Thus

$$|a| = a^+ + a^- = \sum_{\alpha_i > 0} \alpha_i a_i + \sum_{\alpha_k < 0} (-\alpha_k) a_k = \sum_{i=1}^{n} |\alpha_i| a_i.$$

COROLLARY 6. *The linear subspace generated by the set $\{a_i\}_{i \in I}$ of Lemma 3 is a linear sublattice.*

A family $\{a_i\}_{i \in I}$ such as the one described in Lemma 3 will be called a *positive orthogonal family.*

The next theorem is reminiscent of the process of lifting countable orthogonal families of idempotents in an SBI ring. Curiously enough, as in ring theory (**10**, p. 316), it is not generally possible to lift uncountable positive orthogonal families (see Example 6 below).

THEOREM 8. *Let A and B be vector lattices and suppose that $f: A \to B$ is an epimorphism. Let $\{b_i\}_{i=1}^{\infty}$ be a countable positive orthogonal family in B. Then there is a countable positive orthogonal family $\{a_i\}_{i=1}^{\infty}$ in A with $f(a_i) = b_i$, for $i = 1, 2, \ldots$.*

Proof. We proceed by induction. For $n = 1$, the statement is almost trivial, for if $b > 0$ is in B, let $f(x) = b$ and take $a = |x|$, so that $a > 0$ and

$$f(a) = f(|x|) = |f(x)| = b.$$

Suppose then that $a_1, \ldots, a_n$ are already disjoint in A^+ with $f(a_i) = b_i$, $i = 1, \ldots, n$, and choose a_{n+1} with $f(a_{n+1}) = b_{n+1}$. Assume that $a_{n+1} > 0$; otherwise replace a_{n+1} by $|a_{n+1}|$ (which also maps onto b_{n+1}). Set

$$u = \sup\{a_i \wedge a_{n+1} : 1 \leqslant i \leqslant n\}.$$

Then $f(u) = 0$ and $0 \leqslant u < a_{n+1}$. Take $\bar{a}_i = a_i - (a_i \wedge a_{n+1})$ and $\bar{a}_{n+1} = a_{n+1} - u$. Then

$$0 \leqslant \bar{a}_i \wedge \bar{a}_{n+1} \leqslant (a_i - (a_i \wedge a_{n+1})) \wedge (a_{n+1} - (a_i \wedge a_{n+1})) = 0,$$

so $\bar{a}_i \wedge \bar{a}_{n+1} = 0$ for each $i = 1, \ldots, n$ and $f(\bar{a}_{n+1}) = b_{n+1}$. This completes the induction step and the proof.

We recall that a vector lattice is *boundedly complete* if every set bounded from above has a supremum. Call a homomorphism $f: A \to B$ between two boundedly complete vector lattices A and B *normal* if f preserves suprema whenever they exist.

Replacing the cardinality restrictions of Theorem 8 by others, we obtain

THEOREM 9. *Let A and B be boundedly complete vector lattices and suppose $f: A \to B$ is a normal epimorphism. Let $\{b_i\}_{i \in I}$ be any positive orthogonal family in B. Then there is a positive orthogonal family $\{a_i\}_{i \in I}$ in A, indexed by the same set I, with $f(a_i) = b_i$, for each $i \in I$.*

We shall simply remark that Theorem 9 can be proved in much the same way as Theorem 8, replacing the finite induction argument by a transfinite one.

The next proposition will be useful in constructing counterexamples to unrestricted lifting. By a *ring of sets* we shall mean a collection of subsets of some set which is closed under the formation of finite unions and differences. If $\mathfrak{A}$ is a ring of sets, then $[\mathfrak{A}]$ will denote the vector lattice consisting of finite linear combinations of characteristic functions of sets from $\mathfrak{A}$.

