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Mumford's Degree of Contact and Diophantine Approximations

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Abstract. The purpose of this note is to present a somewhat unexpected relation between diophantine approximations and the geometric invariant theory. The link is given by Mumford's degree of contact. We show that destabilizing flags of Chow-unstable projective varieties provide systems of diophantine approximations which are better than those given by Schmidt's subspace theorem, and we give examples of these systems

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Introduction

Let us fix two number fields $K \subseteq L$, let Σ be a finite set of places of L containing all infinite places, and let E be a vector space of rank $N+1$ over K. For each $v \in \Sigma$ let $l_{v,0}, \cdots, l_{v,N}$ be a basis of $E \otimes_K L$ over L, $r_{v,0} \geqslant \cdots \geqslant r_{v,N} \geqslant 0$ be integers and $X \subseteq \mathbb{P}(E^\vee)$ be a closed subvariety. Under which condition do we have the following property?

PROPERTY P(X). *The points $x \in X(K)$ such that*

$$\log\left(\frac{|l_{v,i}(\mathbf{x})|_v}{|\mathbf{x}|_v}\right) \leqslant -r_{v,i}h(\mathbf{x}) \quad v \in \Sigma,\ i = 0, \cdots, N,$$

are contained in finitely may subvarieties of X.

We call these last subvarieties the exceptional subvarieties of P(X). One can gain insight into this problem using a theorem of G. Faltings and G. Wüstholz. ([1], Theorem 7.3 and [2], for a quantitative version). They introduce, for each $v \in \Sigma$, a probability measure whose expected value $E_v(X)$ (see (1.7)) allows to establish a criterion for the truth of $P(X)$. Indeed, if the sum over all $v \in \Sigma$ of these expected values is bigger than 1, then $P(X)$ holds. Schmidt's Subspace Theorem corresponds exactly to the case $X = \mathbb{P}^n$. In the situation where the geometry of X is such that

$$\sum_{v\in\Sigma} E_v(X) \leqslant \sum_{v\in\Sigma}\sum_{i=0}^{N} r_{v,i}/(N+1) = \sum_{v\in\Sigma} E_v(\mathbb{P}(E^\vee)), \tag{0.1}$$

if we impose the condition $\sum_{v\in\Sigma} E_v(X) > 1$ we get $\sum_{v\in\Sigma} E_v(\mathbb{P}(E^\vee)) > 1$, and $P(X)$ follows by intersecting X with the exceptional varieties of $P(\mathbb{P}(E^\vee))$.Hence, the strength of Faltings–Wüstholz' result appears when the geometry of the system gives the inequality (0.1).

When (0.1) is true, the Faltings–Wüstholz method is (up to now) stronger then anything else in the context of diophantine approximations on projective varieties.

The aim of this note is to give examples where (0.1) is true. As we will later see, this happens when the Chow-point of $X \times \cdots \times X$ ($\#\Sigma$-times) is unstable with respect to the weighted bases $\{(l_{v,i}, r_{v,i}) : v \in \Sigma,\ \ i = 0, \cdots, N\}$. Geometric invariant theory tells us that unstable objects are those for which it is not possible to have a nice moduli space. Therefore, in order to find examples of (0.1), we have to look for what is avoided in the construction of modular spaces, and that is what we will do.

Unstability is usually associated with singular varieties. We will see this for hypersurfaces, for unstable local rings, and for a family of elliptic surfaces. However, ruled surfaces, and some blow-ups give examples of nonsingular varieties having an unstable embedding. In general it is quite difficult to find examples of unstable projective varieties. All examples we found have a common style: the destabilizing flags are constructed using sections vanishing along special divisors. Moreover, in order to control all the dimensions involved, it is crucial to use the Riemann–Roch theorem, as well as some results on vanishing of higher cohomology.

1. Degree of Contact

1.1. Let E be a vector space of rank $N+1$ over the number field K and $E^\vee = \mathrm{Hom}(E, K)$. Consider a closed dimension d subvariety $X \subseteq \mathbb{P}(E^\vee)$ in the projective space of lines of $E^\vee$. We choose a basis $l_0, \cdots, l_N \in E$, fix nonnegative real numbers $r_0 \geqslant \cdots \geqslant r_N$ and let $\mathbf{r} = (r_0, \cdots, r_N)$. When m is large enough, say $m \geqslant m_0$, the cup product map

$$\varphi : E^{\otimes m} \to H^0(X, \mathcal{O}(m))$$

is surjective, so that $H^0(X, \mathcal{O}(m))$ is generated by the monomials

$$l_0^{\alpha_0} \cdots l_N^{\alpha_N} = \varphi(l_0^{\otimes\alpha_0} \otimes \cdots \otimes l_N^{\otimes\alpha_N}),$$

with $\alpha_0 + \cdots + \alpha_N = m$. A *special basis* is a basis of $H^0(X, \mathcal{O}(m))$ consisting of such elements. We define the weight of l_i to be r_i, $i = 0, \cdots, N$, the weight of a monomial in $E^{\otimes m}$ to be the sum of the weights of l_i's occurring in it, and the weight of a monomial $u \in H^0(X, \mathrm{O}(m))$ to be the minimum $w_{\mathbf{r}}(u)$ of the weights of the monomials in l_i's mapping to u by φ. The weight of a special basis is the sum of the weights of its elements. Finally, $w_{\mathbf{r}}(m)$ denotes the minimal weight among all special bases of $H^0(X, \mathcal{O}(m))$. Fix a special basis of minimal weight, and denote by $w_1, \cdots, w_M$ the weights of its elements in increasing order. Let $F^0 = \{0\}$, and F^j be equal to the span of all monomials u of weight less or equal to $w_j, j = 1, \cdots, M$. We define a probability measure on $\mathbb{R}$ with the density function:

$$\rho_m(x) = \frac{1}{h^0(X, \mathcal{O}(m))} \sum_{\mathrm{j}=1}^{M} \dim(F^j/F^{j-1})\delta_{w_j/m}(x), \tag{1.1}$$

where $\delta_a(x)$ is the Dirac distribution supported at $a \in \mathbb{R}$. Its expected value $E(\rho_m)$ equals

$$E(\rho_m) = \frac{w_{\mathbf{r}}(m)}{m \cdot h^0(X, \mathcal{O}(m))}. \tag{1.2}$$

