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P.J. RYAN'S PROBLEM IN SEMI-RIEMANNIAN SPACE FORMS

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Abstract. We prove that the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent for hypersurfaces of a 5-dimensional semi-Riemannian space form $N^5(c)$. This solves a problem by P.J. Ryan in the case of hypersurfaces of dimension 4 in semi-Riemannian space forms.

1. Introduction. A semi-Riemannian manifold $(M, g), \dim M \geq 3$, is called *semisymmetric* [**17**] if

$$R \cdot R = 0 \tag{1}$$

holds on M. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. A semi-Riemannian manifold $(M, g), \dim M \geq 3$, is said to be *Ricci-semisymmetric* if the following condition is satisfied

$$R \cdot S = 0 \tag{2}$$

Again, the class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. It is clear that every semisymmetric manifold is Ricci-semisymmetric. However, the converse statement is not true, as can be seen for instance from material in [**5**].

Although the conditions (1) and (2) do not coincide for manifolds in general, it is a long-standing question whether the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent for hyper-surfaces of Euclidean spaces; cf. Problem P 808 of [**16**] by P.J. Ryan, and references therein. More generally, one can ask the same question for hypersurfaces of semi-Riemannian space forms.

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Whereas the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent on any 3-dimensional semi-Riemannian manifold, for $n \geq 3$ we have the following results. It had been proved in [**18**] that (1) and (2) are equivalent for hypersurfaces which have positive scalar curvature in a Euclidean space $\mathbf{E}^{n+1}, n \geq 3$. In [**15**] this result was generalized to hypersurfaces of a Euclidean space $\mathbf{E}^{n+1}$, n≥3, which have non-negative scalar curvature and also to hypersurfaces of constant scalar curvature. [**15**] also proves that (1) and (2) coincide for hypersurfaces of Riemannian space forms with nonzero constant sectional curvature. Further, in [**13**] it was proved that (1) and (2) are equivalent for hypersurfaces of an Euclidean space $\mathbf{E}^{n+1}, n \geq 3$, under the additional global condition of completeness. In [**3**] it was shown that (1) and (2) are equivalent for Lorentzian hypersurfaces of a Minkowski space $\mathbf{E}_1^{n+1}, n \geq 4$, [**3**] also proves that (1) and (2) are equivalent for para-Kähler hypersurfaces of a semi-Euclidean space $\mathbf{E}_s^{2m+1}, m \geq 2$. In [**2**] we proved that (1) and (2) are equivalent for hypersurfaces in a 5-dimensional Euclidean space $\mathbf{E}^5$.

In this paper, we generalize this last result to all 4-dimensional hypersurfaces of a semi-Riemannian space form. We prove the following

THEOREM. *For hypersurfaces of a semi-Riemannian space form $N^5(c)$, the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent.*

The paper is organized as follows. In section 2 we fix the notations and give precise definitions of the used symbols. Since our proof will rely on results for pseudosymmetric manifolds, we also introduce and comment on some elements of pseudosymmetry. In section 3, and for later use in the proof, we derive several identities valid for 4-dimensional hypersurfaces of a 5-dimensional semi-Riemannian space form; we also prove some technical lemmas. In section 4, we prove the theorem.

2. Preliminaries. Let (M, g) be an n-dimensional, $n \geq 3$, semi-Riemannian connected manifold of class C^∞. We denote by ∇, S and κ, the Levi-Civita connection, the Ricci tensor and the scalar curvature of (M, g), respectively. We define on M the endomorphisms $\tilde{R}(X, Y)$, $X \wedge Y$ and $\tilde{\mathcal{C}}(X, Y)$ by

$$\tilde{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, (X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$
$$\tilde{\mathcal{C}}(X, Y) = \tilde{\mathcal{R}}(X, Y) = \tilde{\mathcal{R}}(X, Y) + \frac{1}{n-2}\left(\frac{\kappa}{n-1} X \wedge Y - (X \wedge \mathcal{S}\tilde{Y} + \mathcal{S}\tilde{X} \wedge Y)\right)$$

