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Cohomology of Complex Torus Bundles Associated to Cocycles

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Abstract. Equivariant holomorphic maps of Hermitian symmetric domains into Siegel upper half spaces can be used to construct families of abelian varieties parametrized by locally symmetric spaces, which can be regarded as complex torus bundles over the parameter spaces. We extend the construction of such torus bundles using 2-cocycles of discrete subgroups of the semisimple Lie groups associated to the given symmetric domains and investigate some of their properties. In particular, we determine their cohomology along the fibers.

1 Introduction

Equivariant holomorphic maps of Hermitian symmetric domains into Siegel upper half spaces can be used to construct families of abelian varieties parametrized by locally symmetric spaces (see *e.g.* [9]), which can be regarded as complex torus bundles over the parameter spaces. The goal of this paper is to extend the construction of such torus bundles using 2-cocycles of discrete subgroups of the semisimple Lie groups associated to the given symmetric domains and investigate some of their properties.

Let $\mathcal{H}_n$ be the Siegel upper half space of degree n on which the symplectic group $\mathrm{Sp}(n, \mathbb{R})$ acts as usual. The semidirect product $\mathrm{Sp}(n, \mathbb{R}) \ltimes \mathbb{R}^{2n}$ operates on $\mathcal{H}_n \times \mathbb{C}^n$ by

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\mu, \nu)\right) \cdot (Z, \zeta) = \big((AZ + B)(CZ + D)^{-1}, (\zeta + \mu Z + \nu)(CZ + D)^{-1}\big)$$

for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{R})$, $(\mu, \nu) \in \mathbb{R}^{2n}$ and $(Z, \zeta) \in \mathcal{H}_n \times \mathbb{C}^n$, where elements of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ are considered as row vectors. Let $\Gamma_0 = \mathrm{Sp}(n, \mathbb{Z})$, and consider the discrete subgroup $\Gamma_0 \ltimes \mathbb{Z}^{2n}$ of $\mathrm{Sp}(n, \mathbb{R}) \ltimes \mathbb{R}^{2n}$. The natural projection $\mathcal{H}_n \times \mathbb{C}^n \to \mathcal{H}_n$ induces the fiber bundle

$$\pi_0 \colon \Gamma_0 \ltimes \mathbb{Z}^{2n} \setminus \mathcal{H}_n \times \mathbb{C}^n \to \Gamma_0 \setminus \mathcal{H}_n$$

whose fibers are complex tori of dimension n. In fact, each fiber of this bundle has the structure of a principally polarized abelian variety, and therefore the Siegel modular variety $\Gamma_0 \setminus \mathcal{H}_n$ can be regarded as the parameter space of the family of principally polarized abelian varieties (*cf.* [5]).

In order to consider more general families of abelian varieties, let G be a semisimple Lie group of Hermitian type, and let $\mathcal{D}$ be the associated symmetric domain, which can be identified with the quotient G/K of G by a maximal compact subgroup K. We assume that there is a homomorphism $\rho \colon G \to \mathrm{Sp}(n, \mathbb{R})$ of Lie groups

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and a holomorphic map $\tau\colon \mathcal{D} \to \mathcal{H}_n$ that is equivariant with respect to ρ. If Γ is a torsion-free discrete subgroup of G with $\rho(\Gamma) \subset \Gamma_0$, then τ induces the map $\widetilde{\tau}\colon \Gamma \setminus \mathcal{D} \to \Gamma_0 \setminus \mathcal{H}_n$. By pulling the bundle π_0 back via $\widetilde{\tau}$ we obtain a fiber bundle over $\Gamma \setminus \mathcal{D}$ with fibers that are n-dimensional complex tori. This bundle can also be constructed in the following way. Let L be a lattice in $\mathbb{R}^{2n}$ such that $L \cdot \rho(\Gamma) \subset L$. We consider the semidirect product $\Gamma \ltimes L$ with multiplication given by

$$(1.1)\qquad (\gamma_1, \ell_1) \cdot (\gamma_2, \ell_2) = \bigl(\gamma_1\gamma_2, \ell_1\rho(\gamma_2) + \ell_2\bigr)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $\ell_1, \ell_2 \in L$. Then $\Gamma \ltimes L$ acts on $\mathcal{D} \times \mathbb{C}^n$ by

$$(1.2)\qquad \bigl(\gamma, (\mu, \nu)\bigr) \cdot (z, w) = \Bigl(\gamma z, \bigl(w + \mu\tau(z) + \nu\bigr)\bigl(C_\rho\tau(z) + D_\rho\bigr)^{-1}\Bigr),$$

for all $(\mu, \nu) \in L$ and $\gamma \in \Gamma$ with $\rho(\gamma) = \left(\begin{smallmatrix} A_\rho & B_\rho \\ C_\rho & D_\rho \end{smallmatrix}\right) \in \mathrm{Sp}(n, \mathbb{R})$, and we obtain the torus bundle

$$\pi_1\colon \Gamma \ltimes L \setminus \mathcal{D} \times \mathbb{C}^n \to \Gamma \setminus \mathcal{D}$$

over the locally symmetric space $\Gamma \setminus \mathcal{D}$ whose fibers are in fact polarized abelian varieties (see [4], [9]). The total space of such a bundle is known as a Kuga fiber variety, whose arithmetic and geometric aspects have been studied in numerous papers over the years. A Kuga fiber variety is also an example of a mixed Shimura variety in more modern language (*cf.* [8]).

The torus bundle parametrized by $\Gamma \setminus \mathcal{D}$ described above can further be generalized if a 2-cocycle of Γ is used to modify the action of $\Gamma \ltimes L$ on $\mathcal{D} \times \mathbb{C}^n$. Indeed, given a 2-cocycle $\psi\colon \Gamma \times \Gamma \to L$, by replacing the multiplication operation (1.1) with

$$(\gamma_1, \ell_1) \cdot (\gamma_2, \ell_2) = \bigl(\gamma_1\gamma_2, \ell_1\rho(\gamma_2) + \ell_2 + \psi(\gamma_1, \gamma_2)\bigr),$$

we obtain the generalized semidirect product $\Gamma \ltimes_\psi L$ of Γ and L. We denote by $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ the space of $\mathbb{C}^n$-valued holomorphic functions on $\mathcal{D}$, and let ξ be a 1-cochain for the cohomology of Γ with coefficients in $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ satisfying

$$(1.3)\qquad \delta\xi(\gamma_1, \gamma_2)(z) = \psi(\gamma_1, \gamma_2)\begin{pmatrix} \tau(z) \\ 1 \end{pmatrix}$$

for all $z \in \mathcal{D}$ and $\gamma_1, \gamma_2 \in \Gamma$, where δ is the coboundary operator on 1-cochains. Then an action of $\Gamma \ltimes_\psi L$ on $\mathcal{D} \times \mathbb{C}^n$ can be defined by replacing (1.2) with

$$\bigl(\gamma, (\mu, \nu)\bigr) \cdot (z, w) = \Bigl(\gamma z, \bigl(w + \mu\tau(z) + \nu + \xi(\gamma)(z)\bigr)\bigl(C_\rho\tau(z) + D_\rho\bigr)^{-1}\Bigr).$$

If the quotient of $\mathcal{D} \times \mathbb{C}^n$ by $\Gamma \ltimes_\psi L$ with respect to this action is denoted by $Y_{\psi,\xi}$, the natural projection $\mathcal{D} \times \mathbb{C}^n \to \mathcal{D}$ induces the torus bundle $\pi\colon Y_{\psi,\xi} \to X = \Gamma \setminus \mathcal{D}$.

