J. Austral. Math. Soc. (*Series A*) 40 (1986), 20–33

COCYCLES AND REPRESENTATIONS OF GROUPS OF CAR TYPE

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(Received 10 January 1984)

Communicated by G. Brown

Abstract

Representations of non-type I groups G which may be expressed as an increasing union of type I normal subgroups are considered. Groups with this structure are natural generalisations of the CAR algebra (viewed as a twisted group C^*-algebra) and are also group theoretic analogues of AF algebras. This paper gives a systematic account of their representation theory based on a canonical construction of one-cocycles for the G-action on the dual of a normal subgroup. Some examples are considered showing how to construct inequivalent irreducible representations (non-cohomologous cocycles) and also factor representations by a method which generalises the well-known construction of non-isomorphic factors for the CAR algebra.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 22 A 25.

1. Introduction

The representation theory of non-type I groups is of interest for a variety of reasons. For example on the one hand such groups and their representations arise in solid state physics [4] and on the other hand the theory of von Neumann algebras was greatly influenced by the study of representations of the canonical anticommutation relations (CAR) which may be regarded as representations of a certain twisted group C^*-algebra; cf. [8], [17].

Thus we began in [6] to consider the representation theory of the CAR from a group theoretical viewpoint with the idea of understanding both the existing methods of constructing *CAR* representations and of extending them. Subsequently we recognised that our methods amounted to a systematic study of certain one cocycles for the obvious ergodic action of $\oplus_1^\infty \mathbb{Z}_2$ on $\prod_1^\infty \mathbb{Z}_2$ and moreover that they applied to a much wider class of groups. On the other hand recent work of Baggett *et al.* [3] on a systematic study of various methods of constructing cocycles for the irrational rotation C^*-algebra (viewed as a twisted group algebra) appeared complementary to our considerations and we sought to understand it in our terms.

The other motivations for this paper were firstly that the groups for which our techniques apply are, in a natural sense, group theoretic analogues of *AF* C^*-algebras (they include as special cases all *UHF* algebras [15]) and so one would expect a reasonably non-pathological representation theory to exist by analogy with [18]. Secondly a recent result of Sutherland [19] shows that, if one is to construct factors using representations of amenable groups, then there exist various universal models (in the sense that their representation theory contains that of any non-type I amenable group) of which the *CAR* algebra, and groups of the structure considered here, provide examples.

Consequently the preceding considerations led us to consider a class of non-type I second countable locally compact groups G with the following structure. We suppose there is an increasing sequence $\{G_n\}$ of type I normal subgroups with $G = \bigcup_{n=0}^\infty G_n$ such that G_n is regularly embedded in G_m for all m, n with $m > n$ and such that the stabiliser of any irreducible representation of G_0, under the usual G-action on the dual $\hat{G}_0$, is G_0 itself (see [14] for details of these notions). The assumption of normality is crucial for our arguments although it is clearly the most restrictive assumption (and makes it unclear for example whether all *AF* algebras can arise in this way). We call such groups almost type I (*ATI*) and have considered examples elsewhere [7] and will consider more here. A less restrictive notion which comes up in connection with nilpotent groups is that of a locally *ATI* group G. Here we restrict attention to a given primitive ideal J in the primitive ideal space of $C^*(G)$. Then for any type I normal subgroup G_0 of G, J determines by restriction, a G-quasiorbit $\theta \subseteq \hat{G}_0$ (see [10] for this fact). Here a G-quasiorbit is an equivalence class under the relation $\rho_1 \sim \rho_2$ if ρ_1 lies in the closure of the G-orbit through ρ_2 and vice versa. We say G is locally *ATI* if for any such J we can find an increasing sequence of type I normal subgroups $\{G_n\}_{n=0}^\infty$ each acting smoothly on $\theta \subseteq \hat{G}_0$ and with G_m acting smoothly on the subset of the dual of G_n lying over θ for all pairs m, n with $m > n$. As before we assume the stabiliser in G of any $\rho \in \theta$ is G_0 itself. It will be clear from the ensuing proofs that with little extra effort our methods apply equally to locally *ATI* groups; however we will not bother to go through the details.

Before describing our results we make one further comment. Clearly the assumption that G is not type I means that there are G-quasi-orbits in $\hat{G}_0$ which

https://doi.org/10.1017/S1446788700026471 Published online by Cambridge University Press

are not orbits. However it follows from the theorem of Gootman and Rosenberg [10] that the primitive ideal space of the group C^*-algebra of G is parametrised by the G-quasiorbits in $\hat{G}_0$. Now it is not difficult, using a group measure space construction, to construct factor representation of G for each G-quasi-invariant ergodic measure μ on $\hat{G}_0$ (see Proposition 3.1 of Auslander and Moore [10]); however irreducible representations, and explicit constructions of one cocycles for the G-action on $\hat{G}_0$, are more difficult to manufacture. Consequently our first result is

THEOREM 1.1. *Let G be ATI with $G = \bigcup_n G_n$ and μ a G-ergodic quasi-invariant measure on $\hat{G}_0$. If π_0 denotes a representative of the unique equivalence class of multiplicity free representations of G_0 associated with μ then there is an irreducible representation π of G such that $\pi|_{G_0}$ is π_0.*

The construction of π from π_0 employs a type of "inductive limit" of induced representations and is carried out in Section 2. What it provides for us is a particular unitary operator valued one-cocycle for the G-action on $\operatorname{supp}\mu \subseteq \hat{G}_0$. Of course any such cocycle defines in turn a representation of G and there is some interest in giving constructions of them. Our second result, proved in Section 3, shows that the cocycle of Theorem 1.1 is in a sense canonical.