respectively, where $X, Y, Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M, and the Ricci operator $\mathcal{S}$ of (M, g) is defined by $S(X, Y) = g(X, \mathcal{S}\tilde{Y})$. The (0,4)-tensor G is defined by $G(X_1, X_2 X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$. The Riemann curvature tensor R and the Weyl curvature tensor C of (M, G) are defined by

$$R(X_1, X_2, X_3, X_4) = g\big(\tilde{R}(X_1, X_2)X_3, X_4\big), C(X_1, X_2, X_3, X_4) = g\Big(\tilde{\mathcal{C}}(X_1, X_2)X_3, X_4\Big)$$

respectively. Further, for a symmetric (0,2)-tensor field A on M, we define the endomorphism $X \wedge_A Y$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Z, Y)X - A(Z, X)Y$, where $X, Y, Z \in \Xi(M)$. Evidently, we have $X \wedge_g Y = X \wedge Y$. For a (0,k)-tensor field T on

$M, k \geq 1$, and a symmetric (0,2)-tensor field A on M, we define the $(0, k+2)$-tensor fields $R \cdot T$ and $Q(A, T)$ by

$$\begin{aligned}(R \cdot T)(X_1, \ldots, X_k; X, Y) &= -T(\tilde{\mathcal{R}}(X, Y)X_1, X_2, \ldots, X_k) \\ &\quad - \cdots - T(X_1, \ldots, X_{k-1}, \tilde{\mathcal{R}}(X, Y)X_k), \\ Q(A, T)(X_1, \ldots, X_k; X, Y) &= \quad -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) \\ &\quad - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k)..\end{aligned}$$

Let M be a nondegenerate hypersurface of a semi-Riemannian manifold $(N, \tilde{g})$, $\dim N \geq 4$, and let g be the induced metric tensor on M from the metric $\tilde{g}$. If the ambient space is a semi-Riemannian space of constant curvature $N^{n+1}(c)$, and let E denote the (0,4)-tensor on M defined as $E(X_1, X_2, X_3, X_4) = H(X_1, X_4)H(X_2, X_3)$ $-H(X_1, X_3)H(X_2, X_4)$, then the Gauss equation can be written shortly in the form

$$R = \epsilon E + \frac{\tilde{\kappa}}{n(n+1)} G \tag{3}$$

where $\tilde{\kappa}$ is the scalar curvature of $N^{n+1}(c)$, and $\varepsilon = \tilde{g}(\xi, \xi)$, with ξ a unit normal. The shape operator $\mathcal{A}$ and the second fundamental tensor H are related by $g(\mathcal{A}(X)Y) = H(X, Y)$, where $X, Y \varepsilon \Xi(M)$; H is defined by means of

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X, Y)\epsilon\xi$$

Furthermore, for $k > 1$ we also define that $H^k(X, Y) = g(\mathcal{A}^k X, Y)$, and thus $tr(H^k) = tr(\mathcal{A}^k)$.

Contracting (3) we obtain

$$S = \epsilon\big(tr(H)H - H^2\big) + \frac{(n-1)\tilde{\kappa}}{n(n+1)} g \tag{4}$$

For the proof of the theorem, we will rely on results for certain generalizations of the semisymmetric and Ricci-semisymmetric manifolds, namely the pseudosymmetric and Ricci-pseudosymmetric manifolds, respectively.

A semi-Riemannian manifold M is said to be pseudosymmetric if at every point of M the following condition is satisfied (*) the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

This condition is equivalent with the existence of a real-valued function L_R, defined on the set $U_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, such that

$$R \cdot R = L_R Q(g, R) \tag{5}$$

holds on U_R. The class of pseudosymmetric manifolds contains the semisymmetric manifolds as a proper subset [**5**].