In this paper we determine the cohomology $R^k\pi_*\mathcal{O}_{Y_{\psi,\xi}}$ along the fibers of $Y_{\psi,\xi}$ over X associated to the sheaf $\mathcal{O}_{Y_{\psi,\xi}}$ of holomorphic functions on $Y_{\psi,\xi}$ for $k = 0, 1$. We also discuss some properties of the families of torus bundles $Y_{\psi,\xi}$ for certain sets of ψ and ξ. We note that other aspects of such torus bundles were also considered in [7].

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2 Torus Bundles over Locally Symmetric Spaces

In this section we describe the construction of fiber bundles over locally symmetric spaces whose fibers are complex tori. If Γ is a discrete subgroup of a semisimple Lie group of Hermitian type, such a bundle is determined by a 2-cocycle of Γ with coefficients in a lattice and a certain 1-cochain of Γ with coefficients in the space of vector-valued holomorphic functions on the symmetric domain associated to the given semisimple Lie group.

Let G be a semisimple Lie group of Hermitian type, and let K be a maximal compact subgroup of G. Thus the quotient space $\mathcal{D} = G/K$ has the structure of a Hermitian symmetric domain. Let $\mathrm{Sp}(n, \mathbb{R})$ be the symplectic group of degree n, and let $\mathcal{H}_n$ be the associated Siegel upper half space, that is, the space of complex $n \times n$ matrices with positive definite imaginary part. We assume that there is a homomorphism $\rho\colon G \to \mathrm{Sp}(n, \mathbb{R})$ of Lie groups and a holomorphic map $\tau\colon \mathcal{D} \to \mathcal{H}_n$ that is equivariant with respect to ρ, which means that

$$\tau(gz) = \rho(g)\tau(z)$$

for all $z \in \mathcal{D}$. Equivariant pairs (ρ, τ) of this type satisfying some additional conditions were completely classified by Satake [9].

Let L be a lattice in $\mathbb{R}^{2n}$. We shall often consider L as a subgroup of $\mathbb{R}^n \times \mathbb{R}^n$ and write elements of L in the form (μ, ν), where $\mu, \nu \in \mathbb{R}^n$ are regarded as row vectors. Let Γ be a discrete subgroup of G such that $\ell\rho(\gamma) \in L$ for all $\ell \in L$ and $\gamma \in \Gamma$, where $\ell\rho(\gamma)$ is the matrix product of the row vector ℓ of $2n$ entries and the $2n \times 2n$ matrix $\rho(\gamma)$. Thus L has the structure of a right Γ-module, and therefore we can consider the cohomology $H^*(\Gamma, L)$ of the group Γ with coefficients in L. We denote by $\mathfrak{C}^k(\Gamma, L)$ and $\mathfrak{Z}^k(\Gamma, L)$ the spaces of the associated k-cochains and k-cocycles, respectively, and choose an element ψ of $\mathfrak{Z}^2(\Gamma, L)$. Thus ψ is a map $\psi\colon \Gamma \times \Gamma \to L$ satisfying

$$\psi(\gamma_1, \gamma_2)\rho(\gamma_3) + \psi(\gamma_1\gamma_2, \gamma_3) = \psi(\gamma_2, \gamma_3) + \psi(\gamma_1, \gamma_2\gamma_3)$$
$$\psi(\gamma, 1) = 0 = \psi(1, \gamma)$$

for all $\gamma_1, \gamma_2, \gamma_3, \gamma \in \Gamma$, where 1 is the identity element of Γ. We note that an element $\alpha \in \mathfrak{Z}^2(\Gamma, L)$ is a coboundary if $\alpha = \partial\beta$ for some $\beta \in \mathfrak{C}^1(\Gamma, L)$, where

$$\partial\beta(\gamma_1, \gamma_2) = \beta(\gamma_2) - \beta(\gamma_1\gamma_2) + \beta(\gamma_1)\rho(\gamma_2) \tag{2.1}$$

for all $\gamma_1, \gamma_2 \in \Gamma$.

We now consider the generalized semidirect product $\Gamma \ltimes_\psi L$ associated to ψ, which consists of the elements $\big(\gamma, (\mu, \nu)\big)$ in $\Gamma \times L$ and is equipped with the multiplication operation defined by

$$\big(\gamma_1, (\mu_1, \nu_1)\big) \cdot \big(\gamma_2, (\mu_2, \nu_2)\big) = \big(\gamma_1\gamma_2, (\mu_1, \nu_1)\rho(\gamma_2) + (\mu_2, \nu_2) + \psi(\gamma_1, \gamma_2)\big) \tag{2.2}$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $(\mu_1, \nu_1), (\mu_2, \nu_2) \in L$.

Lemma 2.1 *The generalized semidirect product $\Gamma \ltimes_\psi L$ is a group with respect to the multiplication operation given by* (2.2). *The identity element is* $(1,(0,0))$, *and the element*

$$\big(\gamma^{-1}, -(\mu,\nu)\rho(\gamma)^{-1} - \psi(\gamma,\gamma^{-1})\big)$$

is the inverse of $\big(\gamma,(\mu,\nu)\big) \in \Gamma \ltimes_\psi L$.

Proof This is well-known and can be proved in a straightforward manner. ■

The group $\Gamma \ltimes_\psi L$ essentially depends on the cohomology class $[\psi] \in H^2(\Gamma, L)$ of ψ according to the next lemma.

Lemma 2.2 *Let $\psi, \psi' \colon \Gamma \times \Gamma \to L$ be 2-cocycles that are cohomologous, and let ϕ be an element of $\mathfrak{C}^1(\Gamma, L)$ such that*

$$\psi(\gamma_1,\gamma_2) = \psi'(\gamma_1,\gamma_2) + (\partial\phi)(\gamma_1,\gamma_2).$$

Then the map $\Phi \colon \Gamma \ltimes_\psi L \to \Gamma \ltimes_{\psi'} L$ defined by

$$\Phi\big(\gamma,(\mu,\nu)\big) = \big(\gamma,(\mu,\nu)+\phi(\gamma)\big) \tag{2.3}$$

for $\gamma \in \Gamma$ and $(\mu,\nu) \in L$ is an isomorphism.

Proof The proof is easy and is omitted. ■

The symplectic group $\mathrm{Sp}(n,\mathbb{R})$ acts on the Siegel upper half space $\mathcal{H}_n$ as usual by

$$g\zeta = (a\zeta + b)(c\zeta + d)^{-1} \tag{2.4}$$

for all $z \in \mathcal{H}_n$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{R})$. For such $g \in \mathrm{Sp}(n,\mathbb{R})$ and $\zeta \in \mathcal{H}_n$, we set

$$j(g,\zeta) = c\zeta + d.$$

Then the resulting map $j\colon\ \mathrm{Sp}(n,\mathbb{R}) \times \mathcal{H}_n \to \mathrm{GL}(n,\mathbb{C})$ satisfies

$$j(g'g,\zeta) = j(g',g\zeta)\,j(g,\zeta) \tag{2.5}$$

for all $\zeta \in \mathcal{H}_n$ and $g, g' \in \mathrm{Sp}(n,\mathbb{R})$. Given $z \in \mathcal{D}$ and $\gamma \in \Gamma \subset G$, we set

$$j_{\rho,\tau}(\gamma,z) = j\big(\rho(\gamma),\tau(z)\big). \tag{2.6}$$

Using (2.5) and the fact that (ρ,τ) is an equivariant pair, we see that

$$j_{\rho,\tau}(\gamma'\gamma,z) = j_{\rho,\tau}(\gamma',\gamma z)\,j_{\rho,\tau}(\gamma,z)$$

for all $z \in \mathcal{D}$ and $\gamma,\gamma' \in \Gamma$. Thus $j_{\rho,\tau}\colon \Gamma \times \mathcal{D} \to \mathrm{GL}(n,\mathbb{C})$ is an automorphy factor, and automorphic forms involving the determinant of such an automorphy factor have been studied in a number of papers (see *e.g.* [1] and [6]).

