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PROPER HOLOMORPHIC SELF-MAPS OF QUASI-CIRCULAR DOMAINS IN $\mathbf{C}^2$

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Abstract. In this paper, we prove that every proper holomorphic self-map of a smoothly bounded pseudoconvex circular or Hartogs domain of finite type in $\mathbf{C}^2$ is biholomorphic.

§1. Introduction

In this paper, we study proper holomorphic maps between domains of finite type. A domain Ω is said to be quasi-circular if there exist integers p, q $(p+q \geq 1)$ such that whenever $(z, w) \in \Omega$, $(e^{ip\theta}z, e^{iq\theta}w) \in \Omega$ for $\theta \in [0, 2\pi]$. We observe that when $p = q = 1$, Ω is circular; when $p = 0$ or $q = 0$, Ω is Hartogs. The following is the main result that we shall prove in this paper.

THEOREM 1. *Let Ω be a smoothly bounded pseudoconvex quasi-circular domain of finite type in $\mathbf{C}^2$. Then every proper holomorphic self-map of Ω is a biholomorphism.*

As a consequence for classical domains, we have

COROLLARY. *Let Ω be a smoothly bounded pseudoconvex circular or Hartogs domain of finite type in $\mathbf{C}^2$. Then every proper holomorphic self-map of Ω is a biholomorphism.*

Since the first result of Alexander [1] on the unit ball, it has been an open question whether every proper holomorphic self-map of a smoothly bounded domain in $\mathbf{C}^n$ is a biholomorphism. This problem still remains open in general. However, some positive results have been proved in the case of strictly pseudoconvex domains by Pinchuk [13], and in the case of pseudoconvex domains with real-analytic boundary by Bedford [2], Bedford-Bell [3]. Also, for domains with various symmetries, see [4, 9, 10, 12]. A recent paper of Berteloot [6] solves the problem for complete Reinhardt domains with

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C^2 smooth boundary by studying the Lie algebra of holomorphic tangent vector fields of the boundary. In a recent paper [9], we studied the structure of the branch locus of proper holomorphic maps between domains of finite type. As a consequence, it was proved that every proper holomorphic self-map of a smoothly bounded pseudoconvex complete circular domain in $\mathbf{C}^2$ is a biholomorphism. In this paper we continue to use the main result of [9] on the branch locus to study the case of quasi-circular domains of finite type. Using the method of this paper, we also generalize the result of [9] by dropping the assumption of completeness. In fact, we shall prove a more general result. The paper is organized as follows. In Section 1, some basic facts concerning self-maps are collected. In the case of Hartogs, Theorem 1 is proved in Section 2 using a result concerning fixed points of an analytic function. The remaining case is proved in Section 3 by reducing to one variable and two-variable complex dynamics situation.

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§2. Basic facts

To proceed with the proof of the theorem, we need some basic facts. Let $F : \Omega \to D$ be a proper holomorphic map between smoothly bounded pseudoconvex domains of finite type. It is well known that F extends smoothly up to the boundary of Ω. The branch locus of F is defined to be $V_F = \{z \in \Omega : \det F' = 0\}$. It is well-known that the automorphism group action $\mathrm{Aut}(\Omega) \times \Omega \longrightarrow \Omega$, $(F, z) \mapsto F(z)$ extends smoothly to $\overline{\Omega}$. Thus we can assume that $\mathrm{Aut}(\Omega)$ acts smoothly on $\overline{\Omega}$ and in particular on $\partial\Omega$. We say that a subgroup G of $\mathrm{Aut}(\Omega)$ acts transversally at a boundary point p of Ω if the image of the tangent mapping $(\Psi_p)^* : T_eG \longrightarrow T_p(\partial\Omega)$ associated to the mapping $\Psi_p : G \longrightarrow \partial\Omega$, $F \mapsto F(p)$ is not contained in the holomorphic tangent space $H_p(\partial\Omega)$. We will denote by $\mathbf{T}$ the Lie group of the unit circle. If $\mathbf{T}$ is a subgroup of $\mathrm{Aut}(\Omega)$ and acts transversally near a point $p \in \partial\Omega$, we will simply say that Ω admits a transversal $\mathbf{T}$-action at p.

We recall the main result of [9] concerning the branch locus of proper holomorphic maps.

THEOREM 2. *Let $F : \Omega \to D$ be a proper holomorphic map between bounded pseudoconvex domains of finite type in $\mathbf{C}^2$. Suppose that Ω admits* *a* **T**-*action which is transversal at p. Then for any irreducible component V of the branch locus V_F with $p \in \partial \overline{V} = \overline{V} \cap \overline{\Omega}$, $\partial \overline{V}$ is a finite union of* **T**-*orbits.*

Although this result was not explicitly stated in [9], the proof there would apply for a **T**-action which is locally transversal. It is this form that we shall apply in this paper.