A semi-Riemannian manifold M is said to be Ricci-pseudosymmetric if at every point of M the following condition is satisfied (**) the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

This condition is equivalent with the existence of a real-valued function L_S, defined on the set $U_S = \{x \in M \mid S - \frac{k}{n} g \neq 0 \text{ at } x\}$, with S the Ricci tensor, such that

$$R \cdot S = L_S Q(g, S) \tag{6}$$

holds on U_S. Again, the class of Ricci-pseudosymmetric manifolds includes the set of Ricci-semisymmetric manifolds as a proper subset [**4**]. It is clear that every pseudosymmetric manifold is Ricci-pseudosymmetric. However, the converse statement is not true [**4**].

For more detailed information on the geometric motivation for the introduction of pseudosymmetric manifolds, and for a review of results on different aspects of pseudosymmetric spaces, see e.g. [**5**] and [**9**]. In particular, [**1**] studies the conditions of pseudosymmetry and Ricci-pseudosymmetry, realized on hypersurfaces of semi-Riemannian spaces of constant curvature. [**1**] gives extrinsic characterizations of pseudosymmetric and Ricci-pseudosymmetric hypersurfaces of semi-Riemannian spaces of constant curvature in terms of the shape operator.

Analogously to P.J. Ryan's problem for the conditions (1) and (2), one could ask a similar question for pseudosymmetric and Ricci-pseudosymmetric hypersurfaces, respectively. Although the conditions (5) and (6) do not coincide for manifolds in general, one could state the problem whether or not the conditions $R \cdot R = L_R Q(g, R)$ and $R \cdot S = L_S Q(g, S)$ are equivalent for hypersurfaces of semi-Riemannian spaces of constant sectional curvature. But, it is known that this question has a negative answer in general by the existence of non-pseudosymmetric, Ricci-pseudosymmetric hypersurfaces of $S^{n+1}(c)$. Namely, in [**8**] it was shown that Cartan hypersurfaces of $S^{n+1}(c)$, $n = 6, 12, 24$, are such hypersurfaces. This however does not exclude that the conditions (5) and (6) may be equivalent for hypersurfaces in some special cases (see Proposition 4.1).

3. Hypersurfaces of $N^5(c)$. For later use in the proof, we derive several specific indentities for 4-dimensional hypersurfaces of a 5-dimensional semi-Riemannian space form. General identities for hypersurfaces of semi-Riemannian space forms with arbitrary dimension, can be found in [**1**].

Thus, we assume now that M is a hypersurface of a 5-dimensional semi-Riemannian space of constant sectional curvature $N^5(c)$. Using (3) we can present the local components C_{hijk} of the Weyl conformal curvature tensor C of M in the following form

$$C_{hijk} = \varepsilon E_{hijk} + \left(\frac{\kappa}{6} + \frac{\tilde{\kappa}}{20}\right) G_{hijk} - \frac{1}{2}(g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} - g_{ik}S_{hj}). \tag{7}$$

On the other hand, the following identity is satisfied for every 4-dimensional semi-Riemannian manifold (M, g) [**14**]:

$$\begin{aligned} 0 = & g_{hm}C_{likj} + g_{lm}C_{ihjk} + g_{im}C_{hljk} + g_{hj}C_{likm} + g_{lj}C_{ihkm} + g_{ij}C_{hlkm} \\ & + g_{hk}C_{lijm} + g_{lk}C_{ihmj} + g_{ik}C_{hlmj} \end{aligned} \tag{8}$$

Substituting (7) in (8) we get

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$$
\begin{aligned}
0 =\varepsilon(& g_{hm}E_{likj} + g_{lm}E_{ihkj} + g_{im}E_{hljk} + g_{hj}E_{likm} + g_{lj}E_{ihkm} \\
&+ g_{ij}E_{hlkm} + g_{hk}E_{limj} + g_{lk}E_{ihmj} + g_{ik}E_{hlmj}) \\
&- S_{hm}G_{lijk} - S_{lm}G_{ihjk} - S_{im}G_{hljk} - S_{hj}G_{likm} - S_{lj}G_{ihkm} \\
&- S_{ij}G_{hlkm} - S_{hk}G_{limj} - S_{lk}G_{ihmj} - S_{ik}G_{hlmj} \\
&+ \left(\frac{\kappa}{6} + \frac{\tilde{\kappa}}{20}\right)(g_{hm}G_{lijk} + g_{lm}G_{ihjk} + g_{im}G_{hljk} + g_{hj}G_{likm} \\
&+ g_{lj}G_{ihkm} + g_{ij}G_{hlkm} + g_{hk}G_{limj} + g_{lk}G_{ihmj} + g_{ik}G_{hlmj})].
\end{aligned} \tag{9}
$$