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Let $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ denote the space of $\mathbb{C}^n$-valued holomorphic functions on $\mathcal{D}$. Then $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ has the structure of a double Γ-module by

$$(\gamma \cdot f)(z) = f(z), \quad (f \cdot \gamma)(z) = f(\gamma z) j_{\rho,\tau}(\gamma, z) \tag{2.7}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{D}$, where elements of $\mathbb{C}^n$ are considered as row vectors. Thus we can consider the cohomology of the group Γ with coefficients in $\mathcal{A}(\mathcal{D}, \mathbb{C}^n)$, where its group of k-cochains consists of all functions

$$\eta \colon \Gamma^k \to \mathcal{A}(\mathcal{D}, \mathbb{C}^n)$$

such that $\eta(\gamma_1, \ldots, \gamma_k) = 0$ whenever at least one of the γ_i is 1. Then the coboundary operator

$$\delta \colon \mathfrak{C}^k\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big) \to \mathfrak{C}^{k+1}\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$$

is given by

$$\begin{aligned}\delta\eta(\gamma_1, \ldots, \gamma_{k+1}) = \gamma_1 \cdot \eta(\gamma_2, \ldots, \gamma_{k+1}) + \sum_{i=1}^{k} (-1)^i \eta(\gamma_1, \ldots, \gamma_i\gamma_{i+1}, \ldots, \gamma_{k+1}) \\ + (-1)^{k+1} \eta(\gamma_1, \ldots, \gamma_k) \cdot \gamma_{k+1}\end{aligned}$$

for all $\eta \in \mathfrak{C}^k\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$ (see [2, Chapter 15]). In particular, for $k = 1$ we have

$$\delta\eta(\gamma_1, \gamma_2) = \gamma_1 \cdot \eta(\gamma_2) - \eta(\gamma_1\gamma_2) + \eta(\gamma_1) \cdot \gamma_2 \tag{2.8}$$

for all $\gamma_1, \gamma_2 \in \Gamma$, where the right and left actions of Γ are given by (2.7).

Given a 2-cocycle $\psi \in \mathfrak{Z}^2(\Gamma, L)$, we assume that there is an element ξ of $\mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$ satisfying

$$\delta\xi(\gamma_1, \gamma_2)(z) = \psi(\gamma_1, \gamma_2) \begin{pmatrix} \tau(z) \\ 1 \end{pmatrix} \tag{2.9}$$

for all $z \in \mathcal{D}$, where the right hand side is the matrix multiplication of the row vector $\psi(\gamma_1, \gamma_2) \in L \subset \mathbb{R}^n \times \mathbb{R}^n$ and the complex $2n \times n$ matrix $\left(\begin{smallmatrix} \tau(z) \\ 1 \end{smallmatrix}\right)$. If ψ' is another 2-cocycle that is cohomologous to ψ, the corresponding element of $\mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$ can be obtained as follows. Let $\psi' \in \mathfrak{Z}^2(\Gamma, L)$ satisfy

$$\psi = \psi' + \partial\phi \tag{2.10}$$

for some $\phi \in \mathfrak{C}^1(\Gamma, L)$. Then we define the map $\xi' \colon \Gamma \to \mathcal{A}(\mathcal{D}, \mathbb{C}^n)$ by

$$\xi'(\gamma)(z) = \xi(\gamma)(z) - \phi(\gamma) \begin{pmatrix} \tau(z) \\ 1 \end{pmatrix} \tag{2.11}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{D}$.

Lemma 2.3 *If ψ' and ξ' are as in* (2.10) *and* (2.11), *then we have*

$$\delta\xi'(\gamma_1,\gamma_2)(z) = \psi'(\gamma_1,\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix}$$

for all $\gamma_1,\gamma_2 \in \Gamma$ and $z \in \mathcal{H}$.

Proof For $\gamma_1,\gamma_2 \in \Gamma$ and $z \in \mathcal{H}$, using (2.8) and (2.11), we obtain

$$\begin{aligned}\delta\xi'(\gamma_1,\gamma_2)(z) &= \xi'(\gamma_2)(z) - \xi'(\gamma_1\gamma_2)(z) + \xi'(\gamma_1)(\gamma_2 z)\, j_{\rho,\tau}(\gamma_2,z)\\ &= \delta\xi(\gamma_1,\gamma_2)(z) - \phi(\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix}\\ &\qquad + \phi(\gamma_1\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix} - \phi(\gamma_1)\begin{pmatrix}\tau(\gamma_2 z)\\1\end{pmatrix} j_{\rho,\tau}(\gamma_2,z).\end{aligned}$$

However, we have

$$\begin{aligned}\begin{pmatrix}\tau(\gamma_2 z)\\1\end{pmatrix} j_{\rho,\tau}(\gamma_2,z) &= \begin{pmatrix}(a\tau(z)+b)(c\tau(z)+d)^{-1}\\1\end{pmatrix}\big(c\tau(z)+d\big)\\ &= \begin{pmatrix}a\tau(z)+b\\c\tau(z)+d\end{pmatrix} = \rho(\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix}\end{aligned}$$

if $\rho(\gamma_2) = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$. Using this, (2.9) and (2.1), we see that

$$\delta\xi'(\gamma_1,\gamma_2)(z) = \psi(\gamma_1,\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix} - (\partial\phi)(\gamma_1,\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix} = \psi(\gamma_1,\gamma_2)\begin{pmatrix}\tau(z)\\1\end{pmatrix},$$

and therefore the lemma follows. ■

Let $\psi \in \mathfrak{Z}^2(\Gamma, L)$ and $\xi \in \mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D},\mathbb{C}^n)\big)$ be as in (2.9). Given elements $\big(\gamma,(\mu,\nu)\big) \in \Gamma \ltimes_\psi L$ and $(z,w) \in \mathcal{D}\times\mathbb{C}^n$, we set

$$\big(\gamma,(\mu,\nu)\big)\cdot(z,w) = \Big(\gamma z, \big(w + \mu\tau(z) + \nu + \xi(\gamma)(z)\big)\, j_{\rho,\tau}(\gamma,z)^{-1}\Big), \tag{2.12}$$

where $j_{\rho,\tau}\colon \Gamma\times\mathcal{D} \to \mathrm{GL}(n,\mathbb{C})$ is given by (2.6).

