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QUARTIC ALGEBRAS

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ABSTRACT. In this paper we study the fixed point algebra of the automorphism of the rotation algebra $\mathcal{A}_\theta$, $\theta = p/q$ with p, q coprime positive integers, given by $u \longrightarrow v^{-1}$, $v \longrightarrow u$. We give a general characterization of the fixed point algebra, determine its K-theory and consider the related crossed-product algebra $\mathcal{A}_\theta \rtimes_\tau Z_4$.

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1.1. *Introduction.* In a series of papers [3], [4] the fixed point subalgebra of the rotation algebra, $\mathcal{A}_\theta$, under the action of the automorphism $\sigma: u \longrightarrow u^{-1}$, $v \longrightarrow v^{-1}$, referred to as the 'flip', was extensively studied. We wish to consider another fixed point subalgebra of the rotation algebra. This is defined using the automorphism $\tau: u \longrightarrow v^{-1}$, $v \longrightarrow u$ which has the property that $\tau^2 = \sigma$, hence we refer to this as the 'square root of the flip'. If we let $\mathcal{A}_\theta^\tau$ denote this subalgebra and $\mathcal{A}_\theta^\sigma$ that of the flip, by the property above, we have $\mathcal{A}_\theta^\tau \subset \mathcal{A}_\theta^\sigma$. In this paper we will study the case where θ is rational and in a separate paper in preparation that of general θ [6]. In other papers [7], [8] we will also study the fixed point subalgebras of the finite order automorphisms of $\mathcal{A}_\theta$ induced by SL$(2, Z)$, when θ is rational. The question whether $\mathcal{A}_\theta^\tau$ is an AF algebra still remains unresolved.

Our main result is a description of $\mathcal{A}_\theta^\tau$ in the case $\theta = p/q$. Here we will not give the precise result, see Theorem 3.2.5 for this, but just say the following. Let Ω_0, Ω_1 and Ω_2 be three distinct points on the sphere S^2 and associated to each point Ω_i is a set of projections $\{P_i^j\}$ in M_q, the $q \times q$ complex matrices. Then $\mathcal{A}_\theta^\tau$ is, up to isomorphism, the C^*-algebra of continuous functions from S^2 to M_q such that $f(\Omega_i)$ commutes with $\{P_i^j\}$.

For example, when $q = 1$ $\mathcal{A}_\theta^\tau$ is the algebra of continuous functions on S^2 and when $q = 2$ $\mathcal{A}_\theta^\tau$ is the algebra of continuous functions from S^2 to M_2 such that the functions take values in the subalgebra $M_1 \oplus M_1$ at the three points Ω_i.

Interestingly we made use of the classical result in analytic number theory,

$$\sum_{k=0}^{q-1} e^{2\pi i k^2/q} = \begin{cases} (1+i)\sqrt{q}, & q \equiv 0 \pmod 4, \\ \sqrt{q}, & q \equiv 1 \pmod 4, \\ 0, & q \equiv 2 \pmod 4, \\ i\sqrt{q}, & q \equiv 3 \pmod 4. \end{cases}$$

This was originally proved by Gauss [9], [10], and later by several people using different methods, see [2], [13], [1] and [14] for example. It is referred to as a Gaussian sum and together with some generalizations it played a key role in the proof.

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As with $\mathcal{A}_\theta^\sigma$, note that $\mathcal{A}_\theta^\tau$ is independent of p. A simple corollary of the main result is the calculation of the K-theory of $\mathcal{A}_\theta^\tau$.

COROLLARY 1.1.1. *Let $\theta = p/q$, where p, q are coprime positive integers. Then the K-theory of $\mathcal{A}_\theta^\tau$ is given by*

	$q=1$	$q=2$	$q=3$	$q=4$	$q>4$
$K_0(\mathcal{A}_\theta^\tau)$	Z^2	Z^5	Z^7	Z^8	Z^9
$K_1(\mathcal{A}_\theta^\tau)$	0	0	0	0	0

We also consider the related algebra $\mathcal{A}_\theta \rtimes_\tau Z_4$ and prove the following.

THEOREM 1.1.2. *Let $\theta = p/q$, where p, q coprime positive integers and let Ω_i $i = 0, 1, 2$ be any three distinct points of the 2-sphere S^2. Then the crossed product algebra $\mathcal{A}_\theta \rtimes_\tau Z_4$ is isomorphic to the following subalgebra of the C^*-algebra $C(S^2, M_{4q})$:*

$$\mathcal{A}_\theta \rtimes_\tau Z_4 = \{f \in C(S^2, M_{4q}) \mid f(\Omega_i) \text{ commutes with } P_i^j,\ i, j = 0, 1, 2\},$$

where P_i^j, $i, j = 0, 1, 2$, are self adjoint projections in M_{4q} with $P_0^1 = P_0^2 = 0$. The dimension of P_0^0 is $2q$, while the dimension of P_i^j, $i = 1, 2$, $j = 0, 1, 2$ is q.

The format of the paper is as follows. In Section 2 we give the basic notation and definitions we will use. In Section 3 we introduce the automorphisms and state the main result on the description of $\mathcal{A}_\theta^\tau$. In Section 4 we give the scheme of the proof with Section 5 giving the results from analytic number theory and detailing the calculations necessary. Finally in Section 6 we consider the crossed product algebra $\mathcal{A}_\theta \rtimes_\tau Z_4$.

We would like to take this opportunity to thank Professor George Elliott for suggesting we look at fixed point subalgebras and his helpful comments throughout this work. We would also like to thank the Mathematics Department at the University of Toronto where this work was carried out.

2. The rotation algebra.

2.1. *Introduction.* This section will give a characterization of the rotation algebra $\mathcal{A}_\theta$, θ rational, see for example [4] or [12], which we will use, together with some additional notation.

2.2. *Notation.* Assume that $\theta = p/q$, where p, q are coprime positive integers with $1 \leq p \leq q-1$. Let $\rho = e^{2\pi i\theta}$, $\omega = e^{\frac{2\pi i}{q}}$ and define the following $q \times q$ matrices:

$$U_0 = (\delta_i^j \rho^j)_{i,j=0,\cdots,q-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \rho & 0 & \cdots & 0 \\ 0 & 0 & \rho^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \rho^{q-1} \end{bmatrix}$$

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$$V_0 = (\delta_i^{j-1 \pmod q})_{i,j=0,\cdots,q-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\Gamma_0 = (\delta^j_{q-i \pmod q})_{i,j=0,\cdots,q-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$$W_0 = \frac{1}{\sqrt{q}}(\rho^{ij})_{i,j=0,\cdots,q-1} = \frac{1}{\sqrt{q}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \rho & \rho^2 & \cdots & \rho^{q-1} \\ 1 & \rho^2 & \rho^4 & \cdots & \rho^{2(q-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho^{q-1} & \rho^{2(q-1)} & \cdots & \rho^{(q-1)^2} \end{bmatrix}$$

Now U_0, V_0, W_0 are unitary and Γ_0 is a self-adjoint unitary. Also

$$e_{ij} = e_{ii}V_0^k \text{ where } k \equiv (j-i) \bmod q$$

and

$$e_{ii} = \frac{1}{q}\sum_{n=0}^{q-1} \rho^{-ni}U_0^n, \quad i = 0, \cdots, q-1.$$

So U_0 and V_0 generate M_q, the algebra of $q \times q$ matrices. Hence we can define four automorphisms of M_q, α_i, $i = 0, 1, 2$ and γ_0 by:

$$\alpha_0(U_0) = V_0^{-1}, \quad \alpha_0(V_0) = U_0,$$
$$\alpha_1(U_0) = U_0, \quad \alpha_1(V_0) = \omega V_0,$$
$$\alpha_2(U_0) = \omega U_0, \quad \alpha_2(V_0) = V_0,$$
$$\gamma_0(U_0) = U_0^{-1}, \quad \gamma_0(V_0) = V_0^{-1},$$

Then, if we use the convention that, for a unitary matrix U, $\operatorname{Ad} U$ denotes the automorphism of M_q given by $(\operatorname{Ad} U)(A) = U^*AU$, $A \in M_q$, we have

$$\alpha_i = \operatorname{Ad} W_i, \quad i = 0, 1, 2$$

and

$$\gamma_0 = \operatorname{Ad} \Gamma_0$$

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where

$$W_1 = U_0^{-p'} = (\delta_i^j \omega^j)_{i,j=0,\ldots,q-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \omega^{q-1} \end{bmatrix}$$

with $pp' \equiv -1 \pmod q$, $0 < p' < q$,

$$W_2 = V_0^{-pp''} = \begin{bmatrix} 0 & I_{p''} \\ I_{q-p''} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with $pp'' \equiv 1 \pmod q$, $0 < p'' < q$, and $I_t \in M_t$ is the $t \times t$ identity matrix.