We recall certain general facts about boundary behavior of proper holomorphic mappings. Let $F : D_1 \longrightarrow D_2$ be a proper holomorphic mapping between two smoothly bounded pseudoconvex domains in $\mathbf{C}^2$. We suppose that F is smooth up to the boundary. Let r_j be the defining function of D_j. Following [2], we consider the Levi-determinant $\Lambda_{\partial D_j}$ of D_j defined as [2].

One has then

$$\Lambda_{\partial D_2}(F(p))|J_f(p)|^2 = \Lambda_{\partial D_1}(p)$$

for any $p \in \partial D_1$.

For any boundary point $p \in \partial D_j$ we consider also the order of vanishing of $\Lambda_{\partial D_j}$ at p denoted by $\tau_{\partial D_j}(p)$, which is defined as follows: we choose smooth real coordinates $x = (x_1, x_2, x_3)$ on ∂D_j such that p corresponds to $x = 0$, and the formal power series $\Lambda_{\partial D_j}(x) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} a_\alpha x^\alpha$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We set $\tau_{\partial D_j}(p) = \min\{|\alpha| : a_\alpha \neq 0\}$ (of course, this definition does not depend on the choice of coordinates). The following properties of τ are well known (see [2, 3]):

(1) $\tau_{\partial D_j}(p)$ is an upper-semicontinuous function on ∂D_j.

(2) $\tau_{\partial D_2}(F(p)) \leq \tau_{\partial D_1}(p)$ and the equality holds if and only if $\overline{V_F}$ does not contain p, i.e. F is a diffeomorphism on the boundary near p . The following lemma proves to be very useful.

LEMMA 1. *Let $F : \Omega \to \Omega$ be a proper holomorphic map between smoothly bounded pseudoconvex domains of finite type in $\mathbf{C}^n$. Then if $V_F \neq \emptyset$, there exists an irreducible component L_n of V_{F^n} such that $L_i \neq L_j, i \neq j$, and*

$$L_{n+1} \subset F^{-1}(L_n)$$

for $n = 1, 2, 3 \ldots$, where F^n denotes the n-th iteration of F.

Proof. We claim that there exists a sequence (L_n) of complex varieties such that $L_n \subset V_{F^n}, L_{n+1} \subset F^{-1}(L_n)$.

We will construct the family (L_n) by induction. For every n we have $V_{F^{n+1}} = V_{F^n} \cup F^{-1}(V_{F^n})$. Fix any irreducible component L_1 in V_F. Then

$F^{-1}(L_1)$ is contained in V_{F^2} and contains an irreducible component L_2. Assume that the components $L_1, \dots, L_n$ are defined. Then $F^{-1}(L_n)$ is contained in $F^{-1}(V_{F^n}) \subset V_{F^{n+1}}$. So there exists an irreducible component L_{n+1} such that $L_{n+1} \subset F^{-1}(L_n)$. Note that since every restriction $F : L_{n+1} \longrightarrow L_n$ is proper and $F(L_{n+1}) \subset L_n$, we have $F(L_{n+1}) = L_n$. We note that the varieties (L_n) are distinct. Indeed, suppose by contradiction that m is the first integer such that there exists p with $L_m = L_{m+p}$. If $m \geq 2$, we have $F(L_m) = F(L_{m+p})$ and so $L_{m-1} = L_{m+p-1}$. This contradicts the definition of m. So $m = 1$. Let p be the integer such that $L_1 = L_{1+p}$. Since we have $F^p(L_{1+p}) = L_1$, it follows that $F^p(L_1) = L_1$. Since $L_1 \subset V_{F^p}$, letting $Q \in \overline{L}_1 \cap \Omega$, we see

$$\cdots < \tau_{\partial\Omega}(F^{kp}(Q)) < \tau_{\partial\Omega}(F^{(k-1)p}(Q)) < \cdots < \tau_{\partial\Omega}(F^p(Q)) < \tau_{\partial\Omega}(Q) < \infty,$$

which is obviously a contradiction.

We need a simple result from one complex variable dynamics on fixed points, which is key to the study of the case of Hartogs domains.