Lemma 2.4 *The operation given by* (2.12) *determines an action of the group $\Gamma \ltimes_\psi L$ on the space $\mathcal{D}\times\mathbb{C}^n$.*

Proof This can be proved using calculations similar to the ones in Lemma 2.3 and standard ideas from the theory of group extensions. ■

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Remark 2.5 In [7, Lemma 3.3], there is another action of $\Gamma \ltimes_\psi L$ on $\mathcal{D} \times \mathbb{C}^n$ which apparently looks different from the one in Lemma 2.4. However, it can be shown that these two actions are equivalent. The equivalence of these actions in the case of trivial ψ and ξ can be found in [4] and [9, Chapter IV], and similar arguments can be used to prove the equivalence in the general case.

We assume that the discrete subgroup $\Gamma \subset G$ does not contain elements of finite order, so that the quotient $X = \Gamma \setminus \mathcal{D}$ of $\mathcal{D}$ by the Γ-action given by (2.4) has the structure of a complex manifold, and set

$$Y_{\psi,\xi} = \Gamma \ltimes_\psi L \setminus \mathcal{D} \times \mathbb{C}^n, \tag{2.13}$$

where the quotient is taken with respect to the action in Lemma 2.4. Then the map $\pi \colon Y_{\psi,\xi} \to X$ induced by the natural projections $\mathcal{D} \times \mathbb{C}^n \to \mathcal{D}$ and $\Gamma \ltimes_\psi L \to \Gamma$ has the structure of a fiber bundle over X whose fiber over a point corresponding to $z \in \mathcal{D}$ is isomorphic to the complex torus

$$\mathbb{C}^n \Big/ \left(L \cdot \begin{pmatrix} \tau(z) \\ 1 \end{pmatrix} \right).$$

If $\psi = 0$ and $\xi = 0$, then the the corresponding torus bundle $Y_{0,0}$ has the structure of a complex projective variety and is known as a *Kuga fiber variety* (*cf.* [4], [9]).

Proposition 2.6 *Given $\psi \in \mathfrak{Z}^2(\Gamma, L)$ and $\xi \in \mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$, let ψ' and ξ' be as in* (2.10) *and* (2.11). *Then the map $\Phi \colon \Gamma \ltimes_\psi L \to \Gamma \ltimes_{\psi'} L$ given by* (2.3) *and the identity map on $\mathcal{D} \times \mathbb{C}^n$ induce an isomorphism $Y_{\psi,\xi} \to Y_{\psi',\xi'}$ of bundles over $X = \Gamma \setminus \mathcal{D}$.*

Proof It suffices to show that

$$\Phi\big(\gamma, (\mu, \nu)\big) \cdot (z, w) = \big(\gamma, (\mu, \nu)\big) \cdot (z, w),$$

where the actions on the right and left hand sides are with respect to (ψ, ξ) and (ψ', ξ'), respectively. Indeed, we have

$$\begin{aligned} \Phi&\big(\gamma, (\mu, \nu)\big) \cdot (z, w) \\ &= \left(\gamma z, \left(w + (\mu, \nu) \begin{pmatrix} \tau(z) \\ 1 \end{pmatrix} + \phi(\gamma) \begin{pmatrix} \tau(z) \\ 1 \end{pmatrix} + \xi'(\gamma)(z) \right) j_{\rho,\tau}(\gamma, z)^{-1} \right) \\ &= \left(\gamma z, \left(w + (\mu, \nu) \begin{pmatrix} \tau(z) \\ 1 \end{pmatrix} + \xi(\gamma)(z) \right) j_{\rho,\tau}(\gamma, z)^{-1} \right) \\ &= \big(\gamma, (\mu, \nu)\big) \cdot (z, w). \end{aligned}$$

and therefore the proposition follows. ■

3 Families of Torus Bundles

Let $\Gamma \subset G$, $L \subset \mathbb{R}^{2n}$ and $\mathcal{D} = G/K$ be as in Section 2. In this section we discuss some of the properties of families of torus bundles of the form $Y_{\psi,\xi}$ over $X = \Gamma \backslash \mathcal{D}$ for certain sets of $\psi \in \mathfrak{Z}^2(\Gamma, L)$ and $\xi \in \mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$.

Given a 2-cocycle $\psi \colon \Gamma \times \Gamma \to L$, we denote by Ξ_ψ the set of all $\xi \in \mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$ satisfying (2.9). Thus, if $\psi = 0$, the set Ξ_0 coincides with the space

$$\mathfrak{Z}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big) = \big\{\eta \in \mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big) \bigm| \delta\eta = 0\big\}$$

of 1-cocycles in $\mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$, where $\delta\eta$ is as in (2.8). Each $\xi \in \Xi_\psi$ determines the associated torus bundle $Y_{\psi,\xi}$ over X given by (2.13). We denote by

$$\mathfrak{T}_\psi = \{Y_{\psi,\xi} \mid \xi \in \Xi_\psi\}$$

the family of torus bundles $Y_{\psi,\xi}$ parametrized by Ξ_ψ. Thus, if ψ is the zero map, the torus bundle $Y_{0,0}$ determined by $0 \in \Xi_0$ is a Kuga fiber variety. Given $\xi \in \Xi_\psi$ and $\xi' \in \Xi_{\psi'}$, if δ is the coboundary operator on $\mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$, then by (2.9) we have

$$\delta(\xi + \xi')(\gamma_1, \gamma_2)(z) = (\psi + \psi')(\gamma_1, \gamma_2)\begin{pmatrix} \tau(z) \\ 1 \end{pmatrix}$$

for all $z \in \mathcal{H}_n$ and $\gamma_1, \gamma_2 \in \Gamma$; hence we see that $\xi + \xi' \in \Xi_{\psi+\psi'}$.

Let $(\mathcal{D} \times \mathbb{C}^n) \oplus_{\mathcal{D}} (\mathcal{D} \times \mathbb{C}^n)$ be the Whitney sum of two copies of the trivial vector bundle $\mathcal{D} \times \mathbb{C}^n$ over $\mathcal{D}$, which we identify with $\mathcal{D} \times (\mathbb{C}^n \oplus \mathbb{C}^n)$. Then we can consider the map

$$s \colon \mathcal{D} \times (\mathbb{C}^n \oplus \mathbb{C}^n) \to \mathcal{D} \times \mathbb{C}^n$$

given by

$$s(z, v, v') = (z, v + v') \tag{3.1}$$

for all $z \in \mathcal{D}$ and $v, v' \in \mathbb{C}^n$. Let $\psi, \psi' \in \mathfrak{Z}^2(\Gamma, L)$, and let $\Gamma \ltimes_\psi L \ltimes_{\psi'} L$ be the group consisting of the elements of $\Gamma \times L \times L$ equipped with multiplication given by

$$\begin{aligned}\big(\gamma_1, (\mu_1, \nu_1), (\mu_1', \nu_1')\big) &\cdot \big(\gamma_2, (\mu_2, \nu_2), (\mu_2', \nu_2')\big) \\ &= \big(\gamma_1\gamma_2, (\mu_1, \nu_1)\rho(\gamma_2) + (\mu_2, \nu_2) + \psi(\gamma_1, \gamma_2), (\mu_1', \nu_1')\rho(\gamma_2) \\ &\qquad + (\mu_2', \nu_2') + \psi'(\gamma_1, \gamma_2)\big).\end{aligned}$$