1.2. Let $r_0 \geqslant \cdots \geqslant r_N \geqslant 0$ be integers, and $l_0, \cdots, l_N$ be a basis of E. Define $s_i = r_i - r_N,\ i = 0, \cdots, N$, and put $\mathbf{s} = (s_0, \cdots, s_N)$. The $\mathbf{s}$-weight of a monomial of degree m in l_i's differs form its $\mathbf{r}$-weight by mr_N, so

$$w_{\mathbf{r}}(m) = w_{\mathbf{s}}(m) + mr_N h^0(X, \mathcal{O}(m)).$$

By the usual theory of Hilbert polynomials

$$h^0(X, \mathcal{O}(m)) = \deg(X)\frac{m^d}{d!} + \mathrm{O}(m^{d-1}), \tag{1.3}$$

hence

$$w_{\mathbf{r}}(m) = w_{\mathbf{s}}(m) + r_N \deg(X)(d+1)\frac{m^{d+1}}{(d+1)!} + \mathrm{O}(m^d). \tag{1.4}$$

According to [8], Corollary 3.3, there exists an integer $e_{\mathbf{s}}(X)$ such that when m goes to infinity

$$w_{\mathbf{s}}(m) = e_{\mathbf{s}}(X)\frac{m^{d+1}}{(d+1)!} + \mathrm{O}(m^d).$$

If we define

$$e_{\mathbf{r}}(X) = e_{\mathbf{s}}(X) + r_N \deg(X)(d+1), \tag{1.5}$$

then by (1.4) we have

$$w_{\mathbf{r}}(m) = e_{\mathbf{r}}(X)\frac{m^{d+1}}{(d+1)!} + \mathrm{O}(m^d). \tag{1.6}$$

We call $e_{\mathbf{r}}(X)$ the *degree of contact* of X with respect to the basis $l_0, \cdots, l_N$ and to the

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weights $r_0, \cdots, r_N$. Combining (1.3) with (1.6) we obtain

$$E_\infty(X) := \lim_{m\to\infty} E(\rho_m) = \frac{e_{\mathbf{r}}(X)}{(d+1)\deg(X)}. \tag{1.7}$$

1.3. There is an intersection theoretic formula expressing the degree of contact as the degree of a divisor on a suitable modification of X.

Let $\mathcal{O}_{X\times\mathbb{A}^1}(1) = \mathcal{O}_X(1)\otimes\mathcal{O}_{\mathbb{A}^1}$, and t be the coordinate of $\mathbb{A}^1$. To integers $r_0 \geqslant \cdots \geqslant r_N \geqslant 0$ and a basis $l_0, \cdots, l_N$ of E we associate the $K[t]$-submodule I of $H^0(X\times \mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}(1))$ generated by $\{t^{r_i - r_N} l_i,\ i = 0, \cdots, N\}$, and an ideal sheaf $J \subseteq \mathcal{O}_{X\times\mathbb{A}^1}$ defined by

$$J\cdot \mathrm{O}_{X\times\mathbb{A}^1}(1) = \text{sheaf generated by } I \text{ in } \mathcal{O}_{X\times\mathbb{A}^1}(1).$$

Choose a compactification Y of $X\times\mathbb{A}^1$ on which $\mathcal{O}_{X\times\mathbb{A}^1}(1)$ extends to a line bundle $\mathcal{L}$, and let $\pi : B \to Y$ be the blow-up along the subscheme Z of Y defined by the ideal sheaf J. Then we have

$$e_{\mathbf{r}}(X) - r_N \deg(X)(d+1) = c_1(\pi^*\mathcal{L})^d - (c_1(\pi^*\mathcal{L}) - c_1(\mathcal{O}_B(S)))^d, \tag{1.8}$$

where S is the exceptional divisor, ((1.5); [8], Proposition 2.4; [8], Proposition 3.2). According to [3] §4, one may write the degree of contact in terms of Segre classes:

$$e_{\mathbf{r}}(X) - r_N \deg(X)(d+1) = (1 + c_1(\mathcal{L}))^d \cap s(Z, Y),$$

((1.5); [3], Corollary 4.2.2, and the projection formula). Thus, when Z is set-theoretically a point $w_{\mathbf{r}}(m)$ is the Hilbert–Samuel polynomial of J as an ideal in $\mathrm{O}_{X\times\mathbb{A}^1}$ ([3], §4.3, and Example 4.3.4 for the definition of multiplicity), and $e_{\mathbf{r}}(X) - r_N \deg(X)(d+1)$ the multiplicity at Z ([10], §2 Examples *ii*) p.55).

1.4. The Chow point of X is *semistable* with respect to integers $r_0 \geqslant \cdots \geqslant r_N$ and a basis $l_0, \cdots, l_N$ of E if and only if

$$E_\infty(X) \leqslant \frac{1}{N+1}\sum_{i=0}^{N} r_i \tag{1.9}$$

([10], Theorem 2.9). If this property is satisfied for all bases and weights as above, the Hilbert–Mumford theorem implies that the Chow point of X is semistable with respect to the action of $SL(E)$. Needless to say, the opposite of semistable is *unstable*.