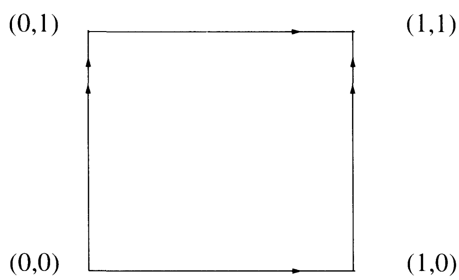
DEFINITION 2.2.1. The rotation algebra $\mathcal{A}_\theta$ is the universal C^*- algebra generated by two unitaries u and v satisfying $vu = \rho uv$ where $\rho = e^{2\pi i\theta}$ and $0 \leq \theta \leq 1$.

LEMMA 2.2.2. *The rotation algebra $\mathcal{A}_\theta$, $\theta = p/q$, where p, q are coprime positive integers, can be described as*

$$\mathcal{A}_\theta = \left\{ f \in C([0,1] \times [0,1], M_q) \;\middle|\; \begin{array}{l} f(x,1) = \alpha_1\big(f(x,0)\big),\ 0 \leq x \leq 1, \\ f(1,y) = \alpha_2\big(f(0,y)\big),\ 0 \leq y \leq 1 \end{array} \right\}$$

with pointwise multiplication and involution.

$[0,1] \times [0,1]$ is folded up according to the arrows:



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REMARK 2.2.3. Using this description the generators u and v of $\mathcal{A}_\theta$ correspond to the functions:

$$U(x,y) = \omega^x U_0,$$
$$V(x,y) = \omega^y V_0 \text{ for } (x,y) \in [0,1] \times [0,1],$$

where $\omega^t = e^{\frac{2\pi i t}{q}}$.

3. Fixed point subalgebras.

3.1. *Introduction.* In this section we will state the main theorem concerning the fixed point algebra of the square root of the flip.

3.2. *The square root of the flip.* The flip σ is the automorphism of $\mathcal{A}_\theta$ defined by $\sigma(u) = u^{-1}$, $\sigma(v) = v^{-1}$.

REMARK 3.2.1. In the description of $\mathcal{A}_\theta$ given in Lemma 2.2.2, σ corresponds to the automorphism

$$(\sigma f)(x,y) = \sigma_0\big(f(1-x, 1-y)\big),$$

where σ_0 is the automorphism of M_q determined by $\sigma_0 = \alpha_1\alpha_2\gamma_0$, that is, $\sigma_0(U_0) = \omega^{-1}U_0^{-1}$, $\sigma_0(V_0) = \omega^{-1}V_0^{-1}$.

The fixed point algebra of the flip, $\mathcal{A}_\theta^\sigma$, is given in the following theorem from [4].

THEOREM 3.2.2. *If p, q are coprime positive integers, then $\mathcal{A}_\theta^\sigma$ is a subalgebra of the C^*-algebra $C(S^2, M_q)$ of continuous functions from the 2-sphere S^2 into M_q. The subalgebra is determined up to isomorphism as follows: there are four distinct points ω_i, $i = 0,1,2,3$ in S^2 and to each point ω_i is associated a self-adjoint projection P_i in M_q. The dimensions of P_i are as follows*

$$\dim P_0 = \begin{cases} \frac{q-2}{2} \text{ if } & q \equiv 0 \pmod 2 \\ \frac{q-1}{2} \text{ if } & q \equiv 1 \pmod 2 \end{cases}$$

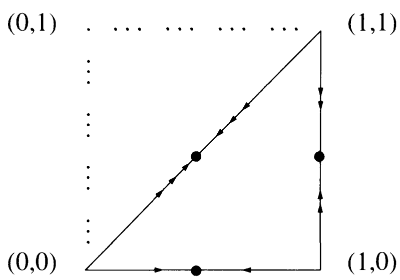
and

$$\dim P_i = \begin{cases} \frac{q}{2} \text{ if } & q \equiv 0 \pmod 2 \\ \frac{q-1}{2} \text{ if } & q \equiv 1 \pmod 2 \end{cases}, \qquad i = 1,2,3.$$

$\mathcal{A}_\theta^\sigma$ is roughly obtained by 'gluing' the triangle

$$T' = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1,\ y \le x\}$$

along its edges according to the arrows in the picture. (For further details see [4].)



The points ω_i, $i = 0, 1, 2, 3$, correspond respectively to the points $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(1, \frac{1}{2})$, $(\frac{1}{2}, 0)$ of T'.

Our goal is to obtain a description for the fixed point algebra of another automorphism of $\mathcal{A}_\theta$.

DEFINITION 3.2.3. The square root of the flip τ is the automorphism of $\mathcal{A}_\theta$ given by

$$\tau(u) = v^{-1}, \quad \tau(v) = u.$$

The reason for its name is the (obvious) property $\tau^2 = \sigma$.

LEMMA 3.2.4. *In the description of $\mathcal{A}_\theta$ given in Lemma 2.2.2 τ corresponds to the automorphism*

$$(\tau f)(x, y) = \tau_0\big(f(1 - y, x)\big),$$

where τ_0 is the automorphism of M_q given by

$$\tau_0 = \alpha_1\alpha_0, \text{ that is }, \tau_0(U_0) = \omega^{-1}V_0^{-1}, \ \tau_0(V_0) = U_0.$$

PROOF. We have

$$U(x, y) = \omega^x U_o = \omega^x \tau_0(V_0) = \tau_0(\omega^x V_0) = \tau_0\big(V(1 - y, x)\big),$$

$$\big(V(x, y)\big)^{-1} = \omega^{-y}V_0^{-1} = \omega^{1-y}\omega^{-1}V_0^{-1} = \omega^{1-y}\tau_0(U_0) = \tau_0\big(U(1 - y, x)\big).$$

∎

We can now state our main theorem, the proof of which will be presented in Section 4 and Section 5.

THEOREM 3.2.5. *Let $\theta = p/q$, with p, q coprime positive integers and let Ω_i, $i = 0, 1, 2$ be any three distinct points of the 2-sphere S^2. Then the fixed point algebra of the automorphism τ, $\mathcal{A}_\theta^\tau$, is isomorphic to the following subalgebra of the C^*-algebra $C(S^2, M_q)$*

$$\mathcal{A}_\theta^\tau = \{f \in C(S^2, M_q) \mid f(\Omega_i) \text{ commutes with } P_i^j, i, j = 0, 1, 2\},$$

where P_i^j, $i, j = 0, 1, 2$, are self-adjoint projections in M_q. The dimensions of P_i^j are given by the following table:

	Ω_0	Ω_1	Ω_2	
$q \equiv 0 \pmod 4$	$\frac{q}{2}$	$\frac{q}{4}$	$\frac{q}{4}$	P_i^0
	0	$\frac{q-4}{4}$	$\frac{q}{4}$	P_i^1
	0	$\frac{q}{4}$	$\frac{q}{4}$	P_i^2
$q \equiv 1 \pmod 4$	$\frac{q-1}{2}$	$\frac{q-1}{4}$	$\frac{q-1}{4}$	P_i^0
	0	$\frac{q-1}{4}$	$\frac{q-1}{4}$	P_i^1
	0	$\frac{q-1}{4}$	$\frac{q-1}{4}$	P_i^2
$q \equiv 2 \pmod 4$	$\frac{q}{2}$	$\frac{q+2}{4}$	$\frac{q-2}{4}$	P_i^0
	0	$\frac{q-2}{4}$	$\frac{q-2}{4}$	P_i^1
	0	$\frac{q-2}{4}$	$\frac{q+2}{4}$	P_i^2
$q \equiv 3 \pmod 4$	$\frac{q-1}{2}$	$\frac{q+1}{4}$	$\frac{q+1}{4}$	P_i^0
	0	$\frac{q-3}{4}$	$\frac{q-3}{4}$	P_i^1
	0	$\frac{q+1}{4}$	$\frac{q+1}{4}$	P_i^2

COROLLARY 3.2.6. *Let $\theta = p/q$, where p, q are coprime positive integers. Then the K-theory of $\mathcal{A}_\theta^\tau$ is given by*

	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q > 4$
$K_0(\mathcal{A}_\theta^\tau)$	Z^2	Z^5	Z^7	Z^8	Z^9
$K_1(\mathcal{A}_\theta^\tau)$	0	0	0	0	0

PROOF. From Theorem 3.2.5, $\mathcal{A}_\theta^\tau$ is a trivial C^*-bundle over the 2-sphere S^2. By applying the formula for $K_*(\mathcal{A}_\theta^\tau)$ given in [5] we obtain the result. ∎

4. **Proof of Theorem 3.2.5: scheme.**

4.1. *Introduction.* This section will begin the proof of the main theorem 3.2.5. After some preliminary results we will give the scheme of the proof leaving the necessary calculations to Section 5.

LEMMA 4.1.1. *Let α_i, $i = 0, 1, 2$ and γ_0 be the automorphisms of M_q defined in 2.2. Then*

$$\alpha_1\alpha_0 = \alpha_0\alpha_2^{-1},\ \alpha_0\alpha_1 = \alpha_2\alpha_0,\ \alpha_0\gamma_0 = \gamma_0\alpha_0,\ \alpha_0^2 = \gamma_0,$$
$$\alpha_1\alpha_2 = \alpha_2\alpha_1,\ \gamma_0\alpha_1 = \alpha_1^{-1}\gamma_0,\ \gamma_0\alpha_2 = \alpha_2^{-1}\gamma_0,\ \gamma_0^2 = 1.$$

PROOF. Straightforward calculation. ∎

As a consequence of Lemmas 3.2.4 and 4.1.1 we obtain that a fixed point of τ satisfies

$$f(x,x) = (\tau f)(x,x) = \tau_0\big(f(1-x,x)\big) = \alpha_1\alpha_0\big(f(1-x,x)\big),$$
$$f(x,0) = (\sigma f)(x,0) = \alpha_1\alpha_2\gamma_0(f(1-x,0) \text{ for } x \in [0,1].$$

The latter equality involves $\sigma = \tau^2$ and thus is already present in the computation of $\mathcal{A}_\theta^\sigma$ ([4]).