Then we see that there is a group homomorphism

$$\widetilde{s} \colon \Gamma \ltimes_\psi L \ltimes_{\psi'} L \to \Gamma \ltimes_{\psi+\psi'} L$$

given by

$$\widetilde{s}(\gamma, \ell_1, \ell_2) = (\gamma, \ell_1 + \ell_2) \tag{3.2}$$

for all $\gamma \in \Gamma$ and $\ell_1, \ell_2 \in L$. If $\xi \in \Xi_\psi$ and $\xi' \in \Xi_{\psi'}$, we let the group $\Gamma \ltimes_\psi L \ltimes_{\psi'} L$ act on the space $\mathcal{D} \times (\mathbb{C}^n \oplus \mathbb{C}^n)$ by

$$\begin{aligned}\big(\gamma, (\mu, \nu), (\mu', \nu')\big) &\cdot (z, w, w') \\ = \Big(\gamma z, &\big(\mu\tau(z) + \nu + w + \xi(\gamma)(z)\big) \cdot j_{\rho,\tau}(\gamma, z)^{-1}, \\ &\big(\mu'\tau(z) + \nu' + w' + \xi'(\gamma)(z)\big) \cdot j_{\rho,\tau}(\gamma, z)^{-1}\Big).\end{aligned}$$

Then the associated quotient space

$$Y_{\psi,\xi} \oplus_X Y_{\psi',\xi'} = \Gamma \ltimes_\psi L \ltimes_{\psi'} L \setminus \mathcal{D} \times (\mathbb{C}^n \oplus \mathbb{C}^n)$$

is the fiber product of the torus bundles $Y_{\psi,\xi} \in \mathfrak{T}_\psi$ and $Y_{\psi',\xi'} \in \mathfrak{T}_{\psi'}$ over X.

Proposition 3.1 *Let $\xi \in \Xi_\psi$ and $\xi' \in \Xi_{\psi'}$ with $\psi, \psi' \in \mathfrak{Z}^1(\Gamma, L)$. Then the map s in* (3.1) *and the morphism $\widetilde{s}$ in* (3.2) *induce a morphism*

$$Y_{\psi,\xi} \oplus_X Y_{\psi',\xi'} \to Y_{\psi+\psi',\xi+\xi'}$$

of torus bundles over X.

Proof By our construction of the torus bundles involved, it suffices to show that

$$s\Big(\big(\gamma, (\mu, \nu), (\mu', \nu')\big) \cdot (z, w, w')\Big) = \widetilde{s}\big(\gamma, (\mu, \nu), (\mu', \nu')\big) \cdot s(z, w, w')$$

for $\gamma \in \Gamma$, $(\mu, \nu), (\mu', \nu') \in L$ and $(z, w, w') \in \mathcal{D} \times (\mathbb{C}^n \oplus \mathbb{C}^n)$, which can be checked easily. ∎

Applying Proposition 3.1 to the special case of $\psi' = 0$, we see that there is a natural morphism

$$Y_{\psi,\xi} \oplus_X Y_{0,\eta} \to Y_{\psi,\xi+\eta}$$

for $Y_{\psi,\xi} \in \mathfrak{T}_\psi$ and $\eta \in \Xi_0$.

Example 3.2 Given an element $h \in \mathcal{A}(\mathcal{D}, \mathbb{C}^n)$, we define $\eta \in \mathfrak{C}^1\big(\Gamma, \mathcal{A}(\mathcal{D}, \mathbb{C}^n)\big)$ by

$$\eta(\gamma)(z) = h(z) - h(\gamma z)\, j_{\rho,\tau}(\gamma, z)$$

for all $z \in D$ and $\gamma \in \Gamma$. Then η is a cocycle; in fact, it is a coboundary. Thus we can consider the associated torus bundle $Y_h = Y_{0,\eta}$ and a morphism $Y_{\psi,\xi} \oplus_X Y_h \to Y_{\psi,\xi+\eta}$ for each $Y_{\psi,\xi} \in \mathfrak{T}_\psi$.

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4 Cohomology

In this section we fix elements $\psi \in \mathfrak{Z}^2(\Gamma, L)$ and $\xi \in \Xi_\psi$, where Ξ_ψ is as in Section 3, and consider the associated torus bundle $\pi \colon Y_{\psi,\xi} \to X$ constructed in Section 2. The cohomology along the fibers of $Y_{\psi,\xi}$ over X can be provided by the direct image functors $R^i\pi_*$, which determine sheaves on X associated to sheaves on $Y_{\psi,\xi}$ (see *e.g.* [3, Section III.8]). We are interested in the images of the sheaf $\mathcal{O}_{Y_{\psi,\xi}}$ of holomorphic functions on $Y_{\psi,\xi}$ under such functors. Given a nonnegative integer k, $R^k\pi_*\mathcal{O}_{Y_{\psi,\xi}}$ is the sheaf on X generated by the presheaf

$$U \mapsto H^k\big(\pi^{-1}(U), \mathcal{O}_{Y_{\psi,\xi}}\big)$$

for open subsets U of X. Note that by Dolbeault's theorem there is a canonical isomorphism

$$H^k\big(\pi^{-1}(U), \mathcal{O}_{Y_{\psi,\xi}}\big) \cong H^{(0,k)}\big(\pi^{-1}(U)\big).$$

Proposition 4.1 *The sheaf $R^0\pi_*\mathcal{O}_{Y_{\psi,\xi}}$ is isomorphic to the sheaf $\mathcal{O}_X$ of holomorphic functions on X.*

Proof Let U be a sufficiently small open ball in X, and let $f \in H^0\big(\pi^{-1}(U), \mathcal{O}_{Y_{\psi,\xi}}\big)$. If $\widetilde{U} \subset \mathcal{D}$ is the inverse image of U under the natural projection map $\mathcal{D} \to X = \Gamma\backslash\mathcal{D}$, then we have

$$\pi^{-1}(U) \cong \Gamma \ltimes_\psi L \backslash \widetilde{U} \times \mathbb{C}^n. \tag{4.1}$$

Thus f may be regarded as a holomorphic function on $\widetilde{U} \times \mathbb{C}^n$ that is invariant under the action of $\Gamma \ltimes_\psi \{0\}$ and satisfies

$$f(z, w) = f\big(z, w + \mu\tau(z) + \nu\big)$$

for all $(z, w) \in \widetilde{U} \times \mathbb{C}^n$ and $(\mu, \nu) \in L$. Hence it follows that f is constant with respect to w and therefore can be identified with a Γ-invariant holomorphic function on $\widetilde{U}$ or a holomorphic function on U. ■

Let $j_{\rho,\tau} \colon \Gamma \times \mathcal{D} \to \mathrm{GL}(n, \mathbb{C})$ be the automorphy factor given by (2.6). Then the discrete subgroup $\Gamma \subset \mathrm{Sp}(n, \mathbb{R})$ acts on $\mathcal{D} \times \mathbb{C}^n$ by

$$\gamma \cdot (z, w) = \big(\gamma z, w \cdot j_{\rho,\tau}(\gamma, z)^{-1}\big)$$

for all $\gamma \in \Gamma$ and $(z, w) \in \mathcal{D} \times \mathbb{C}^n$. If we denote the associated quotient by

$$\mathcal{V} = \Gamma \backslash \mathcal{D} \times \mathbb{C}^n,$$

then the map $p \colon \mathcal{V} \to X = \Gamma \backslash \mathcal{D}$ induced by the natural projection $\mathcal{D} \times \mathbb{C}^n \to \mathcal{D}$ determines the structure of a vector bundle on $\mathcal{V}$ over X. By our construction we see that each holomorphic section $s\colon X \to \mathcal{V}$ of $\mathcal{V}$ over X can be identified with a function $\widetilde{s}\colon \mathcal{D} \to \mathbb{C}^n$ satisfying

$$\widetilde{s}(\gamma z) = \widetilde{s}(z) \cdot j_{\rho,\tau}(\gamma, z)^{-1} \tag{4.2}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{D}$. Now we state the main theorem in this section whose proof will be given in Section 5.