1.5. Let $F(\mathbf{X}_0, \cdots, \mathbf{X}_d)$ be the Chow form of X, with respect to the embedding given by a basis $l_0, \cdots, l_N$ of E. It is a multihomogeneous polynomial of degree $\deg(X)$ in each set of variables $\mathbf{X}_i = (X_{i0}, \cdots, X_{iN})$, $i = 0, \cdots, d$. Let t be an auxiliary variable, and $r_0 \geqslant \cdots \geqslant r_N$ be integers. We consider the decomposition

$$F(t^{r_0}X_{00}, \cdots, t^{r_N}X_{0N}, \cdots, t^{r_N}X_{dN}) = \sum_k t^{\rho_k} F_k(\mathbf{X}_0, \cdots, \mathbf{X}_d),$$

where $F_k(\mathbf{X}_0, \cdots, \mathbf{X}_d)$ are polynomials not containing the variable t. According to [10], Proposition 2.11 we get

$$e_{\mathbf{r}}(X) = \min\{\rho_k;\ F_k \not\equiv 0\}. \tag{1.10}$$

2. Diophantine Approximations

2.1. Let K be an algebraic number field. Denote its ring of integers by O_K, and its collection of places (equivalence classes of absolute values) by M_K. For $v \in M_K$, $x \in K$, we define the absolute value $|x|_v$ by

(1) $|x|_v = |\sigma(x)|^{1/[K:\mathbb{Q}]}$ if v corresponds to the embedding $\sigma : K \hookrightarrow \mathbb{R}$;
(2) $|x|_v = |\sigma(x)|^{2/[K:\mathbb{Q}]} = |\overline{\sigma}(x)|^{2/[K:\mathbb{Q}]}$ if v corresponds to the pair of conjugate embeddings $\sigma, \overline{\sigma} : K \hookrightarrow \mathbb{C}$;
(3) $|x|_v = (N\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)/[K:\mathbb{Q}]}$ if v corresponds to the prime ideal $\mathfrak{p}$ of O_K.

Here $N\mathfrak{p} = \#(\mathrm{O}_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$, and $\mathrm{ord}_{\mathfrak{p}}(x)$ is the exponent of $\mathfrak{p}$ in the prime ideal decomposition of (x), with $\mathrm{ord}_{\mathfrak{p}}(0) := \infty$. In case 1 or 2 we call v real infinite or complex infinite respectively, and write $v|\infty$; in case 3 we call v finite, and write $v \nmid \infty$. These absolute values satisfy the *Product Formula*

$$\prod_{v \in M_K} |x|_v = 1, \text{ for } x \in K^*. \tag{2.1}$$

The height of $\mathbf{x} = (x_0, \cdots, x_N) \in K^{N+1}$ with $\mathbf{x} \neq 0$ is defined as follows: for $v \in M_K$ put

$$|\mathbf{x}|_v = \left(\sum_{i=0}^N |x_i|_v^{2[K:\mathbb{Q}]}\right)^{1/2[K:\mathbb{Q}]} \quad \text{if } v \text{ is real infinite,}$$

$$|\mathbf{x}|_v = \left(\sum_{i=0}^N |x_i|_v^{[K:\mathbb{Q}]}\right)^{1/[K:\mathbb{Q}]} \quad \text{if } v \text{ is complex infinite,}$$

$$|\mathbf{x}|_v = \max\{|x_0|_v, \cdots, |x_N|_v\} \text{ if } v \text{ is finite.}$$

Now define

$$h(\mathbf{x}) = h(x_0, \cdots, x_N) = \sum_{v \in M_K} \log |\mathbf{x}|_v.$$

By Product Formula (2.1) this defines a function on $\mathbb{P}^N(K)$. Further, $h(\mathbf{x})$ depends only on $\mathbf{x}$ and not on the choice of the number field K containing the coordinates

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of $\mathbf{x}$. In other words the function $h(\mathbf{x})$ extends to a function on $\mathbb{P}^N(\overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$.

It is possible to define the height $h(X)$ of a projective variety $X \subseteq \mathbb{P}^N$ defined over K (see [13] for a good reference). This invariant can be computed through the height of the (Cayley–Bertini–van der Waerden –) Chow point of X.

2.2. From now on, we will denote with $L \supseteq K$ a finite field extension, and Σ a finite set of places of L containing all infinite places. For each $v \in \Sigma$ we choose a basis $l_{v,0}, \cdots, l_{v,N}$ of $E \otimes_K L$ and rational numbers $r_{v,0} \geqslant \cdots \geqslant r_{v,N} \geqslant 0$. Since the degree of contact is linear in the weights this defines a real number $E_{v,\infty}(X \otimes_K L)$ as in (1.7).

THEOREM 2.3. *Let us suppose that for some real number* $0 < \delta < 1$ *we have,*

$$\sum_{s\in\Sigma} E_{v,\infty}(X \otimes_K L) = 1 + \delta. \tag{2.2}$$

Then there are effectively computable positive real numbers c_1, c_2, *and* c_3 *depending only on* K, Σ, δ, N, $\deg(X)$, $h(X)$, *and* $\sup_{v\in\Sigma, i=0,\cdots,N} h(l_{v,i})$ *such that all points* $\mathbf{x} \in X(K)$ *with*

$$h(\mathbf{x}) > c_1, \tag{2.3}$$

and

$$\log\left(\frac{|l_{v,i}(\mathbf{x})|_v}{|\mathbf{x}|_v}\right) \leqslant -r_{v,i}h(\mathbf{x}) \quad v \in \Sigma,\ i = 0, \cdots, N, \tag{2.4}$$

are contained in at most c_2 *proper subvarieties of* X *of degree not exceeding* c_3.

Proof. [2] Theorem 10.2, and [1] Theorem 7.3 for a quantitative version. □

Remark 2.4. Theorem 10.2 of [1] claims that if (2.2) holds for some $\delta > 0$, then the solutions $x \in X(K)$ of (2.4) are not Zarisky dense in X. If we further assume that $\delta < 1$, then we can determine explicitly the constants c_1, c_2, c_3 ([2], Theorem 7.3).