Lemma 2. *Let f be an analytic function in a neighborhood U of z_0 with $f(z_0) = z_0$. Suppose that $f^n(V)$ is uniformly bounded for a neighborhood $V \subset\subset U$. Then there exists no sequence z_n in V converging to z_0 such that $f(z_{n+1}) = z_n$ for all n.*

Proof. We consider the Taylor series of f:

$$f(z) = z_0 + \lambda(z - z_0) + \cdots$$

where $\lambda = f'(z_0)$, the multiplier of f at z_0.

If $\lambda = 0$–superattracting, by [8], there is a conformal map $\zeta = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates f to ζ^p for a positive integer $p > 1$, that is

$$g(\zeta) = \phi \circ f \circ \phi^{-1}(\zeta) = \zeta^p.$$

Letting $\zeta_n = \phi(z_n)$, it follows that $\zeta_n \neq \zeta_m$, $n \neq m$, and $\zeta_n \to 0$. We see $g(\zeta_{n+1}) = \zeta_n$, therefore $\zeta_{n+1}^p = \zeta_n$. This implies $\zeta_{n+1} = \zeta_1^{1/p^n}$, a contradiction to $\zeta_n \to 0$.

If $0 < |\lambda| < 1$-attracting, and not superattracting, by [8], there is a conformal map $\zeta = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates f to $\lambda\zeta$, that is

$$g(\zeta) = \phi \circ f \circ \phi^{-1}(\zeta) = \lambda\zeta.$$

Letting $\zeta_n = \phi(z_n)$, it follows that $\zeta_n \neq \zeta_m$, $n \neq m$, and $\zeta_n \to 0$. As before, $g(\zeta_{n+1}) = \zeta_n$, which implies $\lambda\zeta_{n+1} = \zeta_n$, and $\zeta_{n+1} = (1/\lambda)^n\zeta_1$, which converges to ∞, a contradiction.

The case that $|\lambda| > 1$ does not happen by the assumption that $f^n(V)$ is uniformly bounded.

The remaining case is when $|\lambda| = 1$. By the same assumption, and by [8] there is a conformal map $\zeta = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates f to $\lambda\zeta$, that is

$$g(\zeta) = \phi \circ f \circ \phi^{-1}(\zeta) = \lambda\zeta.$$

As before, this also leads to a contradiction.

The following fact is well-known for biholomorphic maps, and its proof for proper maps is included. This lemma is very important for incomplete Hartogs and quasi-circular domains. Denote by $A(r, R) = \{r < |z| < R\}$ an annulus in the complex plane.

LEMMA 3. *If $f(z) : A(r_1, R_1) \to A(r_2, R_2)$ is a proper holomorphic map with multiplicity m, then we have*

$$\frac{R_2}{r_2} = \left(\frac{R_1}{r_1}\right)^m.$$

Proof. It is well-known that f extends continuously to the boundary. Consider $\phi(z) = \ln|f(z)|$. Then $\phi(z)$ is harmonic, continuous up to the boundary. A simple topological argument shows that

$$\lim_{|z|\to r_1} \phi(z) = a$$

where $a = \ln r_2$ or $\ln R_2$, and

$$\lim_{|z|\to R} \phi(z) = \{\ln r_2, \ln R_2\} \setminus \{a\}.$$

Suppose f maps $|z| = r_1$ to $|z| = r_2$. Then we have $\phi(z) = \ln r_2$ when $|z| = r_1$; $\phi(z) = \ln R_2$ when $|z| = R_1$. Therefore ϕ solves the Dirichlet problem. However $c(\ln|z| - \ln r_1) + \ln r_2$ also solves the same Dirichlet problem where

$$c = \frac{\ln \frac{R_2}{r_2}}{\ln \frac{R_1}{r_1}}.$$

By the uniqueness, it follows that $c(\ln|z| - \ln r_1) + \ln r_2 = \ln|f(z)|$, which gives $|f(z)| = |z|^c$. Since f is a proper holomorphic map with multiplicity m, we have $c = m$, which proves the lemma.

Suppose that f maps $|z| = r_1$ to $|z| = R_2$. We only need to consider $1/f$. The same proof then applies.

LEMMA 4. *Let $F : \Omega \to D$ be a proper holomorphic map between smoothly bounded pseudoconvex domains of finite type in $\mathbf{C}^n$. Suppose that E is a totally real manifold of dimension n in the boundary of Ω on which τ is constant. Then we have $\overline{V}_F \cap E = \emptyset$.*

Proof. If not so, we let $p \in \overline{V}_F \cap E$. It follows that $\tau(F(p)) < \tau(p)$, and choosing q in E not in $\overline{V}_F$ near p (we can do so, since $\partial\overline{V}_F$ is smooth at most points [12]), we have $\tau(F(p)) < \tau(p) = \tau(q) = \tau(F(q))$, which contradicts the fact that τ is upper semicontinuous.