Hence it is possible to identify the fixed point algebra $\mathcal{A}_\theta^\tau$ as the algebra

$$\mathcal{A}_\theta^\tau = \left\{ f \in C(T, M_q) \;\middle|\; \begin{array}{l} f(x,x) = \alpha_1\alpha_0\big(f(1-x,x)\big), \\ f(x,0) = \alpha_1\alpha_2\gamma_0\big(f(1-x,0)\big) \end{array} \right\}$$

where T is the triangle

$$T = \{(x,y) \in [0,1] \times [0,1] \mid y \leq \min\{x, 1-x\}\}$$

The identification is by restricting $f \in \mathcal{A}_\theta^\tau$ from the square to the triangle.

REMARK 4.1.2. At the three points of T: $(0,0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0)$, the values of $f \in \mathcal{A}_\theta^\tau$ are restricted to a subalgebra of M_q described as follows:

1. At $(0,0) : \{A \in M_q \mid \alpha_0(A) = A\}$ since

$$\alpha_1\big(f(0,0)\big) = f(0,1) = (\tau f)(0,1) = \tau_0\big(f(0,0)\big) = \alpha_1\alpha_0\big(f(0,0)\big).$$

2. At $(\frac{1}{2}, \frac{1}{2}) : \{A \in M_q \mid \alpha_1\alpha_0(A) = A\}$ since

$$f\Big(\frac{1}{2}, \frac{1}{2}\Big) = (\tau f)\Big(\frac{1}{2}, \frac{1}{2}\Big) = \tau_0\Big(f\Big(\frac{1}{2}, \frac{1}{2}\Big)\Big) = \alpha_1\alpha_0\Big(f\Big(\frac{1}{2}, \frac{1}{2}\Big)\Big).$$

3. At $(\frac{1}{2}, 0) : \{A \in M_q \mid \alpha_2\gamma_0(A) = A\}$ since

$$\begin{aligned} f\Big(\frac{1}{2}, 0\Big) &= (\tau^2 f)\Big(\frac{1}{2}, 0\Big) = \tau\Big(\tau_0\Big(f\Big(1, \frac{1}{2}\Big)\Big)\Big) = \tau_0^2\Big(f\Big(\frac{1}{2}, 1\Big)\Big) \\ &= \tau_0^2\alpha_1 f\Big(\frac{1}{2}, 0\Big) = \alpha_1\alpha_2\alpha_0^2\alpha_1\Big(f\Big(\frac{1}{2}, 0\Big)\Big) = \alpha_2\gamma_0\Big(f\Big(\frac{1}{2}, 0\Big)\Big). \end{aligned}$$

Here we have used the definition of τ and Lemma 4.1.1.

REMARK 4.1.3. For a unitary matrix W, the algebra

$$\{A \in M_q \mid W^* A W = A\}$$

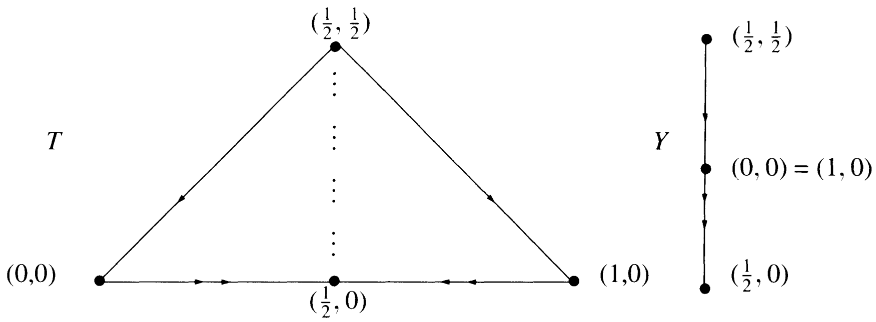
is isomorphic to

$$\{A \in M_q \mid D^* A D = A\},$$

where D is the matrix of the eigenvalues of W. The latter algebra is clearly determined by the dimensions of the eigenspaces of W.

REMARK 4.1.4. Since $\alpha_0^4 = 1$ and $\alpha_0 = \operatorname{Ad} W_0$, $W_0^4 = \mu I_q$, where $\mu \in T$. By direct computation $W_0^2 = \Gamma_0$ hence μ is 1. Similarly $(W_0 W_1)^4 = \rho^{p'p''} I_q$. Therefore the only eigenvalues of W_0 and $\widetilde{W_0 W_1} = \rho^{-\frac{p'p''}{4}} W_0 W_1$ are ± 1 and $\pm i$.

4.2. *Scheme of the proof.* $\mathcal{A}_\theta^T$ can also be described in the following way. Its spectrum can be obtained by folding the triangle T along the axis $x = \frac{1}{2}$ and joining the corresponding edges. $\mathcal{A}_\theta^T$ can then be identified with an algebra of functions from S^2 into M_q, continuous except on a tree Y on S^2, the one corresponding to the edges of T.



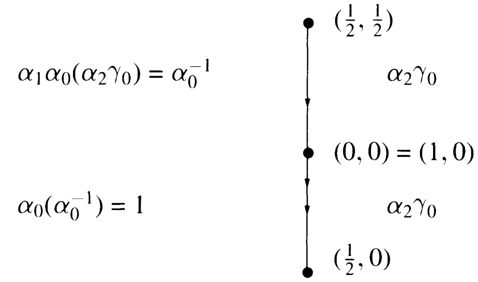
To prove Theorem 3.2.5 it is still necessary to show the following:

(1) The C^*-algebra $\mathcal{A}_\theta^T$ is isomorphic to the C^*-algebra

$$\mathcal{B} = \left\{ f \in C(S^2, M_q) \;\middle|\; \begin{array}{l} \alpha_0\big(f(0,0)\big) = f(0,0),\ \alpha_1\alpha_0\big(f(\frac{1}{2},\frac{1}{2})\big) = f(\frac{1}{2},\frac{1}{2}), \\ \alpha_2\gamma_0\big(f(\frac{1}{2},0)\big) = f(\frac{1}{2},0). \end{array} \right\}$$

(2) The dimensions of the projections at the three points $(0,0)$, $(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0)$ are as specified.

To do (1) we will construct a map $\eta\colon S^2 \longrightarrow U_q/T \cong \mathrm{Aut}(M_q)$ which is continuous in $S^2 - Y$ and has the following limits on the edges of the tree Y

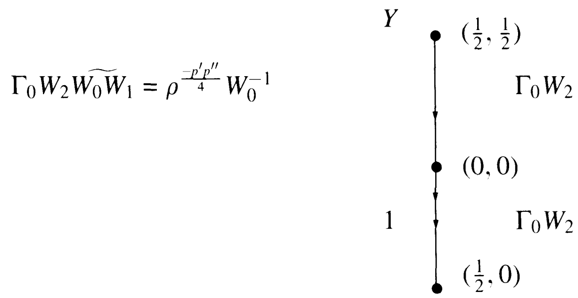


Moreover, to insure the triviality of the bundle over S^2, we need the sum of the winding numbers $\sum_Y$ of $\eta\colon s \longmapsto U(s)$ around the three singular points (*i.e.*, around Y) to be an integer multiple of q. The isomorphism $\tilde{\eta}$ from $\mathcal{B}$ to $\mathcal{A}_\theta^T$ is then given by

$$(\tilde{\eta}f)(s) = \tilde{\eta}(s)f(s), \quad f \in \mathcal{B}, s \in S^2 - Y,$$

where $\tilde{\eta}(s) = \mathrm{Ad}\, U(s)$.

Along the edges of the tree Y, the values of $\eta: s \longmapsto U(s)$ are given by



To go (counterclockwise) around the circle sections we use the following paths

1. $(\frac{1}{2}, 0)$: let P_0 be the spectral projection of the self-adjoint unitary $\Gamma_0 W_2$ corresponding to the eigenvalue -1 . Define $U(t) = (1 - P_0) + e^{2\pi i(t/2 + tn_0)} P_0$, with $t \in [0, 1]$, $n_0 \in Z$ (*cf.* [4], (3.40), (3.41).) Note that $U(0) = 1$ and $U(1) = 1 - 2P_0 = \Gamma_0 W_2$ as required.

2. $(\frac{1}{2}, \frac{1}{2})$: let Q_0, Q_1, Q_2 be the projections onto the -1, $+i$, $-i$ (resp.) eigenspaces of $\widetilde{W_0 W_1} = \rho^{\frac{-p'p''}{4}} W_0 W_1$ (*cf.* 4.1.3). (We introduced the phase factor $\rho^{\frac{-p'p''}{4}}$ to normalize $W_0 W_1$.)