THEOREM 1.2. *Every factor representation ρ of G determines a unitary operator valued one-cocycle for the G-action on $\hat{G}_0$ and this in turn is determined by*

(a) *a positive integer m ($m = \infty$ is permitted) and a G-quasi-invariant ergodic measure μ on $\hat{G}_0$,*

(b) *a sequence $\{b_n\}_{n=1}^{\infty}$ of $\mathcal{U}(H_m)$ valued Borel functions on $\hat{G}_0$ (where H_m is the Hilbert space $m \cdot H = H \oplus \overset{(m)}{\cdots} \oplus H$ with H carrying almost all the representations of G_0 in the support of μ),*

(c) *the cocycle constructed from μ in Theorem* 1.1.

The sequence $\{b_n\}$ of Theorem 1.2 must satisfy a consistency condition which is not particularly restrictive so that a large class of cocycles for the G-action may be constructed for any given m and μ. This method of constructing cocycles generalises that of [9] for the *CAR* algebra and we discuss in Section 4 for this example and one other the production of non-cohomologous cocycles by our methods. We also indicate to some extent how the one-cocycle depends on the choices made in its construction.

For general *ATI* G we have not made much progress on the problem of proving irreducibility or factoriality of the representations associated with these cocycles. Consequently in Section 5 we return to the *CAR* case and some other examples where factor representations are easily shown to be constructable. In particular we indicate using the results of [6] how the Powers factors [16] fit into our approach. Finally we note that the question of irreducible representations, for the CAR constructed by our method has already been settled by Golodets [11]. In fact Section 3 of [6] and in part Section 5 of this paper are a group theoretic elaboration of [11]. (Reference to [11] was inadvertantly omitted from [6].) Finally, this paper may be regarded as a substitute for reference [9] in [6].

2. Proof of Theorem 1.1

In all the examples we have considered which exploit special cases of Theorem 1.1 [6], [7], it has proved convenient to work with "projective" or "multiplier" representations rather than ordinary representations. The representations of the CAR for example correspond to projective representations of an infinite discrete abelian group [8], [17]. Consequently we will state and prove Theorem 1.1 in this section as a theorem about projective representations.

In this context we suppose that σ; $G \times G \to \mathbb{T}$ (= circle group) is a 2-cocycle on G with $\sigma(e, g) = 1$ for all $g \in G$ (e is the identity of G). We may suppose without loss of generality that σ is normalised, i.e. $\sigma(g, g^{-1}) = 1$ for all $g \in G$. Then (G, σ) is ATI if there is an increasing sequence of normal subgroups $G_0 \subset G_1 \subset \cdots \subset G$ such that for each n, G_n is σ-type I (i.e. all factor σ-representations of G_n are type I). We let $(G_n, \sigma)\hat{}$ denote the equivalence classes of irreducible σ-representations of G_n. We suppose that the stabilizer of any $\rho \in (G_0, \sigma)\hat{}$ is G_0 itself and that G_n is regularly embedded in G_m for all m, n with $m > n$. We assume of course that G is not σ-type I.

Our starting point is a G-quasi-invariant strictly ergodic measure μ on $(G_0, \sigma)\hat{}$. We can choose, in a Borel way, for each $\rho \in (G_0, \sigma)\hat{}$ a concrete representative $\beta(\rho)$ and form the multiplicity free representation $\pi_0 = \int_{(G_0,\sigma)\hat{}} \beta(\rho)\, d\mu(\rho)$. Notice that the commutant $\{\pi_0(G)\}'$ is precisely $L^\infty((G_0, \sigma)\hat{}, \mu)$ so that $\{\pi_0(G)\}'' \supseteq L^\infty((G_0, \sigma)\hat{}, \mu)$.

Because G_0 is regularly embedded in G_n for each n, G_n will act smoothly in the $(G_0, \sigma)\hat{}$. Let L_n denote the orbit space of G_n in $(G_0, \sigma)\hat{}$. There are canonical Borel maps p_n: $L_n \to L_{n+1}$ and q_{n+1}: $(G_0, \sigma)\hat{} \to L_{n+1}$ which associate to each point x_n of L_n (or $\rho \in (G_0, \sigma)\hat{}$) the unique G_{n+1} orbit (in $(G_0, \sigma)\hat{}$) containing it. Now L_n and L_{n+1} are standard Borel spaces so there exist, for each n, Borel cross-sections θ_n: $L_{n+1} \to L_n$. Let α_n: $L_n \to L_0$ be the Borel cross-section for q_n given by $\theta_1 \circ \cdots \circ \theta_{n-1}$. Let μ_n be measures on L_n defined inductively by $p_n^*(\mu_{n-1}) = \mu_n$ and $\mu_0 = \mu$. In each case, μ_n is an ergodic quasi-invariant measure on L_n under the action of G_n. By disintegration of measures we may write $\mu = \int_{L_n} \omega_y \, d\mu_n(y)$ where ω_y is (a.e. μ_n) a quasi-invariant measure on the orbit y of the action of G_n on L_0. Thus ω_y is equivalent to the image of Haar measure m_n on G_n/G_0 under the map $G_0 g \rightsquigarrow \alpha_n(y) \cdot g$. In fact the map ξ_n: $(G_0 g, y) \rightsquigarrow \alpha_n(y) \cdot g$ is a Borel isomorphism of $G_n/G_0 \times L_n$ with L_0 and under this map the image of $m_n \times \mu_n$ is equivalent to μ. Let m'_n represent Haar measure on G_n/G_{n-1} and note that in the same way the map ζ_n: $(G_{n-1}g, y) \rightsquigarrow \theta_n(y) \cdot g$ is a Borel isomorphism of $G_n/G_{n-1} \times L_n$ with L_{n-1}, which carries $m'_n \times \mu_n$ to a measure equivalent to μ_{n-1}. The inverse of ζ_n is of the form $y \rightsquigarrow (\delta_n(y), p_{n-1}(y))$ where δ_n: $L_{n-1} \to G_n/G_{n-1}$ is a Borel map. Evidently we may choose a Borel map γ_n: $G_n/G_{n-1} \to G_n/G_0$ so that

