NEW FORMULATION OF DE SITTER'S THEORY OF MOTION FOR JUPITER I–IV. I. EQUATIONS OF MOTION AND THE DISTURBING FUNCTION

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ABSTRACT

A brief discussion is given of the basic features of de Sitter's theory. The main advantage of his theory is that it contains no small divisors, thanks to the use of elliptic rather than circular intermediate orbits in the first approximation. A 50-year extension of the satellite observations available to de Sitter makes it desirable to rederive the elements of his intermediate orbits, whose perijoves have a common retrograde motion. Furthermore, the theory suffers from a convergence problem, which can be avoided by reformulating the theory in terms of canonical variables, a task that is begun here. We adopt a formulation in Poincaré's canonical relative coordinates rather than, as customary, in ordinary relative coordinates or in the Jacobian canonical coordinates. By means of the generalized Newcomb operators devised by Izsak, the disturbing function is expanded in a form that is very convenient for use with the modified Delaunay variables, G, L - G, H - G, $\ell + \omega + \Omega$, ℓ, and Ω and their associated Poincaré variables.

1. INTRODUCTION

In a paper entitled "Outlines of a New Mathematical Theory of Jupiter's Satellites," de Sitter (1918) introduced an entirely new approach to the problem of the motions of the Galilean satellites (Io, Europa, Ganymede, and Callisto). In the many earlier treatments of this problem, whose difficulty arises from the strong mutual attractions, Wargentin, Lagrange, Laplace, Souillart, Sampson, and others adopted circular and coplanar intermediate orbits as a first approximation. The motions of the three inner satellites are characterized by an exact commensurability among their mean motions n_1, n_2, and n_3,

$$n_1 - 3n_2 + 2n_3 = 0 \quad , \tag{1}$$

and the near-commensurabilities

$$n_1 = 2n_2 + \kappa = 4n_3 + 3\kappa \quad , \tag{2}$$

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V. Szebehely (ed.), Dynamics of Planets and Satellites and Theories of Their Motion, 189-206.

with

$$\kappa = n_1 - 2n_2 = n_2 - 2n_3 \approx \frac{1}{68} n_3 \quad .$$

The first relation results in the famous Laplace libration condition on the mean longitudes λ_1, λ_2, and λ_3,

$$\lambda_1 - 3\lambda_2 + 2\lambda_3 = 180° + \theta \quad , \tag{3}$$

where the libration argument θ has a period of about 6 years and nearly zero amplitude. Equation (2) is, in a sense, more troublesome since it gives rise to small divisors and therefore slowly convergent series for the mean longitudes. de Sitter took advantage of the fact that, in a coordinate system having a prograde rotation with the angular velocity κ, the satellites will have mean motions $c_i = n_i - \kappa$, $i = 1, 2, 3, 4$, the first three of which satisfy the relations

$$c_1 - 3c_2 + 2c_3 = 0 \tag{4}$$

and

$$c_1 = 2c_2 = 4c_3 \quad . \tag{5}$$

Now, (4) corresponds exactly to (1), while in (5), the near-commensurabilities expressed by (2) are turned into exact ones. The circular orbits mentioned are periodic orbits (Poincaré's first kind) for the special initial conditions imposed by (1) and (2). de Sitter discovered that a set of elliptic periodic orbits (Poincaré's second kind) is similarly associated with (4) and (5). The prograde rotation is imparted by giving the four perijoves a common retrograde motion, $-\kappa$, in the fixed coordinate system. de Sitter derived numerical values of the elements of these so-called variation orbits by imposing the necessary periodicities on the equations of motion, with the disturbing function limited to its "secular" and "critical" parts. The resulting particular solution of the problem thus limited turns into a general solution through the addition of the purely periodic "variations," derived by the Lagrangian method of varying the arbitrary constants. Finally, the remainder of the disturbing function gives rise to periodic terms that de Sitter simply called "perturbations," also derived by means of the Lagrangian method.

The greatest advantage of de Sitter's approach is that no small divisors appear at any stage of the solution. Furthermore, the elliptic intermediaries, plus the relatively simple variations, include not only the troublesome long-period terms, which in the earlier theories contain the small divisors, but also the short-period "great inequalities" and the libration. However, de Sitter was disappointed to find that, owing to the presence of an infinite secular determinant, exponential terms of the type $\epsilon_1 \exp(\beta t) + \epsilon_2 \exp(-\beta t)$ appeared in the expressions for the perturbations in longitude, and although ϵ_1 and ϵ_2 are very small, this term will ultimately cause divergence. In his Darwin Lecture, de Sitter (1931) discussed this problem and announced the future publication of the complete expressions for the perturbations beyond the first order. Probably because of a subsequent sickness that caused his death in 1934, these expressions were not completed; at least they never appeared in print. This incompleteness of the theory and the lack of convenient tables for ephemeris calculation, such as those published by Sampson (1910), have prevented practical applications of the theory, a very unfortunate circumstance since recent results (Aksnes and Franklin, 1975) indicate that de Sitter's theory, as far as it can be applied, is at least as accurate as that due to Sampson, although both are now in need of revision (Lieske, 1975; Aksnes and Franklin, 1976).