Transvecting this with $H^{hm} = H_{rs}g^{rh}g^{sm}$ we find

$$
\begin{aligned}
0 =& tr(H)(H_{lk}H_{ij} - H_{lj}H_{ik}) + \alpha G_{lijk} \\
&- (H_{ij}H^2_{lk} + H_{lk}H^2_{ij} - H_{ik}H^2_{jl} - H_{jl}H^2_{ik}) \\
&- g_{ij}(H^3_{lk} - tr(H)H^2_{lk} - \beta H_{lk}) - g_{lk}(H^3_{ij} - tr(H)H^2_{ij} - \beta H_{ij}) \\
&+ g_{jl}(H^3_{ik} - tr(H)H^2_{ik} - \beta H_{ik}) + g_{ik}(H^3_{lj} - tr(H)H^2_{lj} - \beta H_{lj}),
\end{aligned} \tag{10}
$$

where

$$
\alpha = \frac{1}{3}\left(tr(H)\left(\varepsilon\frac{1}{2}\kappa - tr(H^2)\right) + tr(H^3)\right), \tag{11}
$$

$$
\beta = \frac{1}{3}\left(tr(H^2) - (trH))^2\right) + \frac{2}{3}\varepsilon\left(\frac{3}{20}\tilde{\kappa} - \frac{1}{4}\kappa\right). \tag{12}
$$

Lemma 3.1. *Let M be a hypersurface of a 5-dimensional semi-Riemannian space of constant curvature $N^5(c)$. If a point $x \in M$ is not umbilical then the following relation holds at x.*

$$
H^4 = tr(H)H^3 + \beta H^2 + \alpha H + \lambda g, \quad \lambda \in \mathbf{R}, \tag{13}
$$

where α and β as defined by (11) *and* (12)*, respectively, now in addition also satisfy the following two relations:*

$$
\alpha = \frac{1}{3}\left(tr(H^3) - tr(H)\left(tr(H^2) - \beta\right)\right), \tag{14}
$$

$$
\beta = \frac{1}{2}\left(tr(H^2) - (tr(H))^2\right). \tag{15}
$$

Proof. Contracting (10) with g^{ij}, we obtain

$$
\left((tr(H))^2 - tr(H^2) + 2\beta\right)H_{lk} + \left(3\alpha - tr(H^3) + tr(H)\left(tr(H^2) + \beta\right)\right)g_{lk} = 0. \tag{16}
$$

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Since x is not an umbilical point of M, (16) implies (14) and (15). Next, transvecting (10) with H^l_h and symmetrising the resulting equality in h, i, we get

$$Q\big(g, H^4 - tr(H)H^3 - \beta H^2 - \alpha H\big) = 0,$$

which implies (13), completing the proof.

LEMMA 3.2. *Let M be a hypersurface of a 5-dimensional semi-Riemannian space of constant curvature $N^5(c)$. If at a point $x \in M$ the relations: $H^3 - tr(H)H^2 - \beta H = 0$ and $\alpha = 0$ are satisfied then the equality $Q(H^2, E) = 0$ holds at x, with E as defined in section 2 and α given by (11).*

Proof. By our assumptions, (10) reduces to

$$tr(H)E_{lijk} = F_{lijk}, \tag{17}$$

where $F_{lijk} = H_{ij}H^2_{lk} + H_{lk}H^2_{ij} - H_{ik}H^2_{jl} - H_{jl}H^2_{ik}$. From (17) there follows that $tr(H)Q(H, E) = Q(H, F)$ holds at x, where F is the (0,4)-tensor at x with local components F_{lijk}. Since the tensor $Q(H, E)$ vanishes identically, we get $Q(H, F) = 0$. On the other hand, it is easy to check that $Q(H, F) = -Q(H^2, E)$. Hence the tensor $Q(H^2, E)$ vanishes at x. But this completes the proof.