Remark 2.5. A priori, it is not clear why one has to choose the coefficients of the linear forms $l_{v,0}, \cdots, l_{v,N}$ in a field L bigger than K. One reason is that this is coherent with the spirit of the original Roth's Theorem, where one considers rational approximations of algebraic numbers. Moreover, Theorem 2.3 can explicitly control the dependence of the solutions from the field L.

3. Hypersurfaces

3.1. Let $l_0, \cdots, l_N$ be a basis of E and for each $v \in \Sigma$ let $r_{v,0} \geqslant \cdots \geqslant r_{v,N} \geqslant 0$ be integers. Let $X \subseteq \mathbb{P}(E^\vee)$ be a hypersurface. Suppose that, with respect to the basis $l_0, \cdots, l_N$, X is given by the polynomial $F(l_0, \cdots, l_N) = \sum_{i_0+\cdots+i_N=g} a_{i_0,\cdots,i_N} l_0^{i_0} \cdots l_N^{i_N}$.

THEOREM 3.2. *For each $v \in \Sigma$ let $d_v > 0$ be real numbers such that*

$$\sum_{v\in\Sigma} \frac{1}{d_v} \min\left\{ \sum_{k=0}^{N} i_k \sum_{j\neq k} r_{v,j};\ a_{i_0,\cdots,i_n} \neq 0 \right\} > N \cdot g. \tag{3.1}$$

Then the points $\mathbf{x} \in X(K)$ with

$$\log\left(\frac{|l_i\mathbf{x})|_v}{|\mathbf{x}|_v}\right) \leqslant -\frac{r_{v,i}}{d_v} h(\mathbf{x}),\ \ v \in \Sigma,\ \ i = 0, \cdots, N$$

are contained in finitely many subvarieties of X of bounded degree.

Proof. The Chow form of X is $F(\cdots, p^{j_0,\cdots j_{k-1}, j_{k+1},\cdots j_N}, \cdots)$, where we used the Plücker coordinates

$$p^{j_0,\cdots j_{k-1}, j_{k+1},\cdots j_N} = (-1)^k \det(X_{ij}),$$

with $i, k \in \{0, \cdots, N-1\}$, and $j \in \{0, \cdots, k-1, k+1, \cdots, N\}$. We compute the degree of contact using (1.10). Then by (3.1) we can apply Theorem 2.3 and finish the proof. □

3.3. If we do not have any singularities on X, the discriminant of the polynomial F does not vanish. Since the discriminant is $SL(E)$-invariant, this means that if X is non-singular, then it is semistable ([10], §1.5). Since we are interested in unstable objects, this means that $X^{\#\Sigma}$ has to have quite bad singularities.

4. Local Rings

4.1. Let E be a vector space of rank $N+1$ over K, and $E^\vee$ =Hom (E, K). Consider a dimension d closed subvariety $X \subseteq \mathbb{P}(E^\vee)$ in the projective space of lines of $E^\vee$. Let $P \in X(L)$ be a closed point, and let $l_0, \cdots, l_N$ be a basis of $E \otimes_K L$ so that $P = (1, 0, \cdots, 0)$ with respect to the coordinates in $\mathbb{P}(E^\vee)$ defined by this basis. For $v \in \Sigma$ we define weights $r_{0,v} = k_v$, $r_{1,v} = \cdots = r_{v,N} = 0$.

PROPOSITION 4.2. *Let $P \in X(L)$ be a closed point, and for $v \in \Sigma$ let $d_v > 0$ be a real number such that*

$$\mathrm{mult}_P X \sum_{v\in\Sigma} \frac{k_v}{d_v} > (\dim X + 1)\ \deg\ X.$$

Then the points $\mathbf{x} \in X(K)$ with

$$\log \frac{|l_0(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant -\frac{k_v}{d_v} h(\mathbf{x}),\ \ v \in \Sigma,$$

lie in finitely many subvarieties of X of bounded degree.

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Proof. For each $v \in \Sigma$ the ideal $J \cdot \mathrm{O}_{X\times\mathbb{A}^1}(1)$ from Proposition 1.3 is generated by $\{t^{k_v}l_0, l_1, \cdots, l_N\}$. Since $\{l_1, \cdots, l_N\}$ generate the maximal ideal $\mathfrak{m}_{P,X}$, and l_0 is a unit at P, $J = (t^{k_v}, \mathfrak{m}_{P,X})\mathrm{O}_{X\times\mathbb{A}^1}(1)$. Hence

$$e_{\mathbf{r}}(X) = k_v \cdot \mathrm{mult}_{(0,P)}(X \times \mathbb{A}^1) = k_v \cdot \mathrm{mult}_P X.$$

The claim follows by the linearity of the expected value in the weights, (1.7), and Theorem 2.3. □

4.3. The attempt to generalize this theorem leads to a numerical measure of the degree of singularity at a point. Let R be a local ring of dimension r and m be a positive integer. Then the mth *flat multiplicity* $e_m(R)$ of R is defined by

$$e_0(R) = \sup\left\{\frac{e(I)}{r!col(I)}:\ I \text{ of finite colength in } R\right\}$$
$$e_m(R) = e_0(R[[t_0, \cdots, t_m]]),$$

where $e(I)$ denotes the multiplicity of the ideal I. Further, a local ring is called *semistable* if $e_1(R) = 1$. Let X be a projective scheme, and $\mathcal{L}$ be an ample line bundle on X. For a sufficiently large n let

$$\phi_n : X \to \mathbb{P}^{h^0(X,\mathcal{L}^{\otimes n})-1}$$

be the embedding defined by $\mathcal{L}^{\otimes n}$. Suppose that there exists a point P on X such that the local ring $\mathrm{O}_{P,X}$ is unstable. Then for every positive integer n, there exists a positive integer $m > n$ such that the Chowpoint and the Hilbert point corresponding to $\phi_m(X)$ are unstable under the natural action of $SL(h^0(X, \mathcal{L}^{\otimes m}))$ ([10], Proposition 3.12, [12], Proposition 1.3). Little is known about the semistability of local rings. See, however, [12] and [10], §3.