The reasons for undertaking a new formulation of de Sitter's theory can be summarized as follows. First, it is desirable to derive new values for the elements of the variation orbits for a current epoch, on the basis of an almost 50-year extension of the satellite observations available to de Sitter. Of particular interest are the series of plates of the Galilean satellites taken by D. Pascu with the Leander-McCormick refractor during the last decade and the highly accurate photometric observations of the mutual satellite events in 1973. Even more accurate observations, in the form of range or range-rate data on the satellites, can be expected in the near future from an on-going experiment with the Arecibo radio telescope. It is vital that the orbital elements be as precise as possible since they enter the theory in numerical form and cannot be changed subsequently without redoing the theory. Second, de Sitter claimed that the afore-mentioned exponential terms can be avoided by using Delaunay's or von Zeipel's perturbation method in terms of canonical elements, instead of the Lagrangian method in terms of Kepler elements. Rather than adopting either of the two first-mentioned methods, we propose to use the more elegant canonical method due to Hori (1966), which is based on Lie series. A canonical formulation has the added advantage of simplifying the equations of motion and the construction of the theory, provided the disturbing function is expanded in an appropriate way.

Thus, the goal of our undertaking is not to revise de Sitter's theory in its original form, but to construct a new theory that will incorporate only the most essential features of the old one. In the two remaining sections, we present the first part of this work on the equations of motion and the expansion of the disturbing function, to be followed by later parts on the derivation of the variation orbits, the variations, and the perturbations, in de Sitter's terminology.

2. THE EQUATIONS OF MOTION

To apply Hori's perturbation method, we need a canonical formulation of the equations of motion with a common Hamiltonian. A formulation of this kind due to Jacobi has been widely used in investigations of the three-body problem, and Marsden (1964) adopted it in his thesis on the short-period terms in the motions of the Galilean satellites. Jacobi's method amounts to choosing a different origin for the coordinates of each body, such that the second body is referred to the first and each succeeding body is referred to the center of mass of all the preceding ones. Unfortunately, the use of the Jacobian coordinates complicates the expansion of the disturbing function considerably. There is a simpler canonical formulation* due to Poincaré (1897) and advocated by Charlier (1902) for use on the three-body problem, although they did not attempt to apply the method. In the following adaptation of the method to the problem at hand, we shall use a notation similar to that introduced by Marsden.

*I am indebted to Dr. Hori for pointing out the existence of this formulation.

Let m_p $(p = 0, 1, \ldots, n)$ be the masses and ξ_p, η_p, and ζ_p be the cartesian coordinates, referred to the center of mass of the system, of $n + 1$ interacting bodies. We take m_0 (Jupiter) to be the central mass to which we wish to refer the coordinates x_p, y_p, z_p $(p = 1, 2, \ldots, n)$ of the remaining masses (satellites plus perturbing bodies):

$$x_p = \xi_p - \xi_0 \quad , \qquad y_p = \eta_p - \eta_0 \quad , \qquad z_p = \zeta_p - \zeta_0 \quad . \tag{6}$$

It is well known that if $X_p = m_p\dot{x}_p$, $Y_p = m_p\dot{y}_p$, and $Z_p = m_p\dot{z}_p$ are taken as the momenta conjugate to the coordinates x_p, y_p, and z_p, only a semi-canonical formulation is achieved in which each body has its own Hamiltonian. If, instead, we define the momenta by

$$X_p = \frac{\partial T}{\partial \dot{x}_p} \quad , \qquad Y_p = \frac{\partial T}{\partial \dot{y}_p} \quad , \qquad Z_p = \frac{\partial T}{\partial \dot{z}_p} \quad , \tag{7}$$

where, in terms of the inertial velocities,

$$T = \frac{1}{2} \sum_{q=0}^{n} m_q(\dot{\xi}_q^2 + \dot{\eta}_q^2 + \dot{\zeta}_q^2) \quad , \tag{8}$$

we have Hamilton's canonical equations,

$$\left.\begin{aligned} \frac{dX_p}{dt} &= \frac{\partial F}{\partial x_p} \;, & \frac{dY_p}{dt} &= \frac{\partial F}{\partial y_p} \;, & \frac{dZ_p}{dt} &= \frac{\partial F}{\partial z_p} \\ \frac{dx_p}{dt} &= -\frac{\partial F}{\partial X_p} \;, & \frac{dy_p}{dt} &= -\frac{\partial F}{\partial Y_p} \;, & \frac{dz_p}{dt} &= -\frac{\partial F}{\partial Z_p} \end{aligned}\right\} p = 1, 2, \ldots, n \quad , \tag{9}$$

where the Hamiltonian F represents (the negative of) the total energy of the system,

$$F = -T + k^2 \sum_{q=1}^{n} \sum_{r=0}^{q-1} \frac{m_q m_r}{r_{qr}} \quad , \tag{10}$$