PROPOSITION 3.1. [7] *(Lemma 1) Let (M, g) be a hypersurface of a semi-Riemannian space of constant sectional curvature $N^{n+1}(c), n \geq 3$. If the shape operator $\mathcal{A}$ of M satisfies at a point $x \in M$ the condition*

$$\mathcal{A}^2 = \alpha\mathcal{A} + \beta Id, \quad \alpha, \beta, \in \mathbb{R}, \tag{18}$$

then the following relation is fulfilled at x

$$R \cdot R = \left(\frac{\tilde{\kappa}}{n(n+1)} - \varepsilon\beta\right)Q(g, R). \tag{19}$$

Let M be a hypersurface of a semi-Riemannian space of constant curvature $N^5(c)$. In addition, we assume that x is a point of M at which the following relation is satisfied

$$S = \frac{\kappa}{4}g. \tag{20}$$

Substituting (20) into (4) and (9), respectively, we obtain

$$H^2_{hm} = tr(H)H_{hm} + \varepsilon\tau g_{hm} \tag{21}$$

and

$$\begin{aligned} &g_{hm}E_{lijk} + g_{lm}E_{ihjk} + g_{im}E_{hljk} + g_{hj}E_{likm} + g_{lj}E_{ihkm} \\ &+ g_{ij}E_{hlkm} + g_{hk}E_{limj} + g_{lk}E_{ihmj} + g_{ik}E_{hlmj} \\ &+ \frac{1}{3}\varepsilon\tau(g_{hm}G_{lijk} + g_{lm}G_{ihjk} + g_{im}G_{hljk} + g_{hj}G_{likm} \\ &+ g_{lj}G_{ihkm} + g_{ij}G_{hlkm} + g_{hk}G_{limj} + g_{lk}G_{ihmj} + g_{ik}G_{hlmj}) = 0, \end{aligned} \tag{22}$$

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where

$$\tau = \frac{3}{20}\tilde{\kappa} - \frac{\kappa}{4} \tag{23}$$

Let V^h be the local components of a vector $V \in T_x(M)$ such that $\rho = g_{ij}V^iV^j \neq 0$. Transvecting (21) and (22) with V^h, we get, respectively

$$W^h H_{hm} = tr(H)W_m + \varepsilon\tau V_m \tag{24}$$

and

$$\begin{aligned} &V_m(E_{lijk} + \varepsilon\tau G_{lijk}) + W_m(g_{lk}H_{ij} + g_{ij}H_{lk} - g_{lj}H_{ik} - g_{ik}H_{lj}) \\ &+ V_j(E_{likm} + \varepsilon\tau G_{likm}) + W_j(g_{lm}H_{ik} + g_{ik}H_{lm} - g_{lk}H_{im} - g_{im}H_{lk}) \\ &+ V_k(E_{limj} + \varepsilon\tau G_{limj}) + W_k(g_{lj}H_{im} + g_{im}H_{lj} - g_{lm}H_{ij} - g_{ij}H_{lm}) = 0, \end{aligned} \tag{25}$$

where $W_j = V^hH_{hj}$, $W^i = g^{ih}W_h$ and $V_j = g_{hj}V^h$. Further, transvecting (24) and (25) with V^m, we find, respectively

$$\tilde{\rho} = W^hW_h = tr(H)\bar{\rho} + \varepsilon\tau\rho, \quad \text{where} \quad \bar{\rho} = V^hW_h \tag{26}$$