§3. The case of Hartogs

In this section, we shall prove Theorem 1 in case of Hartogs in two cases. When the Hartogs domain is complete, we shall make use of the base domain; when the domain is not complete, we shall exploit the incompleteness.

Case 1. Complete Hartogs domains

A domain Ω is said to be a complete Hartogs domain if whenever $(z, w) \in \Omega$, $(z, \lambda w) \in \Omega$ for $|\lambda| < 1$. The base of a complete Hartogs domain Ω is defined as

$$E = \Omega \cap \{w = 0\}.$$

LEMMA 5. *Let Ω be a smoothly bounded pseudoconvex complete Hartogs domain of finite type in $\mathbf{C}^2$. Then there exists a neighborhood U of $\partial\Omega \cap \overline{E}$ so that $\partial\Omega \cap U$ is strictly pseudoconvex at every point except possibly at points of $\partial\Omega \cap \overline{E}$.*

Proof. Let $\rho(z,w)$ be a defining function of Ω. Let $(z_0,0)\in\partial E$. We see that $\nabla\rho(z_0,0)\neq 0$, and the complex z plane is transversal to $\partial\Omega$ at $(z_0,0)$. Then we may assume $\rho_z(z_0,0)\neq 0$. Define

$$r(z,w)=\int_0^{2\pi}\rho(z,e^{i\theta}w)d\theta.$$

Then r is a defining function of Ω, but $r(z,w)=r(z,|w|)$. By the linear change of coordinates, $(z,w)\to(z-z_0,w)$, we have, $r_z(0,0)\neq 0$,

$$r(z,w)=\Re z+\phi(\Im z,|w|).$$

Since Ω is of finite type, we have

$$\phi(\Im z,|w|)=c|w|^{2k}+0(|\Im z|+|\Im zw|+|w|^{2k}).$$

It is easy to see that $\partial\Omega$ is strictly pseudoconvex when $w\neq 0$ and small.

LEMMA 6. *Let $F:\Omega\to D$ be a proper holomorphic map between pseudoconvex domains of finite type in $\mathbf{C}^2$. If Ω is complete Hartogs, then there exist $z_1,z_2,\dots,z_n\in E$ such that the branch locus of F satisfies*

$$V_F\subset\bigcup_{i=1}^{n}\{z=z_i\}\cup\{w=0\}.$$

Proof. First we remark that the $\mathbf{T}$-action on Ω is transversal whenever $(z,w)\notin\partial\Omega\cap\overline{E}$. Let V be an irreducible component of V_F, and let $p\in\partial\overline{V}$, $p=(z_0,w_0)$. If $p\notin\partial\Omega\cap\overline{E}$, By Theorem 2, $V=\{z=z_0\}$. If $p\in\partial\Omega\cap\overline{E}$, it follows $V=\{w=0\}\cap\Omega$. Now it suffices to prove that V_F has finitely many irreducible components. If not so, we have

$$V_F\subset\bigcup_{i=1}^{\infty}\{z=z_i\}\cup\{w=0\}.$$

Then z_j must converges to the boundary of E. By Lemma 5, there exists an neighborhood U of $\partial\Omega\cap\overline{E}$ such that $\partial\Omega$ is strictly pseudoconvex except possibly at $\partial\Omega\cap\overline{E}$; therefore $\partial\Omega\cap\{z=z_i\}\cap U$ must be strictly pseudoconvex points for i large, which contradicts the fact that branch locus never hits strictly pseudoconvex points.

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Now we are ready to prove Theorem 1. Assume $F = (f, g)$. By Lemma 1, we have $L_n \subset V_{F^n}$. By Lemma 6, we have $L_n = \{z = z_n\} \cap \Omega$, for some $z_n \in E$. We claim first the sequence $\{z_n\}$ must stay away from the boundary of E since by Lemma 2, $\partial\Omega \cap U \setminus \partial\overline{E}$ is strictly pseudoconvex and the branch locus does not hit strictly pseudoconvex points. Without loss of generality, we may assume that $\{z_n\}$ converges to $z_0 \in E$. Since $L_{n+1} \subset F^{-1}(L_n)$, $F(L_{n+1}) = L_n$, which implies $f(z_{n+1}, w) = z_n$. It follows that $\frac{\partial f}{\partial w}(z, w)$, being analytic in z in E has infinitely many zeros $\{z_n\}$ accumulating at an interior point z_0 in E, therefore is identically zero. This implies that $f(z, w)$ is independent of w. Furthermore, we have $F = (f(z), g(z, w))$, $f(z_0) = z_0$ and $g(z_0, w) : L_0 \to L_0$ is a proper self map, where of course $L_0 = \{z = z_0\} \cap \Omega$. It is easy to see that f maps E into E, and f fixes z_0 such that $f(z_{n+1}) = z_n$ with $z_n \to z_0$. This is impossible by Lemma 2 and completes the proof of Theorem 1 in the case of complete Hartogs domains.