Then $U(t) = \Gamma_0 W_2 W(t)$, where

$$W(t) = 1 - (Q_0 + Q_1 + Q_2) + e^{2\pi i(t/2 + m_0 t)} Q_0 + e^{2\pi i(t/4 + m_1 t)} Q_1 + e^{2\pi i(3t/4 + m_2 t)} Q_2,$$

with $t \in [0, 1]$, $m_i \in Z$, $i = 0, 1, 2$. Note that $W(0) = 1$ and $W(1) = \widetilde{W_0 W_1}$.

3. $(0, 0)$: let R_0, R_1, R_2 be the projections onto the -1, $+i$, $-i$ (resp.) eigenvalues of W_0^{-1}. Then

$$U(t) = 1 - (R_0 + R_1 + R_2) + e^{2\pi i(\frac{1-t}{2} + \ell_0(1-t))} R_0 + e^{2\pi i(\frac{1-t}{4} + \ell_1(1-t))} R_1 \\ + e^{2\pi i(\frac{3}{4}(1-t) + \ell_2(1-t))} R_2$$

with $t \in [0, 1]$, $\ell_i \in Z$, $i = 0, 1, 2$. Note that $U(0) = W_0^{-1}$, $U(1) = 1$.

To compute Σ_Y (*i.e.*, the winding number of η around Y) we first compute the winding number $\Sigma_{(\frac{1}{2}, 0)}$ of the path $U(t) = (1 - P_0) + e^{2\pi i(t/2 + tn_0)} P_0$, $t \in [0, 1]$, around $(\frac{1}{2}, 0)$:

$$\Sigma_{(\frac{1}{2}, 0)} = \frac{1}{2\pi i} \Big[\text{Trace}\Big(\ln U(1) - \ln U(0) \Big) \Big] = \frac{1}{2\pi i} \text{Trace} \Big[2\pi i \Big(\frac{1}{2} + n_0 \Big) P_0 - 0 \Big] \\ = \Big(\frac{1}{2} + n_0 \Big) \dim P_0,$$

since

$$\ln U(t) = \ln \Big[1 + \Big(-1 + e^{2\pi i(\frac{1}{2} + n_0)t} \Big) P_0 \Big] = 2\pi i (\frac{1}{2} + n_0) t P_0.$$

Analogously one computes the winding numbers around $(\frac{1}{2}, \frac{1}{2})$ and $(0, 0)$. Then we have

$$\Sigma_Y = \Big(\frac{1}{2} + n_0 \Big) \dim P_0 + \Big[\Big(\frac{1}{2} + m_0 \Big) \dim Q_0 + \Big(\frac{1}{4} + m_1 \Big) \dim Q_1 + \Big(\frac{3}{4} + m_2 \Big) \dim Q_2 \Big] \\ - \Big[\Big(\frac{1}{2} + \ell_0 \Big) \dim R_0 + \Big(\frac{1}{4} + \ell_1 \Big) \dim R_1 + \Big(\frac{3}{4} + \ell_2 \Big) \dim R_2 \Big] + \Big(\frac{pp'p''}{4q} + k_0 \Big) q.$$

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The last term in Σ_Y came from unwinding the phase factor $\rho^{\frac{-p'p''}{4}}$. In fact, the path

$$c(t) = e^{\frac{2\pi i}{4q}pp'p''(t-1)+2\pi i t k_0},\ t \in [0,1],\ k_0 \in Z,\ c(0) = \rho^{\frac{-p'p''}{4}},\ c(1) = 1,$$

has winding number

$$\frac{\text{Trace}}{2\pi i}\,[\ln c(1) - \ln c(0)] = \left(\frac{pp'p''}{4q} + k_0\right) q.$$

To finish the proof of Theorem 3.2.5 we need to compute the dimension of the spectral projections of W_0 and $\widetilde{W_0 W_1}$ (as the ones of $\Gamma_0 W_2$ are given in [4]) and to show that Σ_Y is an integer multiple of q for some choice of the integer parameters n_0, k_0, m_i, ℓ_i, $i = 0, 1, 2$.

5. Proof of Theorem 3.2.5: calculations.

5.1. *Introduction.* Here we will present the necessary calculations to complete the proof of theorem 3.2.5. We will consider 4 cases depending on the congruence of $q \bmod 4$, but first detail some results from analytic number theory which we will use.

THEOREM 5.1.1 ([11]). *Let* $G(p,q) = \sum_{m=0}^{q-1} e^{2\pi i p m^2/q}$ *where* p, q *are integers. If* $(p,q) = 1$, *then:*

1. $G(p,1) = 1$, $G(p,2) = 0$ and

$$G(p,2^a) = \begin{cases} (1+i^p)2^{a/2} & \text{for } a \text{ even}, a > 0, \\ e^{\pi i p/4} 2^{(a+1)/2} & \text{for } a \text{ odd}, a > 1. \end{cases}$$

2. For q odd $G(p,q) = (p|q)G(1,q)$, where $(p|q)$ is the Jacobi symbol.
3. For all q, $G(1,q) = \mathcal{E}_q\sqrt{q}$, where

$$\mathcal{E}_q = (1+i)\cdot\frac{(1+i^{-q})}{2} = \begin{cases} 1+i & q \equiv 0 \pmod 4 \\ 1 & q \equiv 1 \pmod 4 \\ 0 & q \equiv 2 \pmod 4 \\ i & q \equiv 3 \pmod 4 \end{cases}$$

4. $(k_1,k_2) = 1 \Rightarrow G(p,k_1k_2) = G(pk_1,k_2)G(pk_2,k_1)$.

COROLLARY 5.1.2. *If* p, q *are positive integers and* $(p,q) = 1$ *then*

i) $G(p,q) = (p|q)G(1,q)$ *if* q *is odd,*

ii) $G(p,q) = 0$ *if* $q \equiv 2 \pmod 4$,

iii)

$$G(p,q) = (p|r)G(1,q)\begin{cases} (1+i^{pr})/(1+i^r) & \textit{if } a \textit{ even}, \\ e^{\pi i(p-1)r/4} & \textit{if } a \textit{ odd}, \end{cases}$$

if $q \equiv 0 \pmod 4$, *where* $q = 2^a r$ *with* r *odd.*

LEMMA 5.1.3. *Let* W_0 *be the matrix defined in 2.2 then,*

$$\text{Trace}(W_0) = \frac{1}{\sqrt{q}}G(p,q).$$

PROOF. $W_0 = \frac{1}{\sqrt{q}}(\rho^{ij})_{i,j}$. So

$$\mathrm{Trace}(W_0) = \frac{1}{\sqrt{q}} \sum_{k=0}^{q-1} \rho^{k^2} = \frac{1}{\sqrt{q}} \sum_{k=0}^{q-1} e^{2\pi i p k^2/q} = \frac{1}{\sqrt{q}} G(p,q).$$

■

LEMMA 5.1.4. *Let $\widetilde{W_0 W_1}$ be the matrix defined in Remark 4.1.4 then,*

$$|\mathrm{Trace}(\widetilde{W_0 W_1})|^2 = \begin{cases} 0, & \text{if } q \equiv 0 \pmod 4, \\ 1, & \text{if } q \equiv 1 \pmod 4, \\ 2, & \text{if } q \equiv 2 \pmod 4, \\ 1, & \text{if } q \equiv 3 \pmod 4, \end{cases}$$

PROOF.

$$\begin{aligned} \mathrm{Trace}(\widetilde{W_0 W_1}) &= \mathrm{Trace}(\rho^{\frac{-p'p''}{4}} W_0 W_1) \\ &= \rho^{\frac{-p'p''}{4}} \mathrm{Trace}(W_0 W_1) \\ &= \frac{\rho^{\frac{-p'p''}{4}}}{\sqrt{q}} \sum_{k=0}^{q-1} \rho^{k^2} \omega^k \\ &= \frac{\rho^{\frac{-p'p''}{4}}}{\sqrt{q}} \sum_{k=0}^{q-1} e^{2\pi i p k^2/q + 2\pi i k/q}. \end{aligned}$$

Hence

$$\begin{aligned} |\mathrm{Trace}(\widetilde{W_0 W_1})|^2 &= \frac{1}{q} \sum_{k=0}^{q-1} e^{2\pi i p k^2/q + 2\pi i k/q} \cdot \sum_{\ell=0}^{q-1} e^{-2\pi i p \ell^2/q - 2\pi i \ell/q} \\ &= \frac{1}{q} \sum_{d=0}^{q-1} \sum_{\ell=0}^{q-1} e^{2\pi i d[p(d+2\ell)+1]/q}, \text{ where } d = k - \ell \\ &= \frac{1}{q} \sum_{d=0}^{q-1} e^{2\pi i p d^2/q + 2\pi i d/q} \sum_{\ell=0}^{q-1} e^{4\pi i d p \ell/q}. \end{aligned}$$

Now the inner sum is zero unless $q|2dp$, or $q|2d$ as $(p,q) = 1$, in which case it is q.

If q is odd then $d = 0$ is the only non-zero contribution and the double sum is equal to q.