PROPOSITION 13. *Let A be a vector lattice, $\mathfrak{A}$ a ring of sets, and $\theta: \mathfrak{A} \to A^+$ a mapping which satisfies the conditions:*

(1) *$\theta(E - F) = \theta(E) - \theta(F)$ if $F \subset E$ with $E, F \in \mathfrak{A}$.*

(2) *$\theta(E \cap F) = \theta(E) \wedge \theta(F)$ for $E, F \in \mathfrak{A}$.*

Then θ has a unique extension to a homomorphism $\theta: [\mathfrak{A}] \to A$. If θ is 1–1 on $\mathfrak{A}$, the extension is a monomorphism. If $\theta(\mathfrak{A})$ generates A linearly, then the extension is an epimorphism.

Remark. The above statements remain valid if $\mathfrak{A}$ is a collection of sets which is closed under the formation of differences (and hence under intersections). It is easily checked that the linear space generated by the characteristic functions of sets from $\mathfrak{A}$ coincides with $[\mathfrak{B}]$, where $\mathfrak{B}$ is the ring generated by $\mathfrak{A}$.

Proof. For simplicity, we confuse sets with their characteristic functions wherever convenient. First observe that if E, F, and $E \cup F$ are in $\mathfrak{A}$ with $E \cap F = \emptyset$, we have $\theta(E \cup F) = \theta(E) + \theta(F)$, by (1). Let χ_E denote the characteristic function of $E \in \mathfrak{A}$. In order that θ have a unique linear extension to $[\mathfrak{A}]$ it is necessary and sufficient that

$$\sum_{i=1}^{n} \alpha_i \chi_{E_i} = 0 \quad \text{implies} \quad \sum_{i=1}^{n} \alpha_i \theta(E_i) = 0.$$

The finite additivity of θ clearly guarantees this (write each E_i as a disjoint union—the fact that $\mathfrak{A}$ is closed under differences permits this—and apply the standard argument for finitely additive measures). Next let $f \in [\mathfrak{A}]$, so that

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}.$$

We may assume that the E_i's are disjoint in pairs. Then

$$\theta(|f|) = \sum_{i=1}^{n} |\alpha_i|\theta(E_i) \quad \text{and} \quad \theta(E_i) \wedge \theta(E_k) = 0 \quad \text{for } i \neq k.$$

By Lemma 3, we then have $\theta(|f|) = |\theta(f)|$. Now assume that θ is 1–1 on $\mathfrak{A}$. For $f \geqslant 0$ write

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$$

with the E_i's disjoint, so that each $\alpha_i \geqslant 0$. Thus if $\theta(f) = 0$, we must have

$$\alpha_1 \theta(E_1) \vee \ldots \vee \alpha_n \theta(E_n) = 0,$$

since the $\theta(E_i)$'s are orthogonal. Hence $\alpha_i \theta(E_i) = 0$, for each i. But $\theta(E_i) \neq 0$ if $E_i \neq \emptyset$ since θ is 1–1 on $\mathfrak{A}$. Thus each $\alpha_i = 0$, so $f = 0$. Finally, suppose $\theta(f) = 0$ for f arbitrary. Then $\theta(|f|) = |\theta(f)| = 0$, so $|f| = 0$ and hence $f = 0$. Thus θ is a monomorphism. The last statement is obvious.

A vector lattice is called *countably decomposable* if every positive orthogonal family is at most countable.

LEMMA 4. *A ring of sets $\mathfrak{A}$ is countably decomposable if and only if the vector lattice $[\mathfrak{A}]$ is countably decomposable.*

Proof. Every non-zero positive step-function dominates some non-zero positive multiple of a characteristic function of a non-void set.

Example 6. Let I be a set having uncountable cardinality $\mathfrak{M}$ and let T' be the Boolean algebra of all finite and cofinite subsets of I. Express T' as a quotient $\theta'\colon S' \to T'$ of the free Boolean algebra S' on $\mathfrak{M}$ generators. It is known (**6**, p. 116) that every free Boolean algebra is countably decomposable. If we let T be the ring of all finite subsets of I and take $S = \theta'^{-1}(T)$, then S is evidently a countably decomposable ring of sets which is mapped homomorphically onto T by $\theta = \theta'|S$. Now let $A = [S]$ and $B = [T]$. By Proposition 13, θ has a unique extension to an epimorphism $\theta: A \to B$ of the vector lattices and by Lemma 4, A is countably decomposable since S is. But B contains a positive orthogonal family of cardinality $\mathfrak{M}$, so it is obviously impossible to lift this family or to find a cross-section to θ. A less sophisticated, but somewhat more intuitive counterexample for the case $\mathfrak{M} = \aleph_1$ can be had by taking S to be the ring of sets generated by all sets of the form $\{t\} \cup [7/4 + t, 9/4 + t)$, where $0 \leqslant t \leqslant 1$. Setting $T = S/N$, where N is the ideal of sets in S which are contained in $[7/4, 13/4)$, we see that T is isomorphic to the ring of all finite subsets of $[0, 1]$ whereas S is countably decomposable.