Case 2. Incomplete Hartogs domains

LEMMA 7. *Let $F : \Omega \to D$ be a proper holomorphic map between pseudoconvex domains of finite type in $\mathbf{C}^2$. If Ω is incomplete Hartogs, then there exist $z_1, z_2, \ldots, z_n, \ldots$ such that the branch locus of F satisfies*

$$V_F \subset \bigcup_{i=1}^{\infty} \{z = z_i\} \cap \Omega\,.$$

Proof. In fact, let V be an irreducible component of V_F. If there exists a point $p = (z_0, w_0)$ in $\overline{V} \cap \partial\Omega$ so that the $\mathbf{T}$-action is transversal at p, then by Theorem 2, $V = \{z = z_0\} \cap \Omega$. Now we assume that the $\mathbf{T}$-action is transversal at no points of $\overline{V} \cap \partial\Omega$. Choose a smooth arc Γ in $\overline{V} \cap \partial\Omega$ so that $\tau_{\partial\Omega}(p)$ is constant for $p \in \Gamma$. This is possible since Ω is of finite type. Considering $E = \Gamma \times \mathbf{T}$, this is a totally real manifold of dimension 2 for which $\tau_{\partial\Omega}$ is constant. By Lemma 4, we have $\overline{V} \cap E = \emptyset$, which is obviously a contradiction.

Now we are ready to prove Theorem 1 when Ω is incomplete. We first claim that $w \neq 0$ on $\overline{\Omega}$. Indeed, this follows from pseudoconvexity and the continuity principle.

Assume $V_F \neq \emptyset$. By Lemma 1, there exists $L_n \subset V_{F^n}$ such that $F(L_{n+1}) = L_n$, and by Lemma 7, there exists z_n such that

$$L_n = \{z = z_n\} \cap \Omega.$$

By the claim, there exist $R_n, r_n, R_n > r_n > 0$, such that

$$L_n = \{(z_n, w) : r_n < |w| < R_n\},$$

i.e., L_n is an annulus in the w plane. Since Ω is bounded, we may assume that $R_n \to R, r_n \to r$. Since $w \neq 0$ on $\overline{\Omega}$, we have $r > 0$. On the other hand, $g(z_{n+1}, w)$ is a proper holomorphic map from $r_{n+1} < |w| < R_{n+1}$ to $r_n < |w| < R_n$; and without loss of generality, we may assume that $g(z_n, w)$ has the same multiplicity m for any n, by Lemma 3, we have

$$\frac{R_{n+1}}{r_{n+1}} = \left(\frac{R_n}{r_n}\right)^m,$$

which implies that

$$\frac{R_{n+1}}{r_{n+1}} = \left(\frac{R_1}{r_1}\right)^{nm}.$$

Letting $n \to \infty$, we see the left hand side goes to R/r, while the right hand side goes to infinity, arriving at a contradiction, and the proof of the theorem is complete.

§4. Basins of attraction of quasi-homogeneous maps

If $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a holomorphic map such that $F(0) = 0$ and the eigenvalues of $F'(0)$ are smaller than one in modulus, we may associate with the basin of attraction at the origin, Ω_F, which is defined by

$$\Omega_F = \{z \in \mathbf{C}^2 | \lim_{k\to\infty} F^k(z) = 0\}.$$

Following [7], if $(p, q) = 1$, we say that a holomorphic map $P : \mathbf{C}^2 \to \mathbf{C}^2$ is quasi-homogeneous map of type (p, q) if the components (P_1, P_2) satisfy $P_1(t^p z, t^q w) = t^{np} P_1(z, w)$ and $P_2(t^p z, t^q w) = t^{nq} P_2(z, w)$. It is easy to see that such a map must be a polynomial map.

Now let P be a quasi-homogeneous polynomial with $P^{-1}(0) = 0$. Let Ω_P be the basin of attraction of P at the origin. One can see that Ω_P is a quasi-circular domain of type (p, q), bounded and pseudoconvex. Moreover, P induces a non-injective proper holomorphic self-map of Ω_P. One also has that if $z \in \Omega_P$, then $\lim_{n\to\infty} P^n(z) = 0$; if $z \notin \overline{\Omega}_P$, then $\lim_{n\to\infty} ||P^n(z)|| = \infty$.