4.4. Suppose that there is a closed point P on $X(L)$ such that the local ring $\mathrm{O}_{P,X}$ is unstable. By [10], Lemma 3.6, there exists a sequence of ideals of finite colength

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_M = \mathrm{O}_{P,X} = I_{M+1} = \cdots$$

such that if I is the ideal $\oplus_{i\geqslant 0} I_i t^i \subseteq \mathrm{O}_{P,X}[[t]]$, then

$$e(I) = (1+\varepsilon)(\dim X + 1)!\mathrm{col}(I), \tag{4.1}$$

where $\varepsilon > 0$. We can choose n sufficiently large so that

(1) $\mathcal{L}^{\otimes n}$ is very ample,
(2) the map $S^m H^0(X, \mathcal{L}^{\otimes n}) \to H^0(X, \mathcal{L}^{\otimes nm})$ is surjective for all $m \geqslant 1$,
(3) the map $\psi_m : H^0(X, \mathcal{L}^{\otimes mn}) \to \mathrm{O}_{P,X}/I_0^m$ is surjective for all $m \geqslant 1$,
([12], p. 334). The vector space $H^0(X, \mathcal{L}^{\otimes n})$ has the induced filtration defined by

$$F^j = \psi_1^{-1}(I_j/I_0),\ j = 0, \cdots, M.$$

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For each $v \in \Sigma$ choose a basis $l_{v,0}, \cdots, l_{v,N}$ of $H^0(X, \mathcal{L}^{\otimes n})$ compatible with the filtration. Assign the weights as in (5.2):

$$r_{v,i} = \min\{j : \ l_{v,i} \in F^j\}. \tag{4.2}$$

THEOREM 4.5. *For each $v \in \Sigma$ let $d_v > 0$ be a real number such that*

$$\sum_{v \in \Sigma} \sum_{i=0}^{N} \frac{r_{v,i}}{d_v} > \frac{\deg(X)}{\dim(X)!}. \tag{4.3}$$

Then the points $\mathbf{x} \in X(K)$ with

$$\log\left(\frac{|l_{v,i}(\mathbf{x})|_v}{|\mathbf{x}|_v}\right) \leqslant -\frac{r_{v,i}}{d_v} h(\mathbf{x}), \ \ v \in \Sigma, \ \ i = 0, \cdots, N$$

are contained in finitely many subvarieties of X of bounded degree.

Proof. Since the colength of I corresponds to the sum of weights ([12], p. 334), the theorem follows by (4.1), (4.3) and Theorem 2.3. □

Remark 4.6. According to (0.1), Theorem 4.5 is stronger than the Schmidt Subspace Theorem. For several examples of destabilizing weighted flags of unstable two-dimensional local rings, we refer to [12], §§4, 5, 6.

5. Ruled Surfaces

5.1. The *Steiner surface* $X \subseteq \mathbb{P}^4$ is the closure of the image of the map $\psi : \mathbb{P}^2 \to \mathbb{P}^4$ defined by

$$(x, y, z) \mapsto (xz, yz, x^2, xy, y^2) = (l_0, \cdots, l_4).$$

This surface is the blow-up of $\mathbb{P}^2$ at the point $P = (0, 0, 1)$, and embedded by the system of conics passing through P. It is a rational ruled surface of type $\mathbb{F}_1$, ruled by the pencil of lines passing through P. Theorem 2.3 implies:

PROPOSITION 5.2. *For each $v \in \Sigma$ choose non-negative integers k_v, and positive rationals d_v with*

$$\sum_{v \in \Sigma} \frac{k_v}{d_v} > \frac{9}{4}.$$

Then the solutions $\mathbf{x} \in X(K)$ of the inequalities

$$\log\left(\frac{|l_i(\mathbf{x})|_v}{|\mathbf{x}|_v}\right) \leqslant -\frac{k_v}{d_v} h(\mathbf{x}), \ \ v \in \Sigma, \ \ i = 0, 1$$

are contained in finitely many curves in X.

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Proof. For each $v \in \Sigma$ let us consider the flag given by choosing the linear forms l_i as above and with weights $r_{v,0} = r_{v,1} = k_v$ and $r_{v,2} = r_{v,3} = r_{v,4} = 0$. Then a computation shows that $e_{\mathbf{r}_v}(X) = 4k_v$ ([8], Example 3.6). The degree of the Steiner surface is 3, the number of free points of intersection of two conics passing through $(0, 0, 1)$. Altogether we get

$$\frac{e_{\mathbf{r}_v}(X)}{(\dim(X)+1)\deg(X)} = \frac{4k_v}{3 \cdot 3},$$

and the proposition follows from Theorem 2.3. □

Remark 5.3. If we consider the same problem for $\mathbf{x} \in \mathbb{P}^4(K)$, then the Schmidt Subspace Theorem implies that its solutions are contained in finitely many subspaces of $\mathbb{P}^4$, hence in finitely many curves of X, if the condition $\sum_{v \in S}(k_v/d_v) > 5/2(> 9/4)$ holds true. However, Vojta's conjecture predicts 2 as the best possible lower bound ([14], Conjecture 3.4.3).

5.4. A good reference for the general properties of ruled surfaces is [5], V.2. Fix a smooth curve C of genus g, and a geometrically ruled surface $p : R \to C$. For one section $\sigma : C \to R$ of p we refer to the divisor S on R which is the image of σ. Fix one section, say S, and let f denote the numerical equivalence class of a fibre of p. When the parity of S^2, which is independent of S, is even, then a convenient basis of $Num(R)$ is given by f, and the element G determined by $G^2 = 0$ and $G \cdot f = 1$. For surfaces of odd parity such a G can be found in $Num(R) \otimes \mathbb{Q}$. We will by abuse of language consider G as a divisor on R. This is no restriction since we will be utilizing only the numerical properties of G.