If q is even then $d = 0, q/2$ give non-zero contributions and the double sum is equal to $q(1 - i^{pq})$.

These facts give the result as stated. ■

LEMMA 5.1.5.

$$\mathrm{Trace}(\widetilde{W_0 W_1})^2 = \begin{cases} 0, & \text{if } q \text{ even}, \\ 1, & \text{if } q \text{ odd}. \end{cases}$$

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PROOF. $(\widetilde{W_0W_1})^2 = \rho^{\frac{-p'p''}{2}}\Gamma_0 W_2 W_1$ and the trace of this matrix is given in [4] ($W_{\frac{1}{2},\frac{1}{2}}$ in the notation of that paper). ∎

5.2. $q \equiv 0 \pmod 4$.

REMARK 5.2.1. For the calculations that follow it is convenient to write $G(p,q) = i^A\sqrt{q}(1+i)$, where $A \in \{0,1,2,3\}$ depending on p and q, rather than the more complicated expression given in Corollary 5.1.2.

PROPOSITION 5.2.2. *The dimensions of the four eigenspaces of W_0 are given in the following table.*

	A = 0	A = 1	A = 2	A = 3
eigenvalue +1	$\frac{q+4}{4}$	$\frac{q}{4}$	$\frac{q}{4}$	$\frac{q+4}{4}$
eigenvalue −1	$\frac{q}{4}$	$\frac{q+4}{4}$	$\frac{q+4}{4}$	$\frac{q}{4}$
eigenvalue $+i$	$\frac{q}{4}$	$\frac{q}{4}$	$\frac{q-4}{4}$	$\frac{q-4}{4}$
eigenvalue $-i$	$\frac{q-4}{4}$	$\frac{q-4}{4}$	$\frac{q}{4}$	$\frac{q}{4}$

PROOF. Let

$$m = \dim +1 \text{ eigenspace of } W_0$$
$$n = \dim -1 \text{ eigenspace of } W_0$$
$$u = \dim +i \text{ eigenspace of } W_0$$
$$v = \dim -i \text{ eigenspace of } W_0$$

Since q is even Trace$(\Gamma_0) = 2$. Now Γ_0 is a self adjoint unitary so the dimension of the +1 eigenspace of Γ_0 is $\frac{q+2}{2}$ and the dimension of the −1 eigenspace of Γ_0 is $\frac{q-2}{2}$. However $W_0^2 = \Gamma_0$, therefore

$$(*) \qquad m + n = \frac{q+2}{2} \text{ and } u + v = \frac{q-2}{2}.$$

We will now demonstrate the result for $A = 0$. The other cases may be shown in a similar fashion. If $A = 0$ then Trace$(W_0) = \frac{1}{\sqrt{q}}G(p,q) = 1 + i$. But Trace$(W_0) = (m-n) + i(u-v)$,

$$(**) \qquad \text{so } m - n = 1 \text{ and } u - v = 1.$$

Solving (*) and (**) gives the result. ∎

PROPOSITION 5.2.3. *The dimensions of the four eigenspaces of $\widetilde{W_0W_1}$, are all equal to $q/4$.*

PROOF. Let

$$m = \dim +1 \text{ eigenspace of } \widetilde{W_0W_1}$$
$$n = \dim -1 \text{ eigenspace of } \widetilde{W_0W_1}$$
$$u = \dim +i \text{ eigenspace of } \widetilde{W_0W_1}$$
$$v = \dim -i \text{ eigenspace of } \widetilde{W_0W_1}$$

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Then from Lemma 5.1.4

$$(m-n)^2+(u-v)^2=0,$$

that is, $m=n$ and $u=v$. From Lemma 5.1.5,

$$(m+n)-(u+v)=0$$

and we also have

$$(m+n)+(u+v)=q.$$

Solving these equations gives the result. ∎

REMARK 5.2.4. This proves the dimensions of the projection at the three distinct points are as given in Theorem 3.2.5.

It remains to be shown.

PROPOSITION 5.2.5. *Σ_Y can be made an integer multiple of q.*

PROOF. Using Proposition 5.2.2 and 5.2.3 together with $\dim(P_0)=\frac{q}{2}$ [4], we have

$$\begin{aligned}\Sigma_Y=&\Big(\frac{1}{2}+n_0\Big)\frac{q}{2}+\Big(\frac{1}{2}+m_0\Big)\frac{q}{4}+\Big(\frac{1}{4}+m_1\Big)\frac{q}{4}+\Big(\frac{3}{4}+m_2\Big)\frac{q}{4}\\&-\Bigg[\Big(\frac{1}{2}+\ell_0\Big)\begin{Bmatrix}q/4, & A=0,3\\ \frac{q+4}{4}, & A=1,2\end{Bmatrix}+\Big(\frac{1}{4}+\ell_1\Big)\begin{Bmatrix}q/4, & A=2,3\\ \frac{q-4}{4}, & A=0,1\end{Bmatrix}\\&+\Big(\frac{3}{4}+\ell_2\Big)\begin{Bmatrix}q/4, & A=0,1\\ \frac{q-4}{4}, & A=2,3\end{Bmatrix}\Bigg]+\Big(\frac{pp'p''}{4q}+k_0\Big)q\end{aligned}$$

Putting $m_i=\ell_i \quad i=0,1,2$ we have

$$\begin{aligned}\Sigma_Y=&\Big(\frac{1}{2}+n_0\Big)q/2-\begin{Bmatrix}0, & A=0,3\\ (\frac{1}{2}+m_0), & A=1,2\end{Bmatrix}\\&+\begin{Bmatrix}0, & A=2,3\\ (\frac{1}{4}+m_1), & A=0,1\end{Bmatrix}+\begin{Bmatrix}0, & A=0,1\\ (\frac{3}{4}+m_2), & A=2,3\end{Bmatrix}+\frac{pp'p''}{4}+k_0q.\end{aligned}$$

We now need to observe the following facts.

(i) If $p\equiv 1 \pmod 4$ then $pp'p''\equiv 3 \pmod 4$.

(ii) If $p\equiv 3 \pmod 4$ then $pp'p''\equiv 1 \pmod 4$.

These follow easily from the definitions of p' and p'', given that p, p' and p'' are all odd in this case. We also note that

$$\begin{aligned}A=0,2&\Rightarrow p\equiv 1 \pmod 4,\\ A=1,3&\Rightarrow p\equiv 3 \pmod 4.\end{aligned}$$

This follows by studying the formula for $G(p,q)$ given in Corollary 5.1.2 together with that in Remark 5.2.1 in more detail.

We will now demonstrate the case $A = 0$. The others follow similarly. So $\Sigma_Y = (\frac{1}{2}+n_0)q/2+(\frac{1}{4}+m_1)+\frac{pp'p''}{4}+k_0q$. However, by the above, $pp'p'' \equiv 3 \pmod 4$ so

$$\frac{pp'p''}{4} = \frac{3}{4}+t, \text{ where } t \text{ is an integer.}$$

Therefore

$$\begin{aligned}\Sigma_Y &= \Big(\frac{1}{2}+n_0\Big)q/2+\Big(\frac{1}{4}+m_1\Big)+\frac{3}{4}+t+k_0q\\ &= \frac{q}{4}+n_0\frac{q}{2}+1+m_1+t+k_0q.\end{aligned}$$

Choose $m_1 = -(\frac{q}{4}+1+t), n_0 = 0$ and we are finished. ∎

5.3. $q \equiv 1 \pmod 4$.

PROPOSITION 5.3.1. *The dimensions of the four eigenspaces of W_0 are as follows:*

$$\begin{aligned}\dim +1 \text{ eigenspace } &= \frac{q+1+2(p|q)}{4}\\ \dim -1 \text{ eigenspace } &= \frac{q+1-2(p|q)}{4}\\ \dim \pm i \text{ eigenspace } &= \frac{q-1}{4},\end{aligned}$$

PROOF. Trace $(\Gamma_0) = 1$, $\text{Trace}(W_0) = \frac{1}{\sqrt{q}}G(p,q) = \frac{1}{\sqrt{q}}(p|q)G(1,q) = (p|q)$. Now argue as in Proposition 5.2.2. ∎

COROLLARY 5.3.2.

$$\text{Det}(W_0) = (p|q)(-i)^{\frac{(q-1)}{2}}.$$

PROOF.

$$\text{Det}(W_0) = (-1)^{\frac{q+1-2(p|q)}{4}} = (-1)^{\frac{q-1}{4}}(-1)^{\frac{2(1-(p|q))}{4}} = (-i)^{\frac{(q-1)}{2}}(p|q).$$

∎

PROPOSITION 5.3.3. *The dimensions of the four eigenspaces of $\widetilde{W_0W_1}$ are as follows:*

$$\begin{aligned}\dim +1 \text{ eigenspace } &= \frac{q+1+2V}{4}\\ \dim -1 \text{ eigenspace } &= \frac{q+1-2V}{4}\\ \dim \pm i \text{ eigenspace } &= \frac{q-1}{4},\end{aligned}$$

where $V = (p|q)(-i)^{pp'p''}$. Note that since q is odd at least one of p, p' and p'' is even so $V = \pm 1$.