As proved in [7], we notice that the dynamics of quasi-homogeneous polynomial maps can be related to that of homogeneous ones. Let $\Phi(z, w) =$

(z^q, w^p). There exists a unique homogeneous polynomial $Q(z, w)$ of degree n such that

$$\Phi \circ P = Q \circ \Phi.$$

Also Φ is a proper map from Ω_P to Ω_Q, where Ω_Q is the basin of attraction of Q. It also follows that

$$\Phi \circ P^n = Q^n \circ \Phi.$$

§5. Non-Hartogs case

In this section, we assume that Ω is quasi-circular of type (p, q) with $p, q \geq 1$. We also assume, without loss of generality, that $(p, q) = 1$. First we use Theorem 2 to study the structure of the branching locus of proper holomorphic maps from quasi-circular domains.

LEMMA 8. *Let $F : \Omega \to D$ be a proper holomorphic map between pseudoconvex domains of finite type in $\mathbf{C}^2$. If Ω is quasi-circular of type (p, q), then there exist $p_1, p_2, \ldots, p_n$ such that the branch locus of F satisfies*

$$V_F = \bigcup_{i=1}^{n} \{(\lambda^p z_i, \lambda^q w_i) : \lambda \in \mathbf{C}\} \cap \Omega,$$

where $p_i = (z_i, w_i) \in \partial\Omega$.

Proof. First the transversality of T-action can be verified as in [9, Lemma 4.1]. It is easy to see by T-action that any irreducible component of V_F is given by $\{(\lambda^p z, \lambda^q w) : \lambda \in \mathbf{C}\} \cap \Omega$ for $(z, w) \in \partial\Omega$, which contains the origin since $p, q \geq 1$. We conclude that V_F cannot have infinitely many components by the compactness of V_F near the origin. This proves the lemma.

A quasi-circular domain of type (p, q) is said to be complete provided that whenever $(z, w) \in \Omega$, $(\lambda^p z, \lambda^q w) \in \Omega$ for $|\lambda| \leq 1$. We will prove the theorem in two cases as done in the Hartogs case.

Case 1. Complete quasi-circular domain of type (p, q)

We shall first prove that if the self-map is branched, then it must be a (p, q) quasi-homogeneous polynomial map.

LEMMA 9. *Let $F:\Omega\to\Omega$ be a proper holomorphic self-map of Ω. If $V_F\neq\emptyset$, then $F=(f(z,w),g(z,w))$ is given by*

$$f(z,w)=\sum_{p\alpha+q\beta=pk}A_{\alpha\beta}z^{\alpha}w^{\beta}$$

$$g(z,w)=\sum_{p\alpha+q\beta=qk}B_{\alpha\beta}z^{\alpha}w^{\beta},$$

for some positive integer k.

Proof. By Lemma 1, there exists $L_n\subset V_{F^n}$ such that $F(L_{n+1})=L_n$, and F is proper from $L_{n+1}\to L_n$ where L_n are distinct. By Lemma 8, there exists $(z_n,w_n)\in\partial\Omega$ such that $L_n=\{(\lambda^p z_n,\lambda^q w_n):\lambda\in\mathbf{C}\}\cap\Omega$. It is easy to see $F(0)=0$. Since $\{L_n\}$ are different we may assume that $z_n\neq 0$ for $n=1,2,3\ldots$. Therefore

$$L_n=\{(z,w):w^p=a_n z^q\}\cap\Omega,$$

where $a_n=w_n^p/z_n^q$. Since $F:L_{n+1}\to L_n$ is proper, it follows that if $|\lambda|=1$, then

$$|f(\lambda^p z_{n+1},\lambda^q w_{n+1})|=|z_n|$$

$$|g(\lambda^p z_{n+1},\lambda^q w_{n+1})|=|w_n|.$$

Therefore, we have the function $\phi_n(\lambda)=f(\lambda^p z_{n+1},\lambda^q w_{n+1})$ maps $|\lambda|<1$ properly to $|\lambda|<|z_n|$, and the function $\psi_n(\lambda)=g(\lambda^p z_{n+1},\lambda^q w_{n+1})$ maps $|\lambda|<1$ properly to $|\lambda|<|w_n|$. It follows that $\phi_n(\lambda)$ is a finite Blaschke product. We claim actually that $\phi_n(\lambda)=c_n\lambda^N$ where c_n is a constant and N is independent of n.