$$\alpha_n(p_{n-1}(y)) \cdot \gamma_n(\delta_n(y)) = \alpha_{n-1}(y) \qquad (y \in L_{n-1}).$$

It follows that the image of $m'_n \times \mu_n$ under $\gamma_n \times \alpha_n$ is equivalent to the image of μ_{n-1} under α_{n-1}.

Now define π_n to be the representation of G_n given by

$$\pi_n = \int_{L_n} \beta(\alpha_n(y)) \uparrow_{G_0}^{G_n} d\mu_n(y) \simeq \int_{L_n} \beta(\alpha_n(y)) \, d\mu_n(y) \uparrow_{G_0}^{G_n}.$$

Then, by the Subgroup Theorem,

$$\begin{aligned}
\pi|_{G_{n-1}} &= \int_{G_n/G_0} \int_{L_n} \left(\beta(\alpha_n(y)) \uparrow_{G_0}^{G_{n-1}}\right)^g d\mu_n(y) \, dm'_n(G_{n-1}g) \\
&= \int_{G_n/G_0} \int_{L_n} \beta(\alpha_n(y))^{\gamma_{n-1}} \uparrow_{G_0}^{G_{n-1}} d\mu_n(y) \, dm'_n(G_{n-1}g) \\
&\simeq \int_{G_n/G_0} \int_{L_n} \beta(\alpha_n(y)\gamma_n(g)) \uparrow_{G_0}^{G_{n-1}} d\mu_n(y) \, dm'_n(G_{n-1}g)
\end{aligned}$$

which by the remarks above is equivalent to π_{n-1}. In particular $\pi_n|_{G_0} \simeq \pi_0$. We need to be more explicit about the latter equivalence.

Let K_n be the space of the representation π_n, so that it consists of functions f: $G_n \times L_n \to H$ satisfying $f(hg, y) = \sigma(h, g)\beta(\alpha_n(y))(h)f(g, y)$ $(h \in G_0, g \in G, y \in L_n)$ and $\int\int_{G_n/G_0} \|f(g, y)\|^2 \, dm_n(G_0 g) \, d\mu_n(y) < \infty$. On K_n, π_n is defined as $\pi_n(g)(f)(g', y) = \bar{\sigma}(g', g)f(g'g, y)$ $(g, g' \in G_n, y \in L_n)$. We define a map T_n: $K \to L^2(\mu, H)$ by

$$T_n(f)(x) = f(\xi_n(x))\left(\left(\frac{d(\xi_n^{-1})^*(m_n \times \mu_n)}{d\mu}\right)(x)\right)^{1/2}.$$

Then T_n is an isometry. Moreover a straightforward calculation shows that $T_n(\pi_n(g)(f))(x) = u_n(g, x)T_n(f)(xg)$ $(x \in L_0, g \in G_n)$, where u_n is a $B(H)$ valued σ-1-cocycle, *viz.*, $u_n(g_1 g_2, x) = \sigma(g_1, g_2)u_n(g_1, x)u_n(g_2, xg_1)$ $(g_1, g_2 \in G_n, x \in L_0)$. Restricting to G_0 gives $u_n(g, x) = \beta(\alpha_n(x))^{\tau_n(x)}(g)$ where τ_n: $G_n/G_0 \to G_n$ is a Borel cross-section, so that $\alpha_n(x) \cdot \tau_n(x) = x$ $(x \in L_0)$. It follows that

https://doi.org/10.1017/S1446788700026471 Published online by Cambridge University Press

$u_n(g, x) = W_n(x)\beta(x)W_n(x)^{-1}(g)$ $(g \in G_0, x \in L_0)$ where W_n is a unitary valued Borel function on L_0.

Write $v_n'(g, x) = W_n(x)^{-1}u_n(g, x)W_n(x)$. Then we define $\rho_n'(g)(f)(x) = v_n'(g, x)f(xg)$ $(f \in L^2(\mu, H), g \in G_n)$. The construction yields $\rho_n' \simeq \pi_n$ and $\rho_n'|_{G_0} = \pi_0$. Finally ρ_n are defined inductively as follows. Suppose $\rho_1, \rho_2, \ldots, \rho_{n-1}$ are defined as representations on $L^2(\mu, H)$ of $G_1, G_2, \ldots, G_{n-1}$ respectively, so that $\rho_k|_{G_{k-1}} = \rho_{k-1}$, $\rho_1|_{G_0} = \pi_0$, $\rho_k \simeq \rho_k'$ for all k, and $\rho_k(g)(f)(x) = v_k(g, x)f(xg)$ $(f \in L^2(\mu, H), g \in G_k)$. Since $\rho_n'|_{G_{n-1}} \simeq \rho_{n-1}$ and $\rho_n|_{G_0} = \rho_{n-1}|_{G_0} = \pi_0$, any unitary implementing this equivalence is in the commutant of π_0 and so is multiplication by a scalar function $\varphi \in L^\infty(\mu)$ with $|\varphi| = 1$. Define $\rho_n(g)(f)(x) = \varphi(xg)\bar{\varphi}(x)v_n'(g, x)f(xg)$ $(f \in L^2(\mu, H), g \in G_n)$ to achieve the induction step.