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PROOF. This is an adaptation of Schur's proof of the formula for $G(1,q)$, (see [13], p. 207 for example).

We know

$$|\operatorname{Trace}(\widetilde{W_0 W_1})|^2 = 1, \text{ by Lemma 5.1.4 and}$$
$$\operatorname{Trace}(\widetilde{W_0 W_1})^2 = 1, \text{ by Lemma 5.1.5.}$$

So if m, n, u, v are defined as the dimensions of the $+1, -1, i, -i$ eigenspaces respectively we have the following

$$(m-n)^2 + (u-v)^2 = 1.$$

Hence either $\left\{\begin{matrix} m-n=\pm 1 \\ u-v=0 \end{matrix}\right\}$ or $\left\{\begin{matrix} m-n=0 \\ u-v=\pm 1 \end{matrix}\right\}$. In either case

$$\operatorname{Trace}(\widetilde{W_0 W_1}) = V\eta \text{ with } V = \pm 1 \text{ and } \eta = 1 \text{ or } i.$$

We therefore have the following equations

$$(1) \qquad m+n+u+v = q$$
$$(2) \qquad m-n+i(u-v) = V\eta$$
$$(3) \qquad m-n-i(u-v) = V\eta^{-1}$$
$$(4) \qquad m+n-(u+v) = 1$$

$[(1)+(4)] - [(2)+(3)]$ gives

$$4n = q+1-V(\eta+\eta^{-1}).$$

Therefore $\eta = 1$, since n is an integer, and

$$m = \frac{q+1+2V}{4}$$
$$n = \frac{q+1-2V}{4}$$
$$u = v = \frac{q-1}{4}$$

It remains to show that $V = (p|q)i^{pp'p''}$. To do this we will examine $\operatorname{Det}(\widetilde{W_0 W_1})$.

Now $\operatorname{Det}(\widetilde{W_0 W_1}) = (-1)^{\frac{q+1-2V}{4}} = (-1)^{\frac{q-1}{4}}(-1)^{\frac{2(1-V)}{4}} = (-i)^{\frac{q-1}{2}}V$.

But

$$\begin{aligned} \operatorname{Det}(\widetilde{W_0 W_1}) &= \operatorname{Det}\left(\rho^{\frac{-p'p''}{4}} W_0 W_1\right) \\ &= \rho^{\frac{-p'p''}{4}q} \operatorname{Det}(W_0)\operatorname{Det}(W_1) \\ &= (-i)^{pp'p''}(p|q)(-i)^{\frac{q-1}{2}}. \end{aligned}$$

Hence $V = (p|q)(-i)^{pp'p''}$. ∎

It remains to show

PROPOSITION 5.3.4. *Σ_Y can be made an integer multiple of q.*

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PROOF. Using Proposition 5.3.1 and 5.3.3 together with $\dim(P_0) = \frac{q-1}{2}$ [4], we have

$$\Sigma_Y = \Big(\frac{1}{2}+n_0\Big)(q-1)/2 + \Big(\frac{1}{2}+m_0\Big)\Big(\frac{q+1-2V}{4}\Big) + \Big(\frac{1}{4}+m_1\Big)\frac{(q-1)}{4} + \Big(\frac{3}{4}+m_2\Big)\frac{(q-1)}{4} - \Big[\Big(\frac{1}{2}+\ell_0\Big)\Big(\frac{q+1-2(p|q)}{4}\Big) + \Big(\frac{1}{4}+\ell_1\Big)\frac{(q-1)}{4} + \Big(\frac{3}{4}+\ell_2\Big)\frac{(q-1)}{4}\Big] + \Big(\frac{pp'p''}{4q}+k_0\Big)q$$

Putting $m_i = \ell_i$ $i = 0, 1, 2$ we have

$$\Sigma_Y = \Big(\frac{1}{2}+n_0\Big)\frac{(q-1)}{2} - \frac{\big(V-(p|q)\big)}{2}\Big(\frac{1}{2}+m_0\Big) + \frac{pp'p''}{4} + k_0q.$$

CASE 1: $pp'p'' \equiv 0 \pmod 4$. So $V = (p|q)$ and $\frac{pp'p''}{4} = \frac{(q-1)}{4} + t$, where t is an integer. Therefore

$$\Sigma_Y = \Big(\frac{1}{2}+n_0\Big)\frac{(q-1)}{2} + \frac{(q-1)}{4} + t + k_0q$$
$$= \frac{(q-1)}{2}(1+n_0) + k_0q + t.$$

However, $(q-1)/2$ and q are coprime thus Σ_Y can take any integral value.

CASE 2: $pp'p'' \equiv 2 \pmod 4$. So $V = -(p|q)$ and $\frac{pp'p''}{4} = \frac{(q+1)}{4} + t$, where t is an integer. Therefore

$$\Sigma_Y = \Big(\frac{1}{2}+n_0\Big)\frac{(q-1)}{2} + (p|q)\Big(\frac{1}{2}+m_0\Big) + \frac{(q+1)}{4} + t + k_0q$$
$$= \frac{q+(p|q)}{2} + n_0\frac{(q-1)}{2} + m_0(p|q) + t + k_0q,$$

and $(q-1)/2$ and q are coprime so we are finished.

Alternatively: let $m_0 = \frac{-(\frac{(q+(p|q))}{2}+1)}{(p|q)}$, $n_0 = 0$ then $\Sigma_Y = k_0q$. ∎

5.4. $q \equiv 2 \pmod 4$.

PROPOSITION 5.4.1. *The dimensions of the four eigenspaces of W_0 are as follows:*

$$\dim \pm 1 \text{ eigenspace} = \frac{q+2}{4}$$
$$\dim \pm i \text{ eigenspace} = \frac{q-2}{4}$$

PROOF. Trace $(\Gamma_0) = 2$, Trace $(W_0) = 0$. Now argue as in Proposition 5.2.2. ∎

COROLLARY 5.4.2.

$$\mathrm{Det}(W_0) = (-1)^{\frac{q+2}{4}}$$

PROPOSITION 5.4.3. *The trace of* $\widetilde{W_0 W_1}$ *is determined up to sign by the congruence of* $pp'p''$ mod 4.

If $pp'p'' \equiv 1 \pmod 4$ then $\mathrm{Trace}(\widetilde{W_0 W_1}) = \pm(1 - i)$.
If $pp'p'' \equiv 3 \pmod 4$ then $\mathrm{Trace}(\widetilde{W_0 W_1}) = \pm(1 + i)$.

Note that p, p' and p'' are all odd as $q \equiv 2 \pmod 4$ so these are the only possibilities.

PROOF. $|\mathrm{Trace}(\widetilde{W_0 W_1})|^2 = 2$, by Lemma 5.1.4, so $(m - n)^2 + (u - v)^2 = 2$ where m, n, u, v are defined as usual.

Therefore $m - n = \pm 1$ and $u - v = \pm 1$. Let $\mathrm{Trace}(\widetilde{W_0 W_1}) = U + iV$ where $U = \pm 1$, $V = \pm 1$. Then

$$m + n + u + v = q,$$
$$m + n - u - v = 0, \text{ since } \mathrm{Trace}(\widetilde{W_0 W_1})^2 = 0,$$
$$m - n + i(u - v) = U + iV.$$

Therefore $m = \frac{q+2U}{4}$, $n = \frac{q-2U}{4}$, $u = \frac{q+2V}{4}$, $v = \frac{q-2V}{4}$. Hence

$$\begin{aligned}\mathrm{Det}(\widetilde{W_0 W_1}) &= (-1)^{\frac{q-2U}{4}} (i)^{\frac{q+2V}{4}} (-i)^{\frac{q-2V}{4}} \\ &= (-1)^{\frac{q-(U+V)}{2}} i^{\frac{q}{2}}\end{aligned}$$

However

$$\begin{aligned}\mathrm{Det}(\widetilde{W_0 W_1}) &= \mathrm{Det}\big(\rho^{\frac{-p'p''}{4}} W_0 W_1\big) \\ &= \rho^{\frac{-p'p''}{4} q} \mathrm{Det}(W_0)\,\mathrm{Det}(W_1) \\ &= (-i)^{pp'p''} (-1)^{\frac{q+2}{4}} . (-1)^{q-1}\end{aligned}$$

i) $pp'p'' \equiv 1 \pmod 4$:

$$q \equiv 2 \pmod 8, -i \cdot (-1)^{\frac{U+V}{2}} = -i$$
$$q \equiv 6 \pmod 8, +i \cdot (-1)^{\frac{U+V}{2}} = +i$$

So in either case, that is $q \equiv 2 \pmod 4$, $\frac{U+V}{2} = 2n$, where n is an integer.
But $U = \pm 1$, $V = \pm 1$ so we must have $n = 0$ and $U = -V$. Therefore we have two possibilities $\mathrm{Trace}(\widetilde{W_0 W_1}) = \pm(1 - i)$.

ii) $pp'p'' \equiv 3 \pmod 4$:

$$q \equiv 2 \pmod 8, -i \cdot (-1)^{\frac{U+V}{2}} = +i$$
$$q \equiv 6 \pmod 8, i \cdot (-1)^{\frac{U+V}{2}} = -i$$

So in either case, $\frac{U+V}{2} = 2n + 1$, where n is an integer.