In order to prove this, we first prove that $F|_{L_{n+1}}^{-1}(0)=0$ for sufficiently large n. In fact if not, there are infinitely many n for which there exist z_n, $0\neq z_n\in L_{n+1}$ such that $F(z_n)=0$. Since L_n are only in common at 0, we see z_n are different, and therefore $F^{-1}(0)$ has infinitely many points, which contradicts the properness of F.

Now we only have to prove $\phi_n^{-1}(0)=0$ as well since ϕ_n is a finite Blaschke product. Indeed, if there is $c\neq 0$, $|c|<1$ such that $\phi_n(c)=0$. Then since $F:L_{n+1}\to L_n$, $F(c^p z_{n+1},c^q w_{n+1})\in L_n$, there exists λ_0 such that

$$F(c^p z_{n+1},c^q w_{n+1})=(\lambda_0^p z_n,\lambda_0^q w_n).$$

Since $F^{-1}(0) = 0$ then $\lambda_0 \neq 0$, and therefore $\phi_n(c) = f(c^p z_{n+1}, c^q w_{n+1}) = \lambda_0^p z_n \neq 0$, which is a contradiction (notice $z_n \neq 0$). Therefore $\phi_n(\lambda) = c\lambda^k$ for some k. where k is at most of the multiplicity of F. We can also assume that all ϕ_n have the same multiplicity, say N, which is at most the multiplicity of F. Therefore $\phi_n(\lambda) = c_n \lambda^N$.

Consider the Taylor series of $f(z,w)$ in a small neighborhood of $(0,0)$

$$f(z,w) = \sum_{\alpha,\beta} A_{\alpha\beta} z^\alpha w^\beta.$$

Rewrite $f(z,w)$ in terms of weight (p,q),

$$f(z,w) = \sum_{m=0}^{\infty} f_m(z,w)$$

where

$$f_m(z,w) = \sum_{p\alpha+q\beta=m} A_{\alpha\beta} z^\alpha w^\beta.$$

We have

$$\phi_n(\lambda) = \sum_{m=0}^{\infty} \lambda^m f_m(z_{n+1}, w_{n+1})$$

From $\phi_n(\lambda) = c_n\lambda^N$, for all n, it follows that $f_j(z_{n+1}, w_{n+1})$ when $j \neq N$ and for all n. However, a simple computation shows, invoking $a_{n+1} = w_{n+1}^p / z_{n+1}^q$

$$\begin{aligned} f_j(z_{n+1}, w_{n+1}) &= \sum_{p\alpha+q\beta=j} A_{\alpha\beta} z_{n+1}^\alpha w_{n+1}^\beta \\ &= z_{n+1}^{j/p} \sum_{p\alpha+q\beta=j} A_{\alpha\beta} a_{n+1}^{\beta/p}. \end{aligned}$$

Since a_n are different, we conclude that $a_n^{1/p}$ are zeros of the polynomial:

$$\sum_{p\alpha+q\beta=j} A_{\alpha\beta} t^\beta = 0,$$

where $j \neq N$. A polymonial cannot have infinitely many zero unless it is zero. Therefore, we conclude that

$$A_{\alpha\beta} = 0$$

whenever $p\alpha + q\beta \neq N$. Therefore we have that f is a quasi-homogeneous polynomial of weight (p, q) of degree N, i.e.,

$$f(z, w) = \sum_{p\alpha+q\beta=N} A_{\alpha\beta} z^{\alpha} w^{\beta}.$$

Similarly, using $\psi_n(\lambda)$ we can prove that $g(z, w)$ is a quasi-homogeneous polynomial of weight (p, q). In fact, we notice

$$g^p(\lambda^p z_{n+1}, \lambda^q w_{n+1}) = a_n f^q(\lambda^p z_{n+1}, \lambda^q w_{n+1}),$$

which implies that $\psi_n(\lambda) = c_n \lambda^l$ for some l and that the degree of g is Nq/p. Since $(p, q) = 1$, we have $N = kp$, and the degree of g is qk. We set

$$f(z, w) = \sum_{p\alpha+q\beta=kp} A_{\alpha\beta} z^{\alpha} w^{\beta}$$

$$g(z, w) = \sum_{p\alpha+q\beta=kq} B_{\alpha\beta} z^{\alpha} w^{\beta}.$$

That is to say F is a quasi-homogeneous polynomial map of weight (p, q) of degree k, so the lemma is proved.