Now we are in a position to define the representation π of G by $\pi(g) = \rho_n(g)$ $(g \in G_n)$.

PROPOSITION 2.1. *π is irreducible.*

PROOF. Any operator A in the commutant of π is also in the commutant of π_0 and so is a multiplication operator corresponding to some $\psi \in L^\infty(\mu)$. Since this operator also commutes with $\pi(g)$ $(g \in G)$, we obtain $\psi(xg) = \psi(x)$ (μ a.e.). The ergodicity of μ implies that ψ is a constant a.e. and hence A is scalar.

This completes the proof of the main result of this section.

THEOREM 2.2. *Let (G, σ) be ATI with $G = \bigcup_n G_n$ and μ a G-ergodic quasi-invariant measure on $(G_0, \sigma)^\wedge$. If π_0 denotes a representative of the unique equivalence class of multiplicity free representations corresponding to μ then there is an irreducible representation π of G such that $\pi|_{G_0}$ is π_0.*

3. General cocycles

The construction of Section 2 isolates a particular σ-1-cocycle on $G \times (G_0, \sigma)^\wedge$ for every G-ergodic quasi-invariant measure on $(G_0, \sigma)^\wedge$. Two questions which immediately arise however are to what extent does this cocycle depend on the choices made in its construction and what other possibilities exist for cocycles?

We relabel the Hilbert space on which π_0 acts as H_0 and suppose now that π is any primary σ-representation of G such that $\pi|_{G_0}$ is equivalent to a multiple of π_0, say $m \cdot \pi_0$ $(m = 1, 2, \ldots, \infty)$. Note that any primary representation of G restricts on G_0 to a multiple of some multiplicity free representation and moreover we can assume that π acts on $L^2(\mu, H_0)$ via

$$(3.1)\quad (\pi(g)f)(x) = \delta_\mu(g, x)a(g, x)f(x, g), \qquad x \in (G_0, \sigma)^\wedge, g \in G,$$

where δ_μ is a Radon–Nikodym factor and a is a $\mathscr{U}(H)$-valued Borel 1-cocycle on $G \times (G_0, \sigma)^\wedge$ (see for example [14]). Moreover it follows from the general Mackey analysis [14] that $\pi|_{G_n}$ is equivalent to $m \cdot \rho_n$ for every n. Thus there is a unitary operator B_n on $L^2(\mu, H_0)$:

$$B_n\pi(g)B_n^{-1} = (m \cdot \rho_n)(g), \qquad \text{for all } g \in G_h.$$

As the B_n must lie in the commuting algebra of $m \cdot \pi_0$ there is a Borel function b_n: $(G_0, \sigma)^\wedge \to \mathscr{U}(H)$ such that $(B_n \cdot f)(x) = b_n(x)f(x)$, $x \in (G_0, \sigma)^\wedge$, $f \in L^2(\mu, H)$. Thus we have

$$(3.2)\qquad \delta_\mu(g \cdot x)b_n(x)a(g, x)b_n(xg)^{-1} = m \cdot v(g, x)$$

for μ a.e. x.

For notational convenience assume $\delta_\mu \equiv 1$. Then from (3.2) we see that a and the Borel cocycle $g, x \to b_n(x)^{-1}m \cdot v(g, x)b_n(xg)$, $x \in X$, $g \in G_n$, determine the same representation of G_n. There is a consistency condition on the b_n's of course, namely

$$(3.3)\qquad b_r(x)^{-1}m \cdot v(g, x)b_r(xg) = b_n(x)^{-1}m \cdot v(g, x)b_n(xg)$$

for all $r \leqslant n$, $g \in G_r$ and μ a.e. x. We summarize this discussion as

PROPOSITION 3.1. *For each primary σ-representation π of G there is*

(a) *a quasi-invariant ergodic measure μ on $(G_0, \sigma)^\wedge$,*

(b) *a Hilbert space H_0 and multiplicity free σ-representation π_0 of G_0 on $L^2(\mu, H_0)$,*

(c) *a positive integer m such that $\pi|G_0 \simeq m \cdot \pi_0$,*

(d) *a sequence $\{b_n\}_{n=1}^\infty$ of $\mathscr{U}(H)$-valued Borel functions on $(G_0, \sigma)^\wedge$, where $H = m \cdot H_0$, such that π is given by* (3.1) *and the sequence $\{b_n\}$ satisfies* (3.2) *and* (3.3), *μ a.e. x*

This last result completes the proof of Theorem 1.2. It shows in particular that the measure class of μ and the multiplicity m are invariants for the equivalence class of π. The only question left to be settled is of what freedom exists for the sequence $\{b_n\}$? Well, any sequence $\{b_n\}$ of $\mathscr{U}(H)$-valued Borel functions satisfying (3.3) can be used to define a σ-1-cocycle on $G_n \times (G_0, \sigma)^\wedge$ by

$$(3.4)\qquad a_n(g, x) = b_n(x)^{-1}m \cdot v(g, x)b_n(xg).$$

Then (3.3) guarantees that by defining

$$(3.5)\qquad a(g, x) = a_n(g, x) \qquad \text{for all } g \in G_n$$

the function a: $G \times (G_0, \sigma)^\wedge \to \mathscr{U}(H)$ becomes a σ-1-cocycle on $(G_0, \sigma)^\wedge$.