THEOREM 10. *Let I be any set. Then the following data describe isomorphic vector lattices of which the cardinality of I is a complete algebraic and order invariant:*

(1) *A is a vector lattice containing a positive orthogonal family $\{a_i\}_{i\in I}$ which generates A linearly.*

(2) *A is the vector lattice of all real-valued functions on I (ordered pointwise) which vanish outside of finite sets.*

(3) *A is a vector lattice with the property: There exist epimorphisms $f_i:A \to R$ and linearly independent elements $\{a_i\}_{i\in I}$ of A such that for all $a \in A$,*

$$a = \sum f_i(a)a_i,$$

where $f_i(a) = 0$ for all but a finite number of indices $i \in I$ (by dropping the words "linearly independent" and replacing R by the ground ring for the module A, we obtain the well-known characterization of projective modules).

Finally, a vector lattice A conforming to one of the above descriptions is projective if and only if the index set I is at most countable.

Proof. (1) implies (2) is immediate from Lemma 3.

(2) implies (3): We define f_i to be the epimorphism which evaluates a function at the point i.

(3) implies (1): Observe first that $a_i > 0$, for $a_i = \sum f_k(a_i)a_k$ and by the uniqueness of this expansion we have $f_k(a_i) = \delta_{ki}$ (Kronecker). Moreover,

$$a_i^- = \sum f_k(a_i^-)a_k = \sum (f_k(a_i))^- a_k = 0$$

since $1^- = (-1) \vee 0 = 0$. Thus $a_i > 0$ (no basis element can be zero). Secondly, for $i \neq k$, we have

$$a_i \wedge a_k = \sum f_n(a_i \wedge a_k)a_n = \sum [f_n(a_i) \wedge f_n(a_k)]a_n = 0.$$

This proves the first assertion.

From example 6 we see that A cannot be projective if I is uncountable. Finally suppose that I is at most countable, let B be any vector lattice and $f:B \to A$ an epimorphism. By Theorem 8 we can find a positive orthogonal family $\{b_i\}_{i\in I}$ in B with $f(b_i) = a_i$ for each $i \in I$. We define a linear map $g:A \to B$ by setting $g(a_i) = b_i$ and extending by linearity. Lemma 3 ensures that g is homomorphic, so that g is a cross-section to f. From Proposition 12 we see that A is projective.

Now that we have described some of the projectives, it is perhaps worth while stating a few consequences of projectivity.

COROLLARY 7. *Let $f:A \to B$ be an epimorphism of vector lattices and suppose B is projective. Let N be the null space of f. Then A admits a direct-sum decomposition (as a vector space):*

$$A = S \oplus N$$

where S is a projective linear sublattice isomorphic to B. Moreover, B is semi-simple, as is S and $\mathrm{Rad}(A) \subset N$.

Proof. Choose a cross-section g to f and let $S = g(B)$. Clearly the sum is direct and $\mathrm{Rad}(A) \subset N$ by Proposition 3 since B is semi-simple by Corollary 5.

The foregoing gives us an analogue to the First Principal Wedderburn Structure Theorem.

THEOREM 11. *Let A be a vector lattice which, modulo its radical, is projective. If $N =$* $\mathrm{Rad}(A)$, *then A splits into the vector space direct sum*

$$A = S \oplus N, \qquad \text{where } S \cong A/N,$$

and S is semi-simple. In particular, if A/N is finite dimensional, then such a decomposition obtains.