5.5. Fix a ruled surface R, a very ample divisor $D \sim aG + bf$ on R, and a section S such that $D - aS$ is effective. Let $\mathbb{P}^N = \mathbb{P}(H^0(R, D)^\vee)$ and let $F^j = \{ h \in H^0(R, D),\ h \in H^0(R, D - jS)\}$. Since a section in D can vanish to order at most a on S,

$$H^0(R, D) = F^0 \supset F^1 \supset \cdots \supset F^a \supset F^{a+1} = \{0\}.$$

Choose a basis $l_0, \cdots, l_N$ of $H^0(R, D)$ compatible with this flag, and fix weights r_i, $i = 0, \cdots, N$ by the condition that

$$r_i = a - j \Longleftrightarrow l_i \in F^j \setminus F^{j+1}.$$

That is, r_i equals a minus the order to which l_i vanishes along S, for $i = 0, \cdots, N$. Clearly, the r_i's decrease to zero. This construction generalizes that from **5.1.** of the Steiner surface in $\mathbb{P}^4$. There, we have $C = \mathbb{P}^1, \mathcal{E} = \mathrm{O}_{\mathbb{P}^1}(1) \oplus \mathrm{O}_{\mathbb{P}^1}(2)$ and $R = \mathbb{P}(\mathcal{E})$. Further, $D = \mathrm{O}_{\mathbb{P}(\mathcal{E})}(1) \sim G + 3/2f$ and S, the exceptional divisor, which is numerical equivalent to $G - 1/2f$ and which is the section associated to the bundle

$O_{\mathbb{P}^1}(2)$ in $\mathcal{E}$, we get the flag of that example. By [8] Theorem 6.5 we obtain a non-trivial generalisation of the Schmidt Subspace Theorem for this embedding:

THEOREM 5.6. *Let S be a section of R. For each $v \in \Sigma$ let $l_{v,0}, \cdots, l_{v,N}$ be a basis compatible with the filtration (5.1) associated to S, and let $r_0, \cdots, r_N$ be the corresponding weights (5.2). Suppose that $H^i(R, D - jS) = 0$ for $i > 0$ and $0 \leqslant j \leqslant a$. Further, for each $v \in \Sigma$ let $d_v > 0$ be real numbers such that*

$$(3a^2 D \cdot S - a^3 S^2) \sum_{v \in \Sigma} \frac{1}{d_v} > 3D^2.$$

Then the points $\mathbf{x} \in R(K)$ with

$$\log \frac{|l_{v,i}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant -\frac{r_i}{d_v} h(\mathbf{x}), \quad v \in \Sigma, \quad i = 0, \cdots, N$$

lie in finitely many curves in R of bounded degree.

Proof. Under the condition $H^i(R, D - jS) = 0$ for $i > 0$ and $0 \leqslant j \leqslant a$, [8] Proposition 6.2 implies

$$e_{\mathbf{r}}(X) = 3a^2 D \cdot S - a^3 S^2.$$

By construction R is embedded in $\mathbb{P}(H^0(R, D)^\vee)$, thus $\deg(X) = D^2$. We get

$$\frac{e_{\mathbf{r}}(X)}{(\dim(X) + 1)\deg(X)} = \frac{3a^2 D \cdot S - a^3 S^2}{3 \cdot D^2},$$

and Theorem 2.3 concludes the proof. □

5.7. If $p : \mathcal{E} \to C$ is a rank 2 vector bundle on C such that $\mathbb{P}(\mathcal{E}) \cong R$ we say that $\mathcal{E}$ represent R. Such $\mathcal{E}$ always exists and R determines $\mathcal{E}$ up to tensoring with a line bundle $\mathcal{L}$ on C. Let $\mathcal{L}$ be a line subbundle of $\mathcal{E}$ of maximal degree. The number $\deg \mathcal{E} - 2 \deg \mathcal{L}$ is independent of the choice of $\mathcal{E}$. Then we say that the ruled surface is *bundle semistable* if $\deg \mathcal{E} \geqslant 2 \deg \mathcal{L}$ and *bundle unstable* when $\deg \mathcal{E} < 2 \deg \mathcal{L}$. In the latter case there exists a unique section S of negative selfintersection.

THEOREM 5.8. *Suppose R is bundle unstable. Let $D \sim aG + bf$ be very ample on R. Let $S \sim G + cf$ be the unique section of R of negative selfintersection. For each $v \in \Sigma$ let $l_{v,0}, \cdots, l_{v,N}$ be a basis compatible with the filtration(5.1) associated to S, and let $r_0, \cdots, r_N$ be the corresponding weights (5.2). Further, for each $v \in \Sigma$ let $d_v > 0$ be real numbers such that*

$$b > a\left(c - \frac{1}{2}(\deg \mathcal{E} - 2 \deg \mathcal{L})\right) + 2g - 2, \tag{5.4}$$

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and

$$\sum_{v\in\Sigma}\frac{1}{d_v} > \frac{6b}{a(ac+2b+2g-2)}. \tag{5.5}$$

Then the points $\mathbf{x}\in R(K)$ *satisfying*

$$\log\frac{|l_{v,i}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant -\frac{r_i}{d_v}h(\mathbf{x}),\quad v\in\Sigma,\quad i=0,\cdots,N$$

lie in finitely many curves of R *of bounded degree.*

Proof. Since $\deg\mathcal{E} < 2\deg\mathcal{L}$, (5.4) implies that for all integers j with $0\leqslant j\leqslant a$ we have

$$(b-jc) > (a-j)(2g+1+\tfrac{1}{2}(\deg\mathcal{E}-2\deg\mathcal{L}))+4g-1.$$

Then by [8], Proposition 5.7, the cohomology groups $H^i(R, D-jS)$ vanish for $i>0$ and $0\leqslant j\leqslant a$ (see also [8], Remark after Proposition 6.2). The degree of R is $D^2=2ab$. According to [8] (6.7) we have $3a^2D\cdot S - a^3S^2 = (a^3/2)(\deg\mathcal{E} - 2\deg\mathcal{L})+3a^2b$. Again, (5.4) implies

$$\frac{3a^2D\cdot S-a^3S^2}{3\cdot D^2} = \frac{\frac{a^2}{2}(\deg\mathcal{E}-2\deg\mathcal{L})+3ab}{6b} > \frac{a}{6b}(ac+2b+2g-2).$$