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Now the questions raised at the beginning of this section may be reinterpreted using the preceding observations. By making different choices in the construction of the cocycle v one chooses a different sequence of $\mathscr{U}(H)$-valued Borel functions $\{b_n\}$ on $(G_0, \sigma)^\wedge$ and so defines a new cocycle via (3.4) and (3.5). Note that the sequence $\{b_n\}$ is always arbitrary to the extent that a second sequence $\{b_n'\}$ will produce the same cocycle a_n at the nth stage provided $c_n = b_n' b_n^{-1}$ defines a multiplication operator C_n say on $L^2(\mu, H_0)$, which commutes with $m \cdot \rho_n(g)$ for all $g \in G_n$. This freedom is important for it may be possible to choose c_n so that the sequence B_n' of unitary multiplication operators corresponding to $b_n' = c_n b_n$ converges weakly to a unitary B. Then for example in the case $m = 1$ it follows easily that B will intertwine the representation with cocycle (3.3) with the representation π constructed in Theorem 2.2. Conversely if the representation with cocycle (3.5) is equivalent to π then there is a choice of b_n such that B_n converges weakly to an intertwining operator.

The other question raised at the beginning of this section may be elaborated on to ask when a sequence $\{b_n\}_{n=1}^\infty$ produces a cocycle (3.5) for which the corresponding representation is irreducible or factorial and whether non-cohomologous cocycles may be constructed. Put another way this last point requires us to describe the equivalence relation, on the set of allowed sequences $\{b_n\}_{n=1}^\infty$ which arises from looking at the equivalence classes of cohomologous cocycles defined by (3.5).

Definitive answers to these questions are not available in this generality. In the next section we give a general construction of sequences $\{b_n\}_{n=1}^\infty$ and investigate some examples.

4. Constructing cocycles

We start this section by showing that it is always possible to construct sequences $\{b_n\}_{n=1}^\infty$ of Borel functions having the properties required in the previous section and hence σ-1-cocycles for arbitrary ergodic quasi-invariant measures μ and multiplicity factors m.

For each n we may choose an arbitrary Borel function $d_n\colon G_n/G_{n-1} \times L_n \to \mathscr{U}(H_m)$ having the property that for all $l \in L_n$, $d_n(-, l)$ takes its values in the commutant of $\{m \cdot v(g, x) | g \in G$ and x lies in the orbit $l\}$. (This may well be a nontrivial restriction when v is not circle valued and $m = 1$.) Then define b_1 by $b_1(x) = d_1(G_0 g, l)$ for $x = \alpha_1(l) \cdot g$, $l \in L_1$, $g \in G_1$. Assume b_k has been defined so that $b_k(x)$ lies in the appropriate commutant and

(4.1)
$$b_m(x)^{-1} b_m(xg) = b_n(x)^{-1} b_n(xg) \quad \text{for all } m, n \text{ with } k \geqslant m \geqslant n, \quad \text{for all } g \in G_n.$$

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Then (3.3) holds for all $m, n \leqslant k$. We first define b_{k+1} on the cross-section $\{\alpha_{k+1}(l) | l \in L_{k+1}\}$ by $b_{k+1}(\alpha_{k+1}(l)) = d_{k+1}(G_k, l)$, $l \in L_{k+1}$. Now for each $l_k \in L_k$ there is an $l_{k+1} \in L_{k+1}$ and $g \in G_{k+1}$ such that $\alpha_{k+1}(l_{k+1})g = \alpha_k(l_k)$. Define $b_{k+1}(\alpha_k(l_k)) = d_{k+1}(G_k g, l_{k+1})$. This defines b_{k+1} on the cross-section $\{\alpha_k(l_k) | l_k \in L_k\}$. It remains to define b_{k+1} on the G_k-orbits through the points of this cross-section. But (4.1) forces the choice, for $x = \alpha_k(l_k) \cdot g$, $g \in G_k$, of $b_{k+1}(x) = b_{k+1}(\alpha_k(l_k)) b_k(\alpha_k(l_k))^{-1} \cdot b_k(x)$. It is now straightforward to verify that (3.3) holds for all $m, n \leqslant k + 1$.

This construction demonstrates the existence of a wide class of cocycles. We isolate from the preceding construction the following result.

Proposition 4.1. *If $m = 1$ and the sequence $\{b_n\}$ is chosen via the construction above then every b_n is $\mathbb{T}$-valued and the representation with cocycle* (3.5) *is irreducible.*

Proof. This follows in similar fashion to Proposition 2.1.

The construction of cocycles by Gärding and Wightman [9] admits a generalisation in the setting of *ATI* groups provided some additional structure is present. Assume the following condition holds:

($*$) the cross-sections $\{\alpha_n\}_{n=1}^{\infty}$ of the orbit space $\{L_n\}_{n=1}^{\infty}$ may be chosen so that there exists a Borel map ξ_n: $G_n/G_{n-1} \to G_n$ with

$$\alpha_{n-1}(L_{n-1}) = \{\alpha_n(l_n) \cdot \xi_n(G_{N-1}g) | l_n \in L_n, g \in G_n\}.$$