Proof. The first statement is clear from the preceding corollary and Corollary 2. But it is well known that a finite-dimensional semi-simple vector lattice (necessarily Archimedean) is isomorphic to R^n, where n is the dimension (**3**, p. 240, Corollary to Theorem **1**).

It appears highly likely that free (and therefore projective) vector lattices are countably decomposable. That this is the case for Boolean algebras follows from the duality theory between Boolean algebras and Boolean spaces; see (**6**) for an excellent account. E. C. Weinberg has recently informed us that his results in (**9**) imply that the free vector lattice on a countable number of generators is countably decomposable.

7. The Frattini sublattice. Let A be a vector lattice. A linear subspace $B \subset A$ is a *linear sublattice* if $a \in B$ implies $|a| \in B$. For a subset $S \subset A$, let $[S]$ denote the linear sublattice of A generated by S. An element $a \in A$ is called an *inessential generator* of A if for every subset $S \subset A$, $[S, a] = A$ implies $[S] = A$.

In analogy with the theory of groups and modules, we define the *Frattini sublattice* to be the intersection of all maximal (proper) linear sublattices. We do not attach a special symbol to this object for the following reason.

THEOREM 12. *The Frattini sublattice is the set of all inessential generators and both are zero.*

Proof. Let A be a vector lattice and let N be the intersection of all maximal linear sublattices of A. If a is inessential and M is a maximal linear sublattice, we must have $a \in M$, for otherwise $M = [M, a] = A$, a contradiction. Hence it is enough to show that $N = 0$. We do this by showing that for any $a \neq 0$ in A, one can find a maximal linear sublattice M avoiding a. First we notice that this is indeed possible if A is totally ordered, for then every linear subspace is a linear sublattice (A is totally ordered if and only if no one-dimensional subspace has zero intersection with A^+). By the first part of the proof of Proposition 11, we can always find a "prime" ideal $P \not\ni a$ so that A/P is totally ordered. If P (a linear sublattice) is a hyperplane, the proof is complete. If not, we proceed as follows. The linear sublattices of A/P are in a natural 1–1 correspondence with the linear sublattices of A which contain P. If $\bar{a}$ denotes the non-zero image of a in A/P, then any hyperplane in A/P which misses $\bar{a}$ lifts back into A giving a linear sublattice which misses a and is a hyperplane.

COROLLARY 8. *Every non-zero element of a vector lattice is an essential generator.*

COROLLARY 9. *Every vector lattice has a family of hyperplane sublattices with zero intersection. Every (proper) ideal is the intersection of all hyperplane sublattices which contain it. Moreover, any linear subspace containing a prime ideal is automatically a linear sublattice.*

Proof. If I is any (proper) ideal and $a \notin I$ we can find (as in Proposition 11) a prime ideal P containing I but not a. Thus I is the intersection of all prime ideals containing it. All statements now follow easily from the proof of Theorem 12.

Further perusal of the proof of Theorem 12 reveals that the Frattini sublattice of a lattice-ordered abelian group coincides with its Frattini subgroup.

In any category, we can quasi-order the epimorphisms onto an object B as follows. Given epimorphisms $f:A \to B$ and $g:C \to B$ we call $f \leqslant g$ if there is an epimorphism $h:C \to A$ with $fh = g$. A *cover* of B is an epimorphism $f:A \to B$ such that for every homomorphism $g:C \to A$, fg is an epimorphism implies g is an epimorphism. Thus every cover is $\leqslant$ every quotient $Y \to B$ with Y projective **(2)**.

THEOREM 13. *In the category of vector lattices and linear lattice homomorphisms, the only covers are the isomorphisms.*

Proof. Suppose $f:A \to B$ is a cover and let S be any linear sublattice of A. Let N be the null space of f. Let us assume that $[S, N] = A$. If we can show that $S = A$, the theorem follows, for then N is inessential and therefore zero.

Let $i:S \subset A$ denote the inclusion map. If fi is not epimorphic, then we can find an element $b \in B$ with $b \notin fi(S)$. Let $f(a) = b$. Then $0 \neq a \in [S, N](= A)$, but $a \notin N$ and $a \notin S$. Thus $[S, N]$ would be properly contained in $[S, N, a]$

(Corollary 8), which is clearly absurd. Hence fi is an epimorphism as is i. In other words, $S = A$.