Theorem 4.2 *The sheaf $R^1\pi_* \mathcal{O}_{Y_{\psi,\xi}}$ is isomorphic to the sheaf $\widetilde{\mathcal{V}}$ of holomorphic sections of the vector bundle $\mathcal{V}$ over X.*

5 Proof of Theorem 4.2

Given a torus bundle $\pi\colon Y_{\psi,\xi} \to X$ and a sufficiently small open ball in X, we consider a $(0,1)$-form ω on $\pi^{-1}(U)$ which determines the cohomology class $[\omega]$ in $H^{(0,1)}\big(\pi^{-1}(U)\big) = H^1\big(\pi^{-1}(U), \mathcal{O}_{Y_{\psi,\xi}}\big)$. Let $\widetilde{U} \subset \mathcal{D}$ be as in (4.1), and let $z = (z_1, \dots, z_N)$ be a local holomorphic system of coordinates on $\widetilde{U}$. Then we have

$$\omega = \sum_{\alpha=1}^{N} A_\alpha(z,w) d\bar{z}_\alpha + \sum_{\beta=1}^{n} B_\beta(z,w) d\overline{w}_\beta$$

for some $\mathbb{C}$-valued C^∞ functions $A_\alpha(z,w)$ and $B_\beta(z,w)$ on $\widetilde{U} \times \mathbb{C}^n$, where $w = (w_1, \dots, w_n)$ is the standard coordinate system for $\mathbb{C}^n$. Let $\ell = (\mu, \nu) \in L$, and set

$$\zeta(z,\ell) = \mu \cdot \tau(z) + \nu$$

for all $z \in \mathcal{D}$. Then by (2.12) the action of ℓ on ω is given by

$$\begin{aligned} \ell^*\omega = &\sum_{\alpha=1}^{N} A_\alpha\big(z, w + \zeta(z,\ell)\big)\, d\bar{z}_\alpha \\ &+ \sum_{\beta=1}^{n} B_\beta\big(z, w + \zeta(z,\ell)\big) \Big(d\overline{w}_\beta + \sum_{\alpha=1}^{N} \frac{\partial \overline{\zeta(z,\ell)}_\beta}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \Big). \end{aligned}$$

Since $\ell^*\omega = \omega$, we obtain

$$A_\alpha(z,w) = A_\alpha\big(z, w + \zeta(z,\ell)\big) + \sum_{\beta=1}^{n} B_\beta\big(z, w + \zeta(z,\ell)\big) \frac{\partial \overline{\zeta(z,\ell)}_\beta}{\partial \bar{z}_\alpha},$$

$$B_\beta(z,w) = B_\beta\big(z, w + \zeta(z,\ell)\big).$$

In particular, for fixed $z \in \widetilde{U} \subset \mathcal{D}$, the $(0,1)$-form

$$\widetilde{\Phi}(z,w) = \sum_{\beta=1}^{n} B_\beta(z,w) d\overline{w}_\beta$$

is L-invariant and satisfies $\overline{\partial}_w \widetilde{\Phi}(z,w) = 0$. Thus we obtain, for each z, a $\overline{\partial}_w$-closed $(0,1)$-form $\Phi(z)$ that is cohomologous to $\widetilde{\Phi}(z,w)$ on the complex torus

$$\mathbb{C}^n \Big/ \left(L \cdot \begin{pmatrix} \tau(z) \\ 1 \end{pmatrix} \right).$$

From harmonic theory we see that there are C^∞ functions $C_\beta(z)$ on U such that

$$\Phi^0(z) = \sum_{\beta=1}^{n} C_\beta(z) d\overline{w}_\beta$$

is a harmonic form in w that, for each fixed z, is cohomologous to $\Phi(z)$ in $H^{(0,1)}\big(\pi^{-1}(z)\big)$. Thus there is a C^∞ function $f(z,w)$ on $\widetilde{U} \times \mathbb{C}^n$ such that $f\big(z, w + \zeta(z,\ell)\big) = f(z,w)$ and

$$\begin{aligned}\Phi(z) - \Phi^0(z) = \overline{\partial}_w f(z,w) &= \sum_{\beta=1}^{n} \frac{\partial f(z,w)}{\partial \overline{w}_\beta} d\overline{w}_\beta \\ &= \overline{\partial} f(z,w) - \sum_{\alpha=1}^{N} \frac{\partial f(z,w)}{\partial \overline{z}_\alpha} d\overline{z}_\alpha.\end{aligned}$$

Hence, if we set

$$\omega^{(1)} = \sum_{\alpha=1}^{N} D_\alpha(z,w) d\overline{z}_\alpha + \sum_{\beta=1}^{n} C_\beta(z) d\overline{w}_\beta \tag{5.1}$$

with

$$D_\alpha(z,w) = A_\alpha(z,w) - \frac{\partial f(z,w)}{\partial \overline{z}_\alpha},$$

we see that $\omega - \omega^{(1)} = \overline{\partial} f$, and therefore $[\omega] = [\omega^{(1)}]$ in $H^{(0,1)}(\pi^{-1}(U))$. Since $\omega^{(1)}$ is a $\overline{\partial}$-closed form, we have

$$\begin{aligned}0 = \overline{\partial}\omega^{(1)} = &\sum_{\alpha=1}^{N}\sum_{\lambda=1}^{N} \frac{\partial D_\alpha(z,w)}{\partial \overline{z}_\lambda} d\overline{z}_\lambda \wedge d\overline{z}_\alpha \\ &+ \sum_{\alpha=1}^{N}\sum_{\epsilon=1}^{n} \frac{\partial D_\alpha(z,w)}{\partial \overline{w}_\epsilon} d\overline{w}_\epsilon \wedge d\overline{z}_\alpha + \sum_{\beta=1}^{n}\sum_{\lambda=1}^{N} \frac{\partial C_\beta(z)}{\partial \overline{z}_\lambda} d\overline{z}_\lambda \wedge d\overline{w}_\beta;\end{aligned}$$

hence we obtain

$$\frac{\partial C_\beta(z)}{\partial \overline{z}_\lambda} = \frac{\partial D_\lambda(z,w)}{\partial \overline{w}_\beta}, \quad \frac{\partial D_\alpha(z,w)}{\partial \overline{z}_\lambda} = \frac{\partial D_\lambda(z,w)}{\partial \overline{z}_\alpha} \tag{5.2}$$

for $1 \leq \alpha, \lambda \leq N$ and $1 \leq \beta \leq n$. Thus we have

$$D_\lambda(z,w) = \sum_{\beta=1}^{n} F_{\lambda,\beta}(z)\overline{w}_\beta + P_\lambda(z,w), \quad F_{\lambda,\beta}(z) = \frac{\partial C_\beta(z)}{\partial \overline{z}_\lambda}, \tag{5.3}$$

where $P_\lambda(z,w)$ is a holomorphic function in w. Since $\ell^*\omega^{(1)} = \omega^{(1)}$ for each $\ell \in L$, by (5.1) we obtain

$$D_\lambda(z,w) = D_\lambda\big(z, w + \zeta(z,\ell)\big) + \sum_{\beta=1}^{n} C_\beta(z) \frac{\partial \overline{\zeta(z,\ell)}_\beta}{\partial \overline{z}_\lambda} \tag{5.4}$$