In our setting the construction of [9] amounts to assuming ($*$) and that $d_{k+1}(G_k, l) = 1$ for all $l \in L_{k+1}$, thus allowing a very simple expression for the cocycle. Every $x \in (G_0, \sigma)^\wedge$ determines a sequence of orbits $\{q_n(x)\}_{n=1}^{\infty}$ (notation as in Section 2) with each $q_n(x) \in L_n$ being the G_n-orbit containing x. It is then easy to check that

$$\begin{aligned} b_n(x)^{-1} b_n(xg) &= b_{n-1}(x)^{-1} b_n(\alpha_{n-1}(q_{n-1}(x)))^{-1} b_n(\alpha_{n-1}(q_{n-1}(xg))) b_n(xg) \\ &= b_1(x)^{-1} b_2(\alpha_1(q_1(x)))^{-1} \cdots b_n(\alpha_{n-1}(q_{n-1}(x)))^{-1} \\ &\quad \cdot b_n(\alpha_{n-1}(q_{n-1}(xg))) \cdots b_1(x), \qquad g \in G_n. \end{aligned}$$

The special form for the d_n now gives

$$\begin{aligned} b_n(x)^{-1} b_n(xg) &= d_1\Big(G_0 h_{q_1(x)}, q_1(x)\Big)^{-1} d_2\Big(G_1 h_{q_2(x)}, q_2(x)\Big)^{-1} \\ &\cdots d_n\Big(G_{n-1} h_{q_n(x)}, q_n(x)\Big)^{-1} d_n\Big(G_{n-1} h_{q_n(xg)}, q_n(xg)\Big) \cdots d_1\Big(G_0 h_{q_1(xg)}, q_1(xg)\Big) \end{aligned}$$

where $\alpha_1(q_1(x)) \cdot h_{q_1(x)} = x$ with $h_{q_1(x)} \in G_1$ and $h_{q_k(x)}$ is the element of G_k determined by ($*$) such that $\alpha_k(q_k(x)) h_{q_k(x)} = \alpha_{k-1}(q_{k-1}(x))$ for $k = 2, 3, \ldots, n$.

EXAMPLE 4.2. Since we have claimed that our construction generalises that of [9] let us consider briefly the example of that paper, that is of the action $\oplus_1^\infty \mathbb{Z}_2$ on $\prod_1^\infty \mathbb{Z}_2$. In this case one takes $G = G_0 \oplus G_0$, with $G_0 = \oplus_1^\infty \mathbb{Z}_2$ and $G_n = \oplus_1^n \mathbb{Z}_2 \oplus G_0$. The 2-cocycle σ on G which makes the twisted group algebra [5] $C^*(G, \sigma)$ isomorphic to the *CAR* algebra is given in [6] and will not concern us here. For each k let V_k be a unitary operator on H_n (it is automatically in the commutant of $\{m \cdot v(g, x) | x \in \hat{G}_0, g \in G_k\}$). Note that an expression for $v(g, x)$ is given in Section 5 and in equation (3) of [9]. Now G satisfies condition $(*)$ (in fact with $\delta_k = (0,\ldots,0, \overset{(k)}{1}, 0, \ldots)$ in $\oplus_1^\infty \mathbb{Z}_2$, $\xi(G_{k-1}\delta_k) = \delta_k$ and $\xi(G_{k-1}) = (0)$). Define

$$d_k(G_{k-1}\delta_k, l_k(x)) = V_k. \tag{4.1}$$

(Note that if $x \in \hat{G}_0$, $x = (x_1, x_2, \ldots, x_n, \ldots)$, $x_n = 0, 1$ and the obvious choices give $\alpha_n(q_n(x)) = (0, \ldots, 0, x_{n+1} x_{n+2}, \ldots)$, $n = 1, 2, \ldots$.)

In [9] it is shown how, by appropriate choice of V_k, non-cohomologous cycles may be constructed (when the multiplicity factor m is finite). Moreover *all cocycles* are obtained by this construction. Furthermore Golodets [11] shows in some detail how to obtain irreducible representations for $m = 1, 2, \ldots, \infty$.

With suitable modifications one may also handle other *UHF* algebras (regarded as twisted group algebras as in [15]) by similar methods. Note however that the product structure of $\hat{G}_0$ is responsible for the especially simple features of this example. To see that similar methods work in the non-product situation we consider the following example.

EXAMPLE 4.3. Let $G = \mathbb{Z} \times \mathbb{D}$ where $\mathbb{D}$ denotes the 2^nth roots of unity ($n = 0, 1, 2, \ldots$). Then we may define a two cocycle on G by

$$\sigma((n, d), (n', d')) = l^{2\pi i n' d}, \qquad n, n' \in \mathbb{Z}, d, d' \in \mathbb{D}.$$

We let $G_0 = \mathbb{Z}$, $G_n = G_0 \oplus \mathbb{D}_n$ where $\mathbb{D}_n$ denotes the 2^kth roots of unity for $k \leqslant n$. Identify $\hat{G}_0$ $(= \mathbb{T})$ with $[0, 2\pi)$ and $\mathbb{D}_n$ with the group generated by $\{2\pi/2^k | k = 1, 2, \ldots\}$ with addition mod 2π so that $\mathbb{D}_n \subseteq \mathbb{T}$ acts on $\mathbb{T}$ as a subgroup. The cross-sections α_n are such that $\alpha_n(x)$ is the representative in $[0, \pi/2^{n-1})$ of x modulo $\pi 2^{-(n-1)}$. It is clear that condition $(*)$ holds here also.