But $U = \pm 1$, $V = \pm 1$, hence $n = 0$ or -1 and $U = V = 1$ or $U = V = -1$. Therefore have two possibilities $\mathrm{Trace}(\widetilde{W_0 W_1}) = \pm(1 + i)$.

COROLLARY 5.4.4. *The dimensions of the four eigenspaces of* $\widetilde{W_0W_1}$ *are determined as follows:*

$$\begin{aligned}\dim +1 \textit{ eigenspace} &= \frac{q+2U}{4}\\ \dim -1 \textit{ eigenspace} &= \frac{q-2U}{4}\\ \dim +i \textit{ eigenspace} &= \frac{q+2V}{4}\\ \dim -i \textit{ eigenspace} &= \frac{q-2V}{4}\end{aligned}$$

where

$$\begin{aligned}U = -V \textit{ if } pp'p'' &\equiv 1 \pmod 4\\ U = +V \textit{ if } pp'p'' &\equiv 3 \pmod 4,\end{aligned}$$

with $U, V = \pm 1$.

REMARK 5.4.5. This does not determine the eigenspaces uniquely however for Theorem 3.2.5 it is sufficient.

PROPOSITION 5.4.6. Σ_Y *can be made an integer multiple of* q.

PROOF. Using Propositions 5.4.1 and 5.4.3, Corollary 5.4.4 together with $\dim(P_0) = \frac{q}{2}$ [4] we have

$$\Sigma_Y = \left(\frac{1}{2}+n_0\right)\frac{q}{2} + \left(\frac{1}{2}+m_0\right)\frac{(q-2U)}{4} + \left(\frac{1}{4}+m_1\right)\frac{(q+2V)}{4} + \left(\frac{3}{4}+m_2\right)\frac{(q-2V)}{4}$$
$$-\left[\left(\frac{1}{2}+\ell_0\right)\frac{(q+2)}{4} + \left(\frac{1}{4}+\ell_1\right)\frac{(q-2)}{4} + \left(\frac{3}{4}+\ell_2\right)\frac{(q-2)}{4}\right] + \left(\frac{pp'p''}{4q}+k_0\right)q$$

Putting $m_i = \ell_i i = 0, 1, 2$ we have

$$\Sigma_Y = \left(\frac{1}{2}+n_0\right)\frac{q}{2} - \left(\frac{1}{2}+m_0\right)\frac{(U+1)}{2} + \left(\frac{1}{4}+m_1\right)\frac{(V+1)}{2} - \left(\frac{3}{4}+m_2\right)\frac{(V-1)}{2} + \frac{pp'p''}{4} + k_0q$$

CASE 1: $pp'p'' \equiv 1 \pmod 4$. So $U = -V$ and $\frac{pp'p''}{4} = \frac{(q-1)}{4} + t$ where t is an integer. Therefore

$$\begin{aligned}\Sigma_Y &= \left(\frac{1}{2}+n_0\right)\frac{q}{2} - \left(\frac{1}{2}+m_0\right)\frac{(-V+1)}{2} + \left(\frac{1}{4}+m_1\right)\frac{(V+1)}{2}\\ &\quad - \left(\frac{3}{4}+m_2\right)\frac{(V-1)}{2} + \frac{(q-1)}{4} + t + k_0q\\ &= (1+n_0)\frac{q}{2} + (m_0-m_2)\frac{(V-1)}{2} + m_1\frac{(V+1)}{2} + t + k_0q\end{aligned}$$

Regardless of the sign of V, n_0, m_0, m_1 and m_2 can be chosen so that

$$(1+n_0)\frac{q}{2} + (m_0-m_2)\frac{(V-1)}{2} + m_1\frac{(V+1)}{2} + t = 0$$

hence Σ_Y is a multiple of q.

CASE 2: $pp'p'' \equiv 3 \pmod 4$. So $U = V$ and $\frac{pp'p''}{4} = \frac{(q+1)}{4} + t$ where t is an integer. Therefore

$$\begin{aligned}\Sigma_Y &= \left(\frac{1}{2}+n_0\right)\frac{q}{2} - \left(\frac{1}{2}+m_0\right)\frac{(V+1)}{2} + \left(\frac{1}{4}+m_1\right)\frac{(V+1)}{2} \\ &\quad - \left(\frac{3}{4}+m_2\right)\frac{(V-1)}{2} + \frac{(q+1)}{4} + t + k_0 q \\ &= (1+n_0)\frac{q}{2} + (m_1 - m_0)\frac{(V+1)}{2} - (m_2+1)\frac{(V-1)}{2} + t + k_0 q,\end{aligned}$$

which is the same as Case 1 so we are finished. ∎

5.5. $q \equiv 3 \pmod 4$.

PROPOSITION 5.5.1. *The dimensions of the four eigenspaces of W_0 are as follows:*

$$\begin{aligned}\dim \pm 1 \textit{ eigenspace } &= \frac{q+1}{4} \\ \dim +i \textit{ eigenspace } &= \frac{q-1+2(p|q)}{4} \\ \dim -i \textit{ eigenspace } &= \frac{q-1-2(p|q)}{4}\end{aligned}$$

PROOF. Trace$(\Gamma_0) = 1$, Trace$(W_0) = (p|q)i$. Now argue as in Proposition 5.2.2. ∎

COROLLARY 5.5.2.

$$\mathrm{Det}(W_0) = (p|q)(-i)^{\frac{(q-1)}{2}}$$

PROOF.

$$\begin{aligned}\mathrm{Det}(W_0) &= (-1)^{\frac{q+1}{4}}(i)^{\frac{q-1+2(p|q)}{4}}(-i)^{\frac{q-1-2(p|q)}{4}} \\ &= (-1)^{\frac{q+(p|q)}{2}}(-i)^{\frac{(q-1)}{2}} = (p|q)(-i)^{\frac{(q-1)}{2}}.\end{aligned}$$

∎

PROPOSITION 5.5.3. *The dimensions of the four eigenspaces of $\widetilde{W_0 W_1}$ are as follows:*

$$\begin{aligned}\dim \pm 1 \textit{ eigenspace } &= \frac{q+1}{4} \\ \dim +i \textit{ eigenspace } &= \frac{q-1+2V}{4} \\ \dim -i \textit{ eigenspace } &= \frac{q-1-2V}{4}\end{aligned}$$

where $V = (p|q)(-i)^{pp'p''}$.

PROOF. Following the same argument as in Proposition 5.3.3 and using the same notation we see $\eta = i$ in this case, so

$$\begin{aligned}m = n &= \frac{q+1}{4} \\ u &= \frac{q-1+2V}{4} \\ v &= \frac{q-1-2V}{4}\end{aligned}$$

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and $V = (p|q)(-i)^{pp'p''}$ by considering $\mathrm{Det}(\widetilde{W_0 W_1})$. ■

PROPOSITION 5.5.4. *Σ_Y can be made an integer multiple of q.*

PROOF. Using Propositions 5.5.1 and 5.5.3 together with $\dim(P_0) = (q-1)/2$ [4], we have

$$\Sigma_Y = \left(\frac{1}{2}+n_0\right)\frac{(q-1)}{2} - \left(\frac{1}{2}+m_0\right)\frac{(q+1)}{4} + \left(\frac{1}{4}+m_1\right)\frac{(q-1+2V)}{4}$$
$$- \left(\frac{3}{4}+m_2\right)\frac{(q-1-2V)}{4} - \left[\left(\frac{1}{2}+\ell_0\right)\frac{(q+1)}{4} + \left(\frac{1}{4}+\ell_1\right)\frac{\big(q-1-2(p|q)\big)}{4}\right.$$
$$\left. + \left(\frac{3}{4}+\ell_2\right)\frac{\big(q-1+2(p|q)\big)}{4}\right] + \left(\frac{pp'p''}{4q}+k_0\right)q.$$

Putting $m_i = \ell_i$, $i = 0, 1, 2$ we have

$$\Sigma_Y = \left(\frac{1}{2}+n_0\right)\frac{(q-1)}{2} + \left(\frac{1}{4}+m_1\right)\frac{\big(V+(p|q)\big)}{2} - \left(\frac{3}{4}+m_2\right)\frac{\big(V+(p|q)\big)}{2} + \frac{pp'p''}{4} + k_0 q.$$

CASE 1: $pp'p'' \equiv 0 \pmod 4$. So $V = (p|q)$ and $\frac{pp'p''}{4} = \frac{(q+1)}{4} + t$, where t is an integer. Therefore

$$\Sigma_Y = \left(\frac{1}{2}+n_0\right)\frac{(q-1)}{2} + (p|q)\left(\frac{1}{4}+m_1\right) - (p|q)\left(\frac{3}{4}+m_2\right) + \frac{(q+1)}{4} + t + k_0 q$$
$$= \frac{\big(q-(p|q)\big)}{2} + (p|q)(m_1-m_2) + t + k_0 q + n_0\frac{(q-1)}{2}$$

Let $m_2 - m_1 = \left[\frac{(q-(p|q))}{2} + t\right]/(p|q)$ and $n_0 = 0$, then $\Sigma_Y = k_0 q$.