Then inequality (5.5) imply (5.3), and Theorem 5.6 proves the assertion. □

5.9. Little is known about the unstability of higher dimensional varieties, and any information on this problem has interesting consequences. For instance, Vojta's Conjectures (see [14]) for the ruled hypersurface of $\mathbb{P}^2\times\mathbb{P}^2$ defined by $a^4x+b^4y+c^4z=0$ imply the *abc*-conjecture (see [9], Conjecture 0.2 for another point of view).

6. Blow-up

6.1. Let X be a normal dimension n variety defined over K, H be a Cartier divisor on X, and S be an effective Cartier divisor. Fix an integer $s>0$. Assume that for a sufficiently large r, say larger than $r_0\geqslant s$, $rH-sS$ is very ample. Put $W_{i,j}=H^0(X\otimes_K L, iH-jS)$ for $i,j\in\mathbb{Z}$. Then $W_{r,s}$ has a filtration $W_{r,s}\supset W_{r,s+1}\supset\cdots\supset W_{r,r}$. For $v\in\Sigma$ let $l_{v,0},\ l_{v,1},\cdots,\ l_{v,N}$ be a basis of $W_{r,s}$ compatible with the filtration. For these bases we consider weights $r_{v,i}=r-j$ if $l_{v,i}\in W_{r,j}$ and $l_{v,i}\notin W_{r,j+1}$.

THEOREM 6.2. *Suppose that for any* $i>0$

$$h^i(X\otimes_K L, rH-jS) = O((r+j)^{n-2})\ \text{on}\{(\mathrm{r},\mathrm{j})\in\mathbb{Z}^2|\ \mathrm{s}\leqslant\mathrm{j}\leqslant\mathrm{r}\}.$$

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For each $v \in \Sigma$ *let* d_v *be positive real numbers with*

$$\begin{aligned}&\sum_{i=0}^{n} H^{n-i}(-S)^i \binom{n+1}{i+1}((i+1)(r-s)r^{n-i}s^i - r^{n+1} + r^{n-i}s^{i+1})\\ &\sum_{v\in\Sigma} \frac{1}{d_v} > (n+1)(rH - sS)^n.\end{aligned} \tag{6.1}$$

Then the points $\mathbf{x} \in X(K)$ *with*

$$\log\left(\frac{|l_{v,i}(\mathbf{x})|_v}{|\mathbf{x}|_v}\right) \leqslant -\frac{r_{v,i}}{d_v} h(\mathbf{x}), \quad v \in \Sigma, \quad i = 0, \cdots, N$$

are contained in finitely many subvarieties of X *of bounded degree.*

Proof. For each $v \in \Sigma$ the filtrations constructed above correspond to the filtration of [6], Lemma 3. In loc. cit. it is proven that for each $v \in \Sigma$ the left hand side of (6.1) is a lower bound for the degree of contact. Notice that α_i as defined in [6] equals $H^{n-i}(-S)^i$. The assertion of the theorem follows then from Theorem 2.3. $\square$

Remark 6.3. Lemma 3 in [6] yields that this theorem is stronger than Schmidt's Subspace Theorem restricted to X. Further, one could observe that all intersection products of (6.1) can be written in terms of Segre classes.

6.4. Here we follow [6], §2. Let V be a normal projective variety over K of dimension $n-1 \geqslant 1$, D an ample divisor on V, t a positive integer, and $Y_t \hookrightarrow \mathbb{P}^{h^0(V\otimes_K L, \mathrm{O}(tD))}$ be the projective cone over $\Phi_{|tD|} : V \hookrightarrow \mathbb{P}^{h^0(V\otimes_K L, \mathrm{O}(tD))-1}$. Let $f_t : X_t \to Y_t$ be the blow-up with center at the vertex. Let $\mathrm{O}_{Y_t}(H_{0,t}) = \mathrm{O}_{\mathbb{P}^{h^0(V\otimes_K L, \mathrm{O}(tD))}}(1)|_{Y_t}$, $H_t = f_t^* H_{0,t}$, and S_t be the exceptional divisor of f_t.

PROPOSITION 6.5. *Choose a large integer* t *such that*

$$h^i(V \otimes_K L, \mathrm{O}(mtD)) = 0, \text{ for all } m > 0 \text{ and all } i > 0. \tag{6.2}$$

For each $v \in \Sigma$ *let* $l_{t,v,i}$, $r_{t,v,i}$, $i = 1, \cdots, N$, *be sections and weights as in* **6.1**. *Further, let* $d_{t,v}$ *be positive real numbers satisfying*

$$\sum_{v\in\Sigma} \frac{1}{d_{t,v}} > \frac{(n+1)(r^n - s^n)}{(n+1)s^n(s-r) + r^{n+1} - s^{n+1}}.$$

Then the points $\mathbf{x} \in X_t(K)$ *with*

$$\log\left(\frac{|l_{t,v,i}(\mathbf{x})|_v}{|\mathbf{x}|_v}\right) \leqslant -\frac{r_{t,v,i}}{d_{t,v}} h(\mathbf{x}), \quad v \in \Sigma, \quad i = 0, \cdots, N$$

are contained in finitely many subvarieties of X *of bounded degree.*

Proof. Condition (6.2) implies vanishing of higher cohomology for each $rH_t - jS_t$, $j = 1, \cdots, r$ ([6], Proposition 4). Moreover, all intersection numbers $H_t^{n-i}(-S_t)^i$ are

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zero except for $i = 0, n$ where $H_t^n = -(-S_t)^n$. Thus, (6.3) implies (6.1) and the proposition follows. □

Remark 6.6. The projective cone over a non-singular conic in $\mathbb{P}^2$ and a non-singular quadratic surface in $\mathbb{P}^3$ satisfy the condition of Proposition 6.5 for each $t \geqslant 1$ ([6], Example 6.)