To construct non-trivial cocycles in this situation is reasonably straightforward although the details are a little messy. We let for example

$$f(x) = \begin{cases} +1, & 0 \leqslant x \leqslant \pi, \\ -1, & \pi \leqslant x \leqslant 2\pi, \end{cases}$$

and let d_n: $G_n/G_{n-1} \times L_n \to \mathbb{T}$ be given by

$$d_n(G_{n-1}g, x) = \begin{cases} 1 & \text{if } g \in G_{n-1}, \\ f(2^n x) & \text{otherwise.} \end{cases}$$

https://doi.org/10.1017/S1446788700026471 Published online by Cambridge University Press

Construct the cocycle as in the paragraph preceding Example 4.2. Now to see that this cocycle is not trivial note that for $g_n = \pi 2^{-(n-1)} \in G_n$,

$$b_n\left(\alpha_{n-1}(q_{n-1}(x))\right)^{-1} b_n\left(\alpha_{n-1}(q_{n-1}(xg_n))\right)^{-1}$$
$$= d_n\left(G_{n-1}h_{q_n(x)}, q_n(x)\right)^{-1} \cdot d_n\left(G_{n-1}h_{q_n(xg_n)}, q_n(xg_n)\right)$$

and since either x or xg_n belongs to the interval $[0, \pi 2^{-(n-1)}]$ modulo $\pi 2^{-(n-2)}$ but not both, this is just the nth Rademacher function $R_n(x) = f(2^n x)$. Now all the other terms in the product formula for $b_n(x)^{-1}b_n(xg)$ are constant on the basic intervals $I_j = [j\pi 2^{-(n-1)}, (j+1)\pi 2^{-(n-1)}]$, $j = 0, 1, 2, \ldots$, so

$$\int_{I_j} b_n(x)^{-1} b_n(xg_n)\, dx = 0$$

for any such interval I_j and hence $\int_{\mathbf{T}} b_n(x)^{-1}b_n(xg_n)\, dx$ is zero for all n.

On the other hand if we could find a Borel function $b\colon \mathbb{T} \to \mathbb{T}$ such that $b_n(x)^{-1}b_n(xg) = b(x)^{-1}b(xg)$ for all n and $g \in G_n$ then $\|b(\cdot) - b(\cdot g_n)\|_{L^1(\mathbf{T})} \to 0$ as $n \to \infty$.

So some subsequence $b(xg_{n_i}) \to b(x)$ a.e. and hence $b(x)^{-1}b(xg_{n_i}) \to 1$ a.e. This contradicts the conclusion of the preceding paragraph so the cocycle constructed above is non-trivial.

The preceding two examples and that of [3] fit into the same framework. In the next section therefore we will consider that situation in more detail.

5. Abelian groups with a multiplier

The two examples of the preceding section are special cases of a more general class of group algebras which includes the irrational rotation algebra. If G is an abelian group with a Borel multiplier $\sigma\colon G \times G \to \mathbb{T}$ (or 2-cocycle) then following [2], [11] we let

$$\tilde{\sigma}(g_1, g_2) = \sigma(g_1, g_2)/\sigma(g_2, g_1), \qquad g_1, g_2 \in G,$$

and note that $\tilde{\sigma}\colon G \times G \to \mathbb{T}$ is a bicharacter. Now the twisted group C^*-algebra [5], $C^*(G, \sigma)$, is simple provided the homomorphism $g \to \tilde{\sigma}(-, g)$ from G to $\hat{G}$ is injective [17] and is not type I whenever this map is not surjective [2], [11]. We assume both of these conditions hold. Moreover we will restrict attention to the case where $G = HK$ where H and K are subgroups such that $\tilde{\sigma}|_{H\times H} \equiv 1$ and $\tilde{\sigma}|_{K\times K} \equiv 1$.

It will be readily apparent from the succeeding calculations that $C^*(G, \sigma)$ is isomorphic to the cross-product C^*-algebra $K \times_{\tilde{\sigma}} C_0(\hat{H})$ where the action of K on $\hat{H}$ is given by $\theta \to \theta \cdot \tilde{\sigma}(-, k) \equiv \theta^k$, $k \in K$, $\theta \in H$. As is well known and noted in [6] the CAR algebra is isomorphic to the cross-product of $\oplus_1^\infty \mathbf{Z}_2$ with $\prod_1^\infty \mathbf{Z}_2$ for a certain multiplier σ on $\oplus_1^\infty \mathbf{Z}_2 \oplus \oplus_1^\infty \mathbf{Z}_2$ ($\equiv H \oplus K$ in additive notation). It follows then that one may for example investigate the extent to which the ideas of [3] and of the preceding sections may be used to construct σ-1-cocycles for such groups. We have already noted in Examples 4.2, 4.3, that one may construct non-cohomologous cocycles and we will now consider how to obtain factor representations.

Assume (by choosing a cohomologous cocycle if necessary) that $\sigma|_{H\times H} = \sigma|_{K\times K} \equiv 1$ and $\sigma(g, g^{-1}) = 1$ for all $g \in G = HK$. Consider the right regular σ-representation of G on $L^2(G)$:

(5.1) $$g \cdot f(g') = f(g'g)/\sigma(g', g).$$

Now define

(5.2) $$\hat{f}(\theta, g) = \int \theta(h) \frac{f(h^{-1}g)}{\sigma(g, h^{-1})}\, dh, \qquad f \in L^2(G).$$

As in pp. 62–64 of [7] the map $f \to \hat{f}$ takes $L^2(G)$ to a certain space of functions on $\hat{H} \times G$ which satisfy

(5.3) $$\hat{f}(\theta, gh) = \theta(h)\sigma(g, h)\hat{f}(\theta, g), \qquad h \in H, g \in G.$$