Past experience in this category prompts us to consider the dual situation. In any category, we can quasi-order the monomorphisms out of an object A as follows. Given monomorphisms $f:A \to B$ and $g:A \to C$ we call $f \leqslant g$ if there is a monomorphism $h:B \to C$ with $hf = g$. An *envelope* of A is a monomorphism $f:A \to B$ such that for every homomorphism $g:B \to C$, gf is a monomorphism implies g is a monomorphism. Thus every envelope is $\leqslant$ every embedding $A \to X$ with X injective **(2)**.

The great abundance of envelopes is evidenced by

PROPOSITION 14. *Let A be a linear sublattice of B. Then B is an envelope of A if and only if for each ideal $I \subset B$, $I \cap A = 0$ implies $I = 0$.*

Proof. Suppose B is an envelope and let $f:A \to B$ be the inclusion map. Given an ideal $I \subset B$ with $I \cap A = 0$, let $g:B \to B/I$ be the canonical epimorphism. Since $I \cap A = 0$, gf is a monomorphism as is g, so that $I = \operatorname{Ker} g = 0$.

Conversely, if the second property holds and if $g:B \to C$ is a homomorphism such that gf is a monomorphism, then $(\operatorname{Ker} g) \cap A = 0$. Thus $\operatorname{Ker} g = 0$ so that g is a monomorphism.

Example 7. Let A be any vector lattice and T any totally ordered vector lattice. As in Proposition 5, we construct the lexicographic sum $E = T \oplus A$ as follows. E is the vector space direct sum of T and A ordered by defining $(t, a) \geqslant 0$ if either $t > 0$ or else $t = 0$ and $a \geqslant 0$. Now for every ideal $I \subset E$, we have $I \subset A$ or $A \subset I$. For if $I \not\subset A$ we can find an $x \in I$ with $x \notin A$, so that $x = (t, a)$ with $t \neq 0$. We assume $t > 0$ (otherwise we work with $-x$). Then for every $y \in A$, $|y| \leqslant x$, so $A \subset I$. Evidently then, E is an envelope of A, since A meets every non-zero ideal of E in a non-zero element.

Example 8. Let X be any locally compact, *non-compact* (Hausdorff) space. Take for E either (1) all bounded continuous real-valued functions on X or (2) all continuous real-valued functions on X which vanish at infinity. Let M be the unique minimal free algebra ideal in E of all continuous functions vanishing outside of compact sets. Note that any (lattice) ideal in E is also an algebra ideal, since E is normed and contains constants. Now any linear sublattice A of E containing M is enveloped by E.

There are many variations on these themes. A rather extreme case is obtained by taking any (large) totally ordered vector lattice T and forming the lexicographic sum $E = T \oplus R$ so that the reals R appear as the unique minimal ideal of E. Since E itself is totally ordered, any linear subspace A is a linear sublattice. Any such A containing R is therefore enveloped by E.

These examples show that there may be no way of controlling the size of envelopes if injectives are absent. Comparison with other categories sheds further light on this situation. Every abelian group has a "largest" envelope (in the quasi-order for monomorphisms)—its divisible hull—which is also injective. In fact, for a fixed object in any category, an injective envelope is $\geqslant$ every other envelope. Thus the existence of an injective envelope (any two are isomorphic) for each object keeps matters in hand.

We remarked earlier on the absence of injectives in the category of rings and ring homomorphisms; here, as in the case of vector lattices, envelopes abound. It is easy to see that the analogue of Proposition 14 for rings is valid (ideal $\equiv$ 2-sided ideal). Any integral domain is enveloped by its field of quotients or any extension thereof. Other examples of envelopes are: (1) B any simple ring, A any non-zero subring of B, then B is an envelope of A; (2) the ring C of all completely continuous operators on a Hilbert space of infinite dimension; C envelops any of its subalgebras which contains the unique minimal ideal of operators having finite-dimensional range.

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