CASE 2: $pp'p'' \equiv 2 \pmod 4$. So $V = -(p|q)$ and $\frac{pp'p''}{4} = \frac{(q-1)}{4} + t$, where t is an integer. Therefore

$$\Sigma_Y = \left(\frac{1}{2}+n_0\right)\frac{(q-1)}{2} + \frac{(q-1)}{4} + t + k_0 q$$
$$= (1+n_0)\frac{(q-1)}{2} + t + k_0 q$$

Using the fact that $(q-1)/2$ and q are coprime we are finished. ■

This completes the proof of Theorem 3.2.5.

6. The crossed product.

6.1. *Introduction.* Here we will give a characterization of the crossed product algebra $\mathcal{A}_\theta \rtimes_\tau Z_4$.

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6.2. *The crossed product algebra.*

THEOREM 6.2.1. *Let $\theta = p/q$, with p, q coprime positive integers and let Ω_i, $i = 0, 1, 2$ be any three distinct points of the 2-sphere S^2. Then the crossed product algebra $\mathcal{A}_\theta \rtimes_\tau Z_4$ is isomorphic to the following subalgebra of the C^*-algebra $C(S^2, M_{4q})$:*

$$\mathcal{A}_\theta \rtimes_\tau Z_4 = \left\{ f \in C(S^2, M_{4q}) \mid f(\Omega_i) \text{ commutes with } P_i^j, i, j = 0, 1, 2. \right\},$$

where P_i^j, $i, j = 0, 1, 2$, are self-adjoint projections in M_{4q} with $P_0^1 = P_0^2 = 0$. The dimension of P_0^0 is $2q$, while the dimension of P_i^j, $i = 1, 2, j = 0, 1, 2$, is q.

PROOF. Let z be the canonical unitary in $\mathcal{A}_\theta \rtimes_\tau Z_4$ implementing τ. Then the left regular representation of

$$\mathcal{A}_\theta \rtimes_\tau Z_4$$

is given by

$$\mathcal{A}_\theta \ni A \longmapsto \begin{bmatrix} A & & & \\ & \tau(A) & & \\ & & \tau^2(A) & \\ & & & \tau^3(A) \end{bmatrix},$$

$$Z_4 \ni z \longmapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

If we denote by

$$T(A) = \begin{bmatrix} A & & & \\ & \tau(A) & & \\ & & \tau^2(A) & \\ & & & \tau^3(A) \end{bmatrix}, A \in \mathcal{A}_\theta,$$

and

$$Z = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

we have that

$$\mathcal{A}_\theta \rtimes_\tau Z_4 = \left\{ G = T(A) + T(B)Z + T(C)Z^2 + T(D)Z^3 \mid A, B, C, D \in \mathcal{A}_\theta \right\},$$

where a generic element of $\mathcal{A}_\theta \rtimes_\tau Z_4$ has the form

$$G = \begin{bmatrix} A & D & C & B \\ \tau(B) & \tau(A) & \tau(D) & \tau(C) \\ \tau^2(C) & \tau^2(B) & \tau^2(A) & \tau^2(D) \\ \tau^3(D) & \tau^3(C) & \tau^3(B) & \tau^3(A) \end{bmatrix}.$$

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REMARK 6.2.2. If we change coordinates in M_4 by means of the automorphism

$$\operatorname{Ad} E, \text{ where } E = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{bmatrix},$$

we have

$$\hat{Z} = (\operatorname{Ad} E)(Z) = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & i & \\ & & & -i \end{bmatrix}$$

and

$$(\operatorname{Ad} E)(G) = \frac{1}{4}\begin{bmatrix} \Sigma_0 F_0 & \Sigma_1 F_1 & \Sigma_3 F_2 & \Sigma_2 F_3 \\ \Sigma_1 F_0 & \Sigma_0 F_1 & \Sigma_2 F_2 & \Sigma_3 F_3 \\ \Sigma_2 F_0 & \Sigma_3 F_1 & \Sigma_0 F_2 & \Sigma_1 F_3 \\ \Sigma_3 F_0 & \Sigma_2 F_1 & \Sigma_1 F_2 & \Sigma_0 F_3 \end{bmatrix}$$

where we put

$$\begin{aligned} F_0 &= A + B + C + D, \\ F_1 &= A - B + C - D, \\ F_2 &= A + iB - C - iD, \\ F_3 &= A - iB - C + iD, \\ \Sigma_0 &= 1 + \tau + \tau^2 + \tau^3, \\ \Sigma_1 &= 1 - \tau + \tau^2 - \tau^3, \\ \Sigma_2 &= 1 + i\tau - \tau^2 - i\tau^3, \\ \Sigma_3 &= 1 - i\tau - \tau^2 + i\tau^3. \end{aligned}$$

Therefore if we denote by

$$\mathcal{A}_\theta^\tau(\mathcal{E}) = \{A \in \mathcal{A}_\theta \mid \tau(A) = \mathcal{E}A\}, \quad \mathcal{E} = -1, \pm i,$$

it follows that a generic element G of $\mathcal{A}_\theta \rtimes_\tau Z_4$ has the form

$$G = \begin{bmatrix} A_0 & B_0 & C_0 & D_0 \\ B_1 & A_1 & D_1 & C_1 \\ D_2 & C_2 & A_2 & B_2 \\ C_3 & D_3 & B_3 & A_3 \end{bmatrix},$$

with

$$A_i \in \mathcal{A}_\theta^\tau, \quad B_i \in \mathcal{A}_\theta^\tau(-1), \quad C_i \in \mathcal{A}_\theta^\tau(i), \quad D_i \in \mathcal{A}_\theta^\tau(-i), \quad i = 0, 1, 2, 3.$$

REMARK 6.2.3. In the representation given in 6.2.1 of $\mathcal{A}_\theta \rtimes_\tau Z_4$ the projection P [15] such that $P(\mathcal{A}_\theta \rtimes_\tau Z_4)P \cong \mathcal{A}_\theta^\tau$, is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can identify $\mathcal{A}_\theta^\tau(\mathcal{E})$, $\mathcal{E} = -1, \pm i$, in the following way (*cf.* Section 4)

$$\mathcal{A}_\theta^\tau(\mathcal{E}) = \left\{ f \in C(T, M_q) \;\middle|\; \begin{array}{l} f(x,x) = \mathcal{E}\alpha_1\alpha_0\big(f(1-x,x)\big) \\ f(x,0) = \mathcal{E}\alpha_1\alpha_2\gamma_0\big(f(1-x,0)\big) \end{array} \right\},$$

where T is the triangle

$$T = \big\{(x,y) \in [0,1] \times [0,1] \mid y \le \min\{x, 1-x\}\big\}.$$

Therefore

$$\mathcal{A}_\theta \rtimes_\tau Z_4 \cong \left\{ f \in C(T, M_{4q}) \;\middle|\; \begin{array}{l} f(x,x) = (\alpha_1\alpha_0 \otimes \operatorname{Ad}\hat{Z})\big(f(1-x,x)\big) \\ f(x,0) = \big(\alpha_1\alpha_2\gamma_0 \otimes (\operatorname{Ad}\hat{Z})^2\big)\big(f(1-x,0)\big) \end{array} \right\}.$$

The fiber over each point of T is M_{4q} with the exception of the three points $\Omega_0 = (\frac{1}{2}, 0)$, $\Omega_1 = (\frac{1}{2}, \frac{1}{2})$, $\Omega_2 = (0,0)$ where we have the subalgebras

$$\{A \in M_{4q} \mid A(U_i \otimes \hat{Z}^{Y_i}) = (U_i \otimes \hat{Z}^{Y_i})A\}, \quad i = 0, 1, 2,$$

where $Y_i = \begin{cases} 2, & i = 0 \\ 1, & i = 1, 2, \end{cases}$ and

$$\begin{aligned} U_0 &= \Gamma_0 W_2 (\text{at } \Omega_0), \\ U_1 &= W_0^{-1} (\text{at } \Omega_1), \\ U_2 &= \widetilde{W_0 W_1} (\text{at } \Omega_2). \end{aligned}$$

This proves that the dimensions of the projections P_i^j at the points Ω_i, $i = 0, 1, 2$, are as stated. We can now proceed as in the proof of Theorem 3.2.5 to show that the bundle over S^2 is trivial. Define a map $\hat{\eta}\colon S^2 - Y \longrightarrow M_{4q}$ by

$$\hat{\eta} = \eta \otimes \Gamma$$

where

$$\Gamma\colon S^1 \longrightarrow U_4/T = \operatorname{Aut} M_4$$

which goes from I_4 to $\hat{Z}^2$ on the circle around $(\frac{1}{2}, 0)$, from $\hat{Z}^2$ to $\hat{Z}$ on the circle around $(\frac{1}{2}, \frac{1}{2})$, and back from $\hat{Z}$ to I_4 on the circle around $(0,0)$.

Since the winding number of the map Γ is zero, by the same argument used for $\mathcal{A}_\theta^\tau$ (see Sections 4 and 5) we can prove that the winding number of $\hat{\eta}$ is an integral multiple of $4q$.

This finishes the proof of Theorem 6.2.1. ∎

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