6.7 Let $f : X \to Y$ be the blow-up of a normal projective variety Y along Z, where Z is a not necessarily irreducible or reduced subscheme. Assume that X is non-singular of dimension n. Hence, X is a resolution of the singularities of Y. Let S be the exceptional divisor. Put $H = f^*H_0$ for an ample divisor H_0 on Y, and fix an integer $s \geqslant 0$.

PROPOSITION 6.8. *Take a sufficiently large integer t such that $tH - S$ is ample on X and tH_0 is very ample on Y. Assume (6.1), then the assertion of Theorem 6.2 holds.*

Proof. Vanishing for the higher cohomology spaces is assured by the proof of [6], Proposition 8. □

7. Elliptic Surfaces

7.1. Following [7], we call a flat proper map of $\overline{\mathbb{Q}}$-schemes $p : X \to \mathbb{P}^1$ a rational *Weierstrass fibration* if X is reduced and irreducible rational surface over $\overline{\mathbb{Q}}$, every geometric fibre of p is an irreducible curve of genus 1, and a section $s:\mathbb{P}^1 \to X$ is given, which does not pass through the nodes or the cusps of the fibres. Moreover, we will assume that X is normal, and that the generic fibre of p is smooth. In this case, we may resolve singularities of X and obtain an elliptic surface $\overline{p} : \overline{X} \to \mathbb{P}^1$ (with a section) which we call the induced elliptic surface. In this situation one may represent X in Weierstrass form by the equation

$$y^2 = x^3 + A(t)x + B(t),$$

where A is a quartic and B is a sextic polynomial in the parameter t of $\mathbb{P}^1$.

7.2. Let us consider the following situation: $\overline{X}$ is a minimal rational elliptic surface with a section and a fibre of type IV^* (in the Kodaira classification), X is the associated Weierstrass fibration, S is the given section of $\overline{X}$, f is the numerical class of a fibre of $\overline{X}$, R is the unique rational component of the singular IV^* fibre having multiplicity 3, $D = 3kS + 6kf$ is a very ample divisor on X for some positive integer k. We have a filtration on $H^0(\overline{X}, \mathrm{O}(D))$ defined by

$$H^0(\overline{X}, \mathrm{O}(D)) \supset H^0(\overline{X}, \mathrm{O}(D - R)) \supset \cdots \supset H^0(\overline{X}, \mathrm{O}(D - 18kR)) \supset \{0\}.$$

Suppose K is a number field defining $\overline{X}$ as a geometrically irreducible variety. For each $v \in \Sigma$, we choose a basis $l_{v,0}, \cdots, l_{v,N}$ compatible with this filtration, and define weights $r_{v,i} = 18k - j$ if $l_{v,i}$ is in $H^0(\overline{X}, \mathrm{O}(D - jR))$ but not in $H^0(\overline{X}, \mathrm{O}(D - (j+1)R))$, $i = 0, \cdots, N$.

THEOREM 7.3. *For each $v \in \Sigma$ let d_v be positive real numbers such that*

$$\sum_{v \in \Sigma} \frac{k}{d_v} > \frac{3}{25}.$$

Then all $\mathbf{x} \in X(K)$ satisfying

$$\log\left(\frac{|l_{v,i}(\mathbf{x})|_v}{|\mathbf{x}|_v}\right) \leqslant -\frac{r_{v,i}}{d_v} h(\mathbf{x}), \;\; v \in \Sigma, \;\; i = 0, \cdots, N$$

are contained in finitely many curves in X of bounded degree.

Proof. According to [7], (6.7) we know that the degree of contact of the filtration (7.1) is bounded below by $675k^3$. On the other hand, the degree of X with respect to D equals $27k^2$ ([7], (4.1)). Hence,

$$\frac{e_{\mathbf{r}}(X)}{(\dim X + 1)\deg(X)} \geqslant \frac{675k^3}{3 \cdot 27k^2} = \frac{25}{3}k.$$

Theorem 2.3 implies then the theorem. □

Remark 7.4. Since D is very ample, by Riemann–Roch we can compute the dimension $N+1$ of the space $H^0(\overline{X} \otimes_K L, \mathrm{O}(D))$ which equals $(27k^2 + 3k + 2)/2$ ([7], (4.2)). For each $v \in \Sigma$

$$\sum_{i=0}^{N} r_{v,i} = \frac{225}{2}k^3 + 9k^2 + \frac{23}{2}k,$$

([7], (5.10)). Moreover, for all $k \geqslant 1$ we have

$$\frac{25}{3}k > \frac{1}{\frac{27k^2+3k+2}{2}}\left(\frac{225}{2}k^3 + 9k^2 + \frac{23}{2}k\right).$$

This means that the Schmidt subspace theorem restricted to X is weaker than Theorem 7.3.

Remark 7.5. Filtration (7.1) is clearly given by the order of vanishing of sections of D along the curve R. Calculations of the same type can be carried through for Weierstrass fibrations with singularities of type II* and III*. The filtrations used in the latter cases are also given by the order of vanishing along the unique curve of maximum multiplicity in the singular fibre in question. This curve is the multiplicity six component in the III* fibre and the multiplicity four component in the II* fibre.

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