for all $\ell \in L$. Hence, if we set

$$\widetilde{P}^0_\lambda(z) = \sum_{\beta=1}^{n} \left(C_\beta(z) \frac{\partial \overline{\zeta(z,\ell)}_\beta}{\partial \overline{z}_\lambda} + F_{\lambda,\beta}(z)\overline{\zeta(z,\ell)}_\beta \right)$$

for $1 \leq \lambda \leq N$, by (5.3) and (5.4) we have

$$P_\lambda(z,w) - P_\lambda\big(z, w + \zeta(z,\ell)\big) = \widetilde{P}^0_\lambda(z), \tag{5.5}$$

which is a function of z only. Thus $P_\lambda(z,w)$ must be of the form

$$P_\lambda(z,w) = P^0_\lambda(z) + \sum_{\beta=1}^{n} P^1_{\lambda,\beta}(z) w_\beta \tag{5.6}$$

for each λ. Using (5.5) and (5.6) for $w = \zeta(z,\ell)$, we see that the functions $P^1_{\lambda,\beta}(z)$ satisfy

$$\sum_{\beta=1}^{n} P^1_{\lambda,\beta}(z)\zeta(z,\ell)_\beta = -\sum_{\beta=1}^{n} \left(F_{\lambda,\beta}(z)\overline{\zeta(z,\ell)}_\beta + C_\beta(z) \frac{\partial \overline{\zeta(z,\ell)}_\beta}{\partial \overline{z}_\lambda} \right). \tag{5.7}$$

Since $\zeta(z,\ell) = \mu \cdot \tau(z) + \nu$ for $\ell = (\mu,\nu)$, using $\mu = 0$ and $\nu = (\nu_1, \dots, \nu_n)$ with $\nu_j \neq 0$ and $\nu_k = 0$ for $k \neq j$, from (5.7) we obtain

$$P^1_{\lambda,j}(z) = -F_{\lambda,j}(z) \tag{5.8}$$

for each $j \in \{1, \dots, n\}$. Thus (5.7) reduces to

$$\sum_{\beta=1}^{n} F_{\lambda,\beta}(z)\big(\mu \cdot \tau(z)\big)_\beta = \sum_{\beta=1}^{n} \left(F_{\lambda,\beta}(z)\big(\mu \cdot \overline{\tau(z)}\big)_\beta + C_\beta(z) \frac{\partial(\mu \cdot \overline{\tau(z)})_\beta}{\partial \overline{z}_\lambda} \right)$$

for $\ell = (\mu, 0)$. By considering μ with only one nonzero entry μ_j for each j we see that

$$F_\lambda(z)\tau(z) = F_\lambda(z)\overline{\tau(z)} + C(z)\frac{\partial \overline{\tau(z)}}{\partial \overline{z}_\lambda},$$

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where the $F_\lambda(z)$ and $C(z)$ are row vectors given by

$$F_\lambda(z) = \big(F_{\lambda,1}(z), \dots, F_{\lambda,n}(z)\big), \quad C(z) = \big(C_1(z), \dots, C_n(z)\big)$$

and the products are matrix products. Thus we have

$$2iF_\lambda(z)\big(\operatorname{Im}\tau(z)\big) = C(z)\frac{\partial\overline{\tau(z)}}{\partial\overline{z}_\lambda}. \tag{5.9}$$

By (5.2) and (5.3) we have

$$\begin{aligned}\frac{\partial F_\alpha(z)}{\partial\overline{z}_\lambda}\,{}^t\overline{w} + \frac{\partial P_\alpha(z,w)}{\partial\overline{z}_\lambda} &= \frac{\partial D_\alpha(z,w)}{\partial\overline{z}_\lambda} = \frac{\partial D_\lambda(z,w)}{\partial\overline{z}_\alpha}\\ &= \frac{\partial F_\lambda(z)}{\partial\overline{z}_\alpha}\,{}^t\overline{w} + \frac{\partial P_\lambda(z,w)}{\partial\overline{z}_\alpha},\end{aligned}$$

where $\overline{w}$ is regarded as a row vector and ${}^t\overline{w}$ is its tranpose. Hence it follows that

$$\frac{\partial F_\alpha(z)}{\partial\overline{z}_\lambda} = \frac{\partial F_\lambda(z)}{\partial\overline{z}_\alpha}, \quad \frac{\partial P_\alpha(z,w)}{\partial\overline{z}_\lambda} = \frac{\partial P_\lambda(z,w)}{\partial\overline{z}_\alpha}.$$

Thus, using this and (5.6), we obtain

$$\frac{\partial P^0_\alpha(z)}{\partial\overline{z}_\lambda} + \frac{\partial P^1_\alpha(z)}{\partial\overline{z}_\lambda}\,{}^t w = \frac{\partial P^0_\lambda(z)}{\partial\overline{z}_\alpha} + \frac{\partial P^1_\lambda(z)}{\partial\overline{z}_\alpha}\,{}^t w$$

with $P^1_\epsilon = (P^1_{\epsilon,1}, \dots, P^1_{\epsilon,n})$ for $\epsilon = \alpha, \lambda$, which implies that

$$\frac{\partial P^0_\alpha(z)}{\partial\overline{z}_\lambda} = \frac{\partial P^0_\lambda(z)}{\partial\overline{z}_\alpha}.$$

By (5.1), (5.3), (5.6) and (5.8) we see that

$$\begin{aligned}\omega^{(1)} &= \sum_{\alpha=1}^N \big(F_\alpha(z)\,{}^t\overline{w} + P^0_\alpha(z) + P^1_\alpha(z)\,{}^t w\big)\,d\overline{z}_\alpha + \sum_{\beta=1}^n C_\beta(z)d\overline{w}_\beta\\ &= \sum_{\alpha=1}^N \big(F_\alpha(z)({}^t\overline{w} - {}^t w) + P^0_\alpha(z)\big)\,d\overline{z}_\alpha + C(z)d\,{}^t\overline{w}.\end{aligned}$$

Hence, if we set

$$\omega^{(2)} = \sum_{\alpha=1}^N F_\alpha(z)({}^t\overline{w} - {}^t w)d\overline{z}_\alpha + C(z)d\,{}^t\overline{w}, \tag{5.10}$$

we obtain

$$\omega^{(1)} = \omega^{(2)} + \sum_{\alpha=1}^N P^0_\alpha(z)d\overline{z}_\alpha.$$

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Since $\sum_{\alpha=1}^{N} P_\alpha^0(z)d\overline{z}_\alpha$ is a closed 1-form on $\pi^{-1}(U)$, it is exact by Poincaré's lemma; hence it follows that $[\omega] = [\omega^{(1)}] = [\omega^{(2)}]$ in $H^{(0,1)}\big(\pi^{-1}(U)\big)$. By (5.3) and (5.9) we have

$$\begin{aligned}\frac{\partial C(z)}{\partial \overline{z}_\lambda}\big(\operatorname{Im}\tau(z)\big) &= F_\lambda(z)\big(\operatorname{Im}\tau(z)\big)\\ &= \frac{1}{2i}C(z)\frac{\partial\overline{\tau(z)}}{\partial\overline{z}_\lambda} = \frac{1}{2i}C(z)\frac{\partial}{\partial\overline{z}_\lambda}\big(\overline{\tau(z)}-\tau(z)\big)\\ &= -C(z)\frac{\partial}{\partial\overline{z}_\lambda}\big(\operatorname{Im}\tau(z)\big).\end{aligned}$$