and

$$\begin{aligned} &\rho E_{lijk} + (V_lW_j + V_jW_l - \bar{\rho}g_{jl})H_{ik} + (V_iW_k + V_kW_i - \bar{\rho}g_{ik})H_{lj} \\ &- (V_lW_k + V_kW_l - \bar{\rho}g_{lk})H_{ij} - (V_iW_j + V_jW_i - \bar{\rho}g_{ij})H_{lk} \\ &+ g_{lj}(\varepsilon\tau V_iV_k + W_iW_k) - g_{ij}(\varepsilon\tau V_lV_k + W_lW_k) - g_{lk}(\varepsilon\tau V_iV_j + W_iW_j) \\ &+ g_{ik}(\varepsilon\tau V_jV_l + W_jW_l) + \varepsilon\tau\rho G_{lijk} = 0. \end{aligned} \tag{27}$$

From (27), by transvection with W^l, we obtain

$$tr(H)\big((\rho W_k - \bar{\rho}V_k)H_{ij} - (\rho W_j - \bar{\rho}V_j)H_{ik} + (W_jV_k - W_kV_j)W_i\big) = 0 \tag{28}$$

Proposition 3.2. *Let M be a hypersurface of a semi-Riemannian space of constant curvature $N^5(c)$. If at a point $x \in M$ we have: $R \neq \frac{\kappa}{12}G$ and $S = \frac{\kappa}{4}g$ (i.e. $x \in U_R - U_S$) then*

$$tr(H) = 0 \tag{29}$$

holds at x.

Proof. We suppose that $tr(H)$ is nonzero at x. Now (28) turns into

$$(\rho W_k - \bar{\rho}V_k)H_{ij} - (\rho W_j - \bar{\rho}V_j)H_{ik} + (W_jV_k - W_kV_j)W_i = 0. \tag{30}$$

Transvecting (30) with W^k and using (24) and (26) we find

$$(\rho\tilde{\rho} - \bar{\rho}^2)H_{ij} = (\rho tr(H) - \bar{\rho})W_iW_j - \varepsilon\bar{\rho}\tau V_iV_j + \varepsilon\bar{\rho}\tau(W_jV_i + W_iV_j). \tag{31}$$

We note that the vector Z, with local components $Z_k = \rho W_k - \bar{\rho}V_k$ must be nonzero.

Indeed, if we had $\rho W_k - \bar{\rho} V_k = 0$, then would also

$$V^h(H_{hk} - \frac{\bar{\rho}}{\rho} g_{hk}) = 0 \tag{32}$$

hold at x. Since V is an arbitrary nonzero vector at x, with $\rho = g_{ij}V^iV^j \neq 0$, (32) implies that the tensor H is proportional to g at x. Now (3) yields $R = \frac{\kappa}{12}G$, a contradiction.

Let X^i be the local components of a nonzero vector $X \in T_x(M)$ orthogonal to the vectors V and W (with local components W^i). Transvecting (30) with X^j we find $(\rho W_k - \bar{\rho} V_k)X^j H_{ij} = 0$, whence

$$X^j H_{ij} = 0. \tag{33}$$

Furthermore, transvecting (3) with X^h and using (33), we obtain

$$X^h R_{hijk} = \frac{\tilde{\kappa}}{20}(X_k g_{ij} - X_j g_{ik}), \; X_k = g_{hk}X^h.$$

Contracting this with g^{ij} and using (20) and (23) we get $\tau X_k = 0$, whence $\tau = 0$. Now, by the last equality, (31) reduces to

$$(\rho\tilde{\rho} - \bar{\rho}^2)H_{ij} = \big(\rho tr(H) - \bar{\rho}\big)W_i W_j. \tag{34}$$

We note that $\rho\tilde{\rho} = \bar{\rho}^2$.

Indeed, if we had $\rho\tilde{\rho} \neq \bar{\rho}^2$ then (34) gives *rank* $H \leq 1$ and (3) again reduces to $R = \frac{\kappa}{12}G$, a contradiction.

Thus (34) turns into

$$(\rho tr(H) - \bar{\rho})W_i = 0. \tag{35}$$

But from the equality $W_i = V^s H_{si} = 0$ it follows that H vanishes at x, a contradiction.