To complete the proof of the theorem, we have to use some facts from two variables complex dynamics. Let Ω_F be the basin of attraction of F. Then there exists a unique homogeneous polynomial map Q given in the last section such that

$$\Phi \circ F = Q \circ \Phi,$$

where $\Phi = (z^q, w^p)$ as before. We first claim that $\Omega_F = \Omega$. Indeed, assume that F^n converges to G on Ω. For every $\lambda \in \mathbf{C}$, $|\lambda| < 1$, we have $F^n(\lambda^p z, \lambda^q w) \to G(\lambda^p z, \lambda^q w)$ as $n \to \infty$. However, F^n is a quasi-homogeneous polynomial map of weight (p, q) of degree k^n, therefore

$$F^n(\lambda^p z, \lambda^q w) = \lambda^{k^n} F^n(z, w)$$

which converges to zero. Hence $G = 0$ and $\Omega \subset \Omega_F$. But F is proper and $F(\partial\Omega) = \partial\Omega$. Hence Ω_F is contained in Ω.

Now we notice that $\Phi(\Omega) = \Omega_Q$. Therefore the boundary of Ω_Q is smooth except possibly when $z = 0$ or $w = 0$. Therefore Ω_Q contains strictly pseudoconvex points.

We denote by $\pi : \mathbf{C}^2 \backslash \{0\} \longrightarrow \mathbf{C}P$ the canonical projection. Since Q is nondegenerate, it takes lines to lines in $\mathbf{C}^2$ and naturally induces a rational

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mapping $\varphi : \mathbf{C}P \longrightarrow \mathbf{C}P$ on the projective space. We claim that its Julia set J_φ does not coincide with $\mathbf{C}P$. For the proof we apply an argument of [6]. Suppose by contradiction it does. It is known (see [8, pp. 56–58]) that in this case for every point $a \in J_\varphi$ there exists a neighborhood U and a positive integer n such that $\cup_{k=1}^n \varphi^k(U)$ covers $\mathbf{C}P$. Take a such that $\pi^{-1}(a)$ contains a strictly pseudoconvex point p in $\partial\Omega_Q$. Then there exists a neighborhood W of p in $\mathbf{C}^2$ such that $\cup_{k=1}^n Q^k(W)$ covers $\partial\Omega_Q$. Since Q takes any strictly pseudoconvex point to a strictly pseudoconvex one, we get that Ω is strictly pseudoconvex and by [13] V_Q is empty: a contradiction. Thus, J_ϕ is different from $\mathbf{C}P$. But by the classical results J_ϕ is a closed subset of $\mathbf{C}P$ with empty interior. Therefore $\partial\Omega_Q \cap \pi^{-1}(J_\phi)$ is a nonempty open subset of $\partial\Omega_Q$ which in view of [11], Proposition 7.1, is foliated by Riemann surfaces; this is impossible since Ω_Q is a proper image of the finite type domain Ω_Q. This completes the proof of the theorem.

Case 2. Incomplete quasi-circular domain of type (p,q).

Now we assume that Ω is not complete. We first observe that $(0,0) \notin \overline{\Omega}$. Indeed, this follows from using continuity principle and pseudoconvexity.

Given a point $p = (z,w)$ in $\partial\Omega$, we consider the complex curve

$$L_p = \{(\lambda^p z, \lambda^q w) : \lambda \in \mathbf{C}\}.$$

Now we go back to the proof of the theorem. By Lemma 1, we have $L_n \subset V_{F^n}$ and $F(L_{n+1}) = L_n$, and F is proper from L_{n+1} to L_n with multiplicity m for all n.. By the fact proved above $L_n = L_{p_n} \cap \Omega$ where p_n is a boundary point. Since Ω is incomplete, there exist $r_n, R_n, R_n > r_n > 0$ such that

$$L_n = \{(\lambda^p z_n, \lambda^q w_n) : r_n < |\lambda| < R_n\}.$$

We assume $r_n \to r, R_n \to R$. Since $\overline{\Omega}$ does not contain $(0,0)$, we have $r > 0$. Consider the map $\phi \circ F \circ \pi(\lambda) : \{r_{n+1} < |\lambda| < R_{n+1}\} \to \{r_n < |\lambda| < R_n\}$ where $\phi : \lambda \to (\lambda^p z_{n+1}, \lambda^q w_{n+1})$, and π is the projection from $\mathbf{C}^2 \to \mathbf{C}$. Then it is a proper map. By Lemma 3, we have

$$\frac{R_{n+1}}{r_{n+1}} = \left(\frac{R_n}{r_n}\right)^m,$$

which implies that

$$\frac{R_{n+1}}{r_{n+1}} = \left(\frac{R_1}{r_1}\right)^{nm}.$$

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Letting $n \to \infty$, we see the left hand side goes to R/r, while the right hand side goes to infinity, arriving at a contradiction, and the proof of the theorem is complete.

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