By defining ϕ: $\hat{H} \times K \to \mathbf{C}$ by $\phi(\theta, k) = \hat{f}(\theta, k)$, $\theta \in \hat{H}$, $k \in K$ for $\hat{f}$ defined by (5.2), it is not difficult to verify that the composite map $f \to \phi$ takes $L^2(G)$ to $L^2(\hat{H} \times K)$ where $\hat{H}$ and K are equipped with Haar measure. Finally the right regular σ-representation acts by

(5.4) $$R_{hk}\phi(\theta, k_1) = \theta^k(h)\sigma(k, h)\phi(\theta^k, k_1 k)$$

for $h \in H$, $k, k_1 \in K$ and $\phi \in L^2(\hat{H} \times K)$. This defines a σ-1-cocycle v: $\hat{K} \times H \to \mathbb{T}$ by $v(hk, \theta) = \theta^k(h)\sigma(k, h)$. For the special case of the CAR algebra it is not difficult to show that v is exactly the σ-1-cocycle constructed by Theorem 1.1 for μ the Haar measure on $\hat{H}$. Note that v has the pleasant property of being universally defined no matter what measure is chosen on $\hat{H}$.

Now if T_k denotes right translation by $k \in K$ than (5.4) is just $R_{hk}\phi(\theta, k_1) = v(hk, \theta)T_k(\phi)(\theta, k_1)$. The cocycle $(hk, \theta) \to v(hk, \theta)T_k$ on $G \times \hat{H}$ arises via the construction of example (4.2) for the case of the CAR algebra by choosing in equation (4.1) $V_n = T_{\delta_n}$.

Two results which are more or less immediate from combining the preceding remarks with the analysis of Sections 3 and 4 of [6] are that any other k-quasi-invariant ergodic measure, ν say, on $\hat{H}$ gives a σ-representation on $L^2(\hat{H} \times K, \nu)$ by

(5.5) $$R_{hk}(\nu)\phi(\theta, k_1) = \delta(k, \theta)v(hk, \theta)T_k\phi(\theta, k_1),$$

where $\phi \in L^2(H \times K, \nu)$ and δ is the Radon–Nikodyn factor for ν. In the case of the CAR algebra, by choosing ν to be an appropriate product measure on $\prod_1^\infty \mathbf{Z}_2 \equiv \hat{H}$ and using the product measure calculations in Section 3 of [6] (cf also [13]), one finds that the corresponding CAR representation is one of the type III_λ infinite tensor product factors of Powers [16] (Haar measure of course gives rise to the hyperfinite II_1 factor).

The main object of this section is to note that we may exploit the fact that v gives a 1-cocycle independent of the measure on $\hat{H}$ to give a construction of other factor representations for these examples. Notice that if a: $K \times \hat{H} \to \mathbb{T}$ is a Borel 1-cocycle then by setting

$$(5.6) \qquad R_{hk}(\nu, a)\phi(\eta, k_1) = \delta(k, \theta)a(k, \theta)v(hk, \theta)T_k\phi(\theta, k_1)$$

one has the following fact.

PROPOSITION 5.1. *If ν is a G-quasi-invariant ergodic measure on $\hat{H}$ and $hk \to R_{hk}(\nu, a)$ is the σ-representation* (5.6) *then $R(\nu, a)$ is a factor σ-representation.*

PROOF. For each $hk \in G$ let L_{hk} be the unitary operator on $L^2(\hat{H} \times K, \nu)$:

$$(5.7) \qquad L_{hk}\phi(\theta, k_1) = \theta^{k_1^{-1}}(h^{-1})\sigma(k, h^{-1})\phi(\theta, k^{-1}k_1).$$

(If $\nu \equiv$ Haar measure, then $hk \to L_{hk}$ is the left regular σ-representation.) It is easy to check that L_{hk} is in the commutant of $R(\nu, a)$ for each $hk \in G$. Any unitary operator S in the centre of the von Neumann algebra generated by $R(\nu, a)$ is necessarily a multiplication operator: $S\phi(\theta, k) = (s(\theta)\phi)(\theta, k)$ for some Borel function s on $\hat{H}$ taking its values in the unitary operators on $L^2(K)$. Moreover s satisfies $T_k s(\theta)T_k^{-1} = s(\theta^k)$, $k \in K$, (ν a.e. θ), from (5.6) while from (5.7) by putting h equal to the identity one has $T_{k^{-1}}s(\theta)T_k = s(\theta)$, $k \in K$, (ν a.e. θ). These two conditions force s to be a constant convolution operator on $L^2(K)$. Putting k equal to the identity in (5.7) one immediately deduces that s must commute with the action of the continuous functions on K on $L^2(\hat{K})$ since the functions $k_1 \to \theta^{k_1^{-1}}(h')^{-1}ll = \theta(h')^{-1}\tilde{\sigma}(h'^{-1}, k_1^{-1})$ form a dense set of characters of K for each θ as h' ranges over H. Thus S is a multiple of the identity.

Of course (5.6) is not the most general construction possible in this setting for one still has the freedom to vary the representation (or cocycle) $k \to T_k$ on K. However before making an excursion in that direction we feel it would be more fruitful to decide whether one may deduce from properties of cocycles such as that in (5.6), invariants for the corresponding factor algebras. We intend to pursue this and related questions elsewhere.

Acknowledgements

This paper was written partly at the Institute of Advanced Studies, Australian National University and partly at the University of New South Wales. We thank Derek Robinson and Gavin Brown respectively for their invitations to visit these institutions. We would also like to thank Colin Sutherland for numerous conversations and comments.

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