Thus (35) must reduce to

$$\rho tr(H) = \bar{\rho}, \tag{36}$$

whence it follows that $\bar{\rho}$ must be nonzero. Transvecting (27) with X^l and using (33) we get

$$X_j(H_{ik} - \frac{1}{\bar{\rho}} W_i W_k) = X_k(H_{ij} - \frac{1}{\bar{\rho}} W_i W_j).$$

Without loss of generality we can choose the vector X in such a way that $X^k X_k \neq 0$. The last equality, by transvection with X^j, yields

$$X^j X_j(H_{ik} - \frac{1}{\bar{\rho}} W_i W_k) = 0,$$

whence $H_{ik} = \frac{1}{\rho} W_i W_k$, which reduces (3) to $R = \frac{\kappa}{12} G$, a contradiction. Thus we see that (29) holds at x, which completes the proof.

From the last proposition we immediately deduce the following

Corollary 3.1. *Let M be an Einsteinian hypersurface of a semi-Riemannian space of constant curvature $N^5(c)$. If the set $U_R - U_S \subset M$ is a dense subset of M then the mean curvature of M vanishes identically on M*

4. Proof of the theorem. In this section, we prove the theorem that the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent for hypersurfaces of a semi-Riemannian space form $N^5(c)$. This generalizes the result of [**2**] to indefinite space forms.

We first prove that there are no properly Ricci-pseudosymmetric hypersurfaces in a 5-dimensional semi-Riemannian space form. Otherwise stated, we prove that the conditions (*) and (**) are equivalent for 4-dimensional hypersurfaces of a semi-Riemannian space of constant sectional curvature.

Proposition 4.1. *Every Ricci-pseudosymmetric hypersurface M of a 5-dimensional semi-Riemannian space of constant sectional curvature $N^5(c)$ is pseudosymmetric.*

Proof. In view of Lemma 4.1 of [**1**], our assertion is true at all points of M at which the tensor H^2 is a linear combination of H and g. Let now x be a point of M at which the tensor H^2 is not a linear combination of H and g. Thus, in view of Theorem 3.1 of [**1**], the relation $H^3 = tr(H)H^2 + \tau H$, $\tau \in \mathbf{R}$, holds at x. From the last equality we get $H^4 = tr(H)H^3 + \tau H^2$, $\tau \in \mathbf{R}$. Combining this with (13) we obtain $\beta = \tau$ and $\alpha = \lambda = 0$, where α, β and λ are defined by (11), (12) and (13), respectively. Thus we have $H^3 = tr(H)H^2 + \beta H$. Now Lemma 3.2, together with Theorem 4.1 of [**1**], completes the proof.

Using the above result we can prove the following

Proposition 4.2. *The conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent on the subset U_S of a hypersurface M of a 5-dimensional semi-Riemannian space of constant sectional curvature $N^5(c)$.*

Proof. It is clear that $R \cdot R = 0$ implies $R \cdot S = 0$. We assume now that $R \cdot S = 0$ holds at a point $x \in U_S$. In view of Proposition 4.1 we have at x

$$R \cdot R = LQ(g, R), \tag{37}$$

for some function $L : M^4 \to \mathbb{R}$. It is clear that this implies $R \cdot S = LQ(g, S)$, which yields $LQ(g, S) = 0$. Since the tensor S is not proportional to the metric tensor, L vanishes at x. Thus (37) reduces to $R \cdot R = 0$, completing the proof.