Thus we obtain

$$\frac{\partial}{\partial\overline{z}_\lambda}\big(C(z)\operatorname{Im}\tau(z)\big) = 0,$$

and therefore we see that the function $C(z)\operatorname{Im}\tau(z)$ is holomorphic. Now we define the vector-valued holomorphic function ϕ on $\widetilde{U}\subset\mathcal{D}$ by

$$\phi(z) = \big(\phi_1(z),\dots,\phi_n(z)\big) = -2iC(z)\operatorname{Im}\tau(z). \tag{5.11}$$

Using this, (5.3) and (5.10), we obtain

(5.12)

$$\begin{aligned}\omega^{(2)} &= \sum_{\alpha=1}^{N}\frac{\partial C(z)}{\partial\overline{z}_\alpha}({}^t\overline{w}-{}^tw)d\overline{z}_\alpha + C(z)d\,{}^t\overline{w}\\ &= -\frac{1}{2i}\sum_{\alpha=1}^{N}\phi(z)\frac{\partial\big(\operatorname{Im}\tau(z)\big)^{-1}}{\partial\overline{z}_\alpha}({}^t\overline{w}-{}^tw)d\overline{z}_\alpha - \frac{1}{2i}\phi(z)\big(\operatorname{Im}\tau(z)\big)^{-1}d\,{}^t\overline{w}\\ &= -\frac{1}{2i}\phi(z)\Big(\Big(\overline{\partial}\big(\operatorname{Im}\tau(z)\big)^{-1}\Big)({}^t\overline{w}-{}^tw) + \big(\operatorname{Im}\tau(z)\big)^{-1}\overline{\partial}({}^t\overline{w}-{}^tw)\Big)\\ &= \phi(z)\overline{\partial}\Big(\big(\operatorname{Im}\tau(z)\big)^{-1}\operatorname{Im}{}^tw\Big).\end{aligned}$$

We shall now show that ϕ in (5.11) corresponds to a holomorphic section of the bundle $\mathcal{V}$ over U. By (4.2) it suffices to show that the function $\phi\colon\widetilde{U}\to\mathbb{C}^n$ in (5.12) satisfies

$$\phi(\gamma z) = \phi(z)\,j_{\rho,\tau}(\gamma,z)^{-1} \tag{5.13}$$

for all $\gamma\in\Gamma$ and $z\in\widetilde{U}$. If $\gamma\in\Gamma$, using (2.12), we see that the action of $(\gamma,0)\in\Gamma\ltimes_\psi L$ on $d\overline{w}$ is given by

$$\begin{aligned}(\gamma,0)^*d\overline{w} &= d\Big(\overline{\big(w+\xi(\gamma)(z)\big)\,j_{\rho,\tau}(\gamma,z)^{-1}}\Big)\\ &= d\overline{w}\cdot\overline{j_{\rho,\tau}(\gamma,z)}^{-1} + \overline{w}\cdot d\big(\overline{j_{\rho,\tau}(\gamma,z)}^{-1}\big) + d\big(\overline{\xi(\gamma)(z)}\cdot\overline{j_{\rho,\tau}(\gamma,z)}^{-1}\big)\\ &= d\overline{w}\cdot\overline{j_{\rho,\tau}(\gamma,z)}^{-1} + (\text{terms in } d\overline{z}_\alpha),\end{aligned}$$

where we used the fact that the functions $\xi(\gamma)(z)$ and $j_{\rho,\tau}(\gamma, z)$ are holomorphic in z. Since $\omega^{(2)}$ in (5.10) can be written in the form

$$\omega^{(2)} = C(z)d\,{}^t\overline{w} + (\text{terms in } d\overline{z}_\alpha)$$

and since $(\gamma, 0)^*$ takes terms in $d\overline{z}_\alpha$ to themselves, we see that

$$(\gamma, 0)^*\omega^{(2)} = C(\gamma z)\,{}^t\overline{j_{\rho,\tau}(\gamma, z)}^{-1}d\,{}^t\overline{w} + (\text{terms in } d\overline{z}_\alpha).$$

We now compare terms in $d\overline{w}_\beta$ in the relation $(\gamma, 0)^*\omega^{(2)} = \omega^{(2)}$ to obtain

$$C(\gamma z)\,{}^t\overline{j_{\rho,\tau}(\gamma, z)}^{-1} = C(z).$$

Using this, (5.11), and the fact that

$$\operatorname{Im}\tau(\gamma z) = {}^t\overline{j_{\rho,\tau}(\gamma, z)}^{-1} \cdot \operatorname{Im}\tau(z) \cdot j_{\rho,\tau}(\gamma, z)^{-1},$$

we obtain

$$\begin{aligned}\phi(\gamma z) &= -2iC(\gamma z)\operatorname{Im}\tau(\gamma z)\\ &= -2iC(z)\operatorname{Im}\tau(z)\,j_{\rho,\tau}(\gamma, z)^{-1}\\ &= \phi(z)\,j_{\rho,\tau}(\gamma, z)^{-1}.\end{aligned}$$

Hence it follows that ϕ can be regarded as a holomorphic section of $\mathcal{V}$ over U.

If $\widehat{\varphi}$ is a holomorphic section of $\mathcal{V}$ over U represented by a vector-valued holomorphic function $\varphi\colon \widetilde{U} \to \mathbb{C}^n$, we denote by $\omega_{\widehat{\varphi}}$ the $(0, 1)$-form on $\widetilde{U} \times \mathbb{C}^n$ given by

$$\omega_{\widehat{\varphi}} = \varphi(z)\overline{\partial}\Big(\big(\operatorname{Im}\tau(z)\big)^{-1}\operatorname{Im}{}^t w\Big).$$

Denoting by $\Gamma(U, \mathcal{V})$ the space of holomorphic sections of $\mathcal{V}$ over U and using (5.12), we see that the map

$$\Gamma(U, \mathcal{V}) \to H^{(0,1)}\big(\pi^{-1}(U)\big)$$

sending $\widehat{\varphi}$ to the cohomology class $[\omega_{\widehat{\varphi}}]$ of $\omega_{\widehat{\varphi}}$ is surjective. Thus we obtain the correspoding surjective map

$$\mathfrak{F}\colon \widetilde{\mathcal{V}} \to R^1\pi_*\mathcal{O}_{Y_{\psi,\xi}}$$

of sheaves on X. In order to show that $\mathfrak{F}$ is injective, given $x \in X$, we denote by T_x and $\mathcal{V}_x$ the fibers of the bundles $Y_{\psi,\xi}$ and $\mathcal{V}$, respectively, over x. Then $\mathcal{V}_x$ and $H^1(T_x, \mathcal{O}) = H^{(0,1)}(T_x)$ are the fibers of the sheaves $\widetilde{\mathcal{V}}$ and $R^1\pi_*\mathcal{O}_{Y_{\psi,\xi}}$, respectively. Thus, using the fact that both $\mathcal{V}_x$ and $H^1(T_x, \mathcal{O})$ are isomorphic to the n-dimensional space $\mathbb{C}^n$, we see that the surjectivity of $\mathfrak{F}$ implies its injectivity. Hence it follows that $\mathfrak{F}$ is an isomorphism of sheaves on X, and the proof of Theorem 4.2 is complete.

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