Let M be a Ricci-semisymmetric hypersurface of a semi-Riemannian space form $N^5(c)$. The above result states that $R \cdot R = 0$ holds on the set U_S. Furthermore, we note that (3) reduces on the set $M - U_R$ to $R = \frac{\tilde{\kappa}}{n(n+1)} G$, which implies that $R \cdot R = 0$ is fulfilled trivially on the set $M - U_R$. To finish the proof of our main theorem, we must prove that the condition $R \cdot R = 0$ is fulfilled on the set $U_R - U_S$. Let x be a point of $U_R - U_S$. From Proposition 4.2 there follows that we can assume without loss of generality that there exists a neighbourhood U of x contained in $U_R - U_S$. Thus the manifold (U, g) is an Einstein hypersurface of the 5-dimensional semi-Riemannian space of constant sectional curvature $N^5(c)$. It is clear that (21) holds on U. Further, in view of Proposition 3.2, (21) reduces on U to

$$H^2_{hm} = \varepsilon\tau g_{hm}.$$

[**10**] classifies the possible shape operators $\mathcal{A}$ for an Einstein hypersurface in an indefinite space form. The shape operator $\mathcal{A}$ is either diagonalizable or satisfies either $\mathcal{A}^2 = 0$, or $\mathcal{A}^2 = -b^2\mathrm{Id}(b \in \mathbb{R}, b \neq 0)$. We consider these 3 cases separately.

1. Let the condition $\mathcal{A}^2 = -b^2\mathrm{Id}, b = const.$, be satisfied on U. [**12**] gives an explicit description of the Einsteinian hypersurfaces in semi-Riemannian space forms with shape operator satisfying $\mathcal{A}^2 = -b^2\mathrm{Id}$, with $b = const. \neq 0$. The shape operator of any such hypersurface is parallel ([**12**], Proposition 2). Thus in particular we have

$$(R \cdot H)(X_1, X_4; X, Y) = 0. \tag{38}$$

But on the other hand, from (3) we get immediately the following identity

$$\begin{aligned}(R \cdot R)(X_1, X_2X_3, X_4; X, Y) &= H(X_2, X_3)(R \cdot H)(X_1, X_4; X, Y)\\ &+ H(X_1, X_4)(R \cdot H)(X_2, X_3; X, Y) - H(X_2, X_4)(R \cdot H)(X_1, X_3; X, Y)\\ &- H(X_1, X_3)(R \cdot H)(X_2, X_4; X, Y),\end{aligned}$$

which by (38) reduces to $R \cdot R = 0$. So, Einsteinian hypersurfaces with shape operator satisfying $\mathcal{A}^2 = -b^2\mathrm{Id}, b = const. \neq 0$, are semisymmetric.

2. Let the condition $\mathcal{A}^2 = 0$ be satisfied on U. From Proposition 3.1 it follows that

$$R \cdot R = \frac{\tilde{\kappa}}{n(n+1)} Q(g, R) \tag{39}$$

holds on U. [**11**] considers Einsteinian hypersurfaces in semi-Riemannian spaces of constant sectional curvature with shape operator satisfying $\mathcal{A}^2 = 0$. [**11**] proves that if $\mathcal{A}$ has maximal rank (this is rank $\mathcal{A} = 2$ in our case), then $c = 0$, hence also $\tilde{\kappa} = 0$. Note that if rank $\mathcal{A} \leq 1$ then (3) reduces to $R = \frac{\tilde{\kappa}}{n(n+1)} G$. So, Einsteinian hypersurfaces with shape operator satisfying $\mathcal{A}^2 = 0$ also are semisymmetric.

3. Finally, we assume that $\mathcal{A}^2 = b^2\mathrm{Id}, b = const. \neq 0$, holds on U. From [**10**] it follows that $\mathcal{A}$ must be diagonalizable at every point of U. The Einstein hypersurfaces with diagonalizable shape operator were classified in [**9**]. From Theorem 7.1 of [**9**] it follows that any such hypersurface is a space of constant curvature or a Cartesian product of two spaces of constant curvatures, and thus is semisymmetric.

Summarizing all subcases, we have proved that every Ricci-semisymmetric hypersurface M of a semi-Riemannian space form $N^5(c)$ also satisfies $R \cdot R = 0$. Our main theorem is thus proved.

From the above considerations we also immediately have the following result.

Corollary 4.1. *Any Einsteinian hypersurface of a semi-Riemannian space of constant curvature $N^5(c)$ is semisymmetric*

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