k^2 being the constant of gravitation and $r_{qr}^2 = (x_q - x_r)^2 + (y_q - y_r)^2 + (z_q - z_r)^2$. In order to derive explicit expressions for the momenta from (7), we must express the kinetic energy T in terms of the relative velocities. In deriving the transformation from the inertial frame to the relative frame, we consider only the x components, with the understanding that the y and z components transform in the same way. By means of (6) and the relation

$$\sum_{q=0}^{n} m_q \xi_q = 0 \quad ,$$

we readily deduce that

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$$\xi_0 = -\frac{1}{M}\sum_{q=1}^{n} m_q x_q \quad ,$$

$$\xi_p = x_p - \frac{1}{M}\sum_{q=1}^{n} m_q x_q \quad , \qquad p = 1, 2, \ldots, n \quad , \tag{11}$$

where M is the total mass of the system. If we differentiate the last two equations, and the corresponding ones for the η and ζ components, with respect to t and substitute the result in (8), we find, after some straightforward manipulation,

$$\begin{aligned} T &= \frac{1}{2}\sum_{q=1}^{n} m_q(\dot{x}_q^2 + \dot{y}_q^2 + \dot{z}_q^2) - \frac{1}{2M}\left[\left(\sum_{q=1}^{n} m_q\dot{x}_q\right)^2 + \left(\sum_{q=1}^{n} m_q\dot{y}_q\right)^2 \right. \\ &\quad \left. + \left(\sum_{q=1}^{n} m_q\dot{z}_q\right)^2\right] = \frac{1}{2M}\left[\sum_{q=1}^{n} m_q(M - m_q)(\dot{x}_q^2 + \dot{y}_q^2 + \dot{z}_q^2) \right. \\ &\quad \left. - 2\sum_{q=1}^{n}\sum_{r=1}^{q-1} m_q m_r(\dot{x}_q\dot{x}_r + \dot{y}_q\dot{y}_r + \dot{z}_q\dot{z}_r)\right] \end{aligned} \tag{12}$$

From (7), (11), and (12), it then follows that

$$X_p = m_p\dot{x}_p - \frac{m_p}{M}\sum_{q=1}^{n} m_q\dot{x}_q = m_p\dot{\xi}_p \quad ; \qquad p = 1, 2, \ldots, n \quad , \tag{13}$$

i.e., in Poincaré's canonical formulation, the momenta conjugate to the relative coordinates are related to the inertial velocities in the same way that they are related to the relative velocities in the semicanonical formulation. If we again make use of (6) and the relation

$$m_0\dot{\xi}_0 = -\sum_{q=1}^{n} m_q\dot{\xi}_q = -\sum_{q=1}^{n} X_p \quad ,$$

the desired form of T becomes

$$\begin{aligned} T &= \frac{1}{2}\left[\sum_{q=1}^{n}\frac{1}{m_q}(X_q^2 + Y_q^2 + Z_q^2) + \frac{1}{m_0}\left\{\left(\sum_{q=1}^{n} X_q\right)^2 + \left(\sum_{q=1}^{n} Y_q\right)^2 + \left(\sum_{q=1}^{n} Z_q\right)^2\right\}\right] \\ &= \frac{1}{2}\sum_{q=1}^{n}\frac{1}{m_q^*}(X_q^2 + Y_q^2 + Z_q^2) + \frac{1}{m_0}\sum_{q=2}^{n}\sum_{r=1}^{q-1}(X_qX_r + Y_qY_r + Z_qZ_r) \quad , \end{aligned} \tag{14}$$

where

$$\frac{1}{m_p^*} = \frac{1}{m_p} + \frac{1}{m_0} \quad , \qquad p = 1, 2, \ldots, n \quad . \tag{15}$$

If m_0 is much larger than m_p ($p = 1, 2, \ldots, n$) or if the latter masses do not come very close to each other, we can obtain an approximate solution by neglecting the last term in (14) and by limiting the potential to its main part with $r = 0$ in (10). The resulting intermediate orbits are thus obtained with a Hamiltonian,

$$F_0 = \sum_{q=1}^{n} \left[-\frac{1}{2m_q^*} (X_q^2 + Y_q^2 + Z_q^2) + \beta_q^2 \frac{m_q^*}{r_q} \right] , \tag{16}$$

where β_q^2 is a constant to be suitably chosen, and $r_q^2 = r_{q0}^2 = x_q^2 + y_q^2 + z_q^2$. With $F = F_0$, it follows from (9) that

$$\dot{x}_p = \frac{X_p}{m_p^*} \tag{17}$$

and

$$\ddot{x}_p = \frac{1}{m_p^*} \frac{\partial F_0}{\partial x_p} = -\beta_p^2 \frac{x_p}{r_p^3} ,$$

i.e., elliptic motion. Hence, there exists an energy integral,

$$\frac{1}{2} (\dot{x}_p^2 + \dot{y}_p^2 + \dot{z}_p^2) = \beta_p^2 \left(\frac{1}{r_p} - \frac{1}{2a_p} \right) , \tag{18}$$

where a_p is the semimajor axis, by means of which (16) can be written as

$$F_0 = \sum_{q=1}^{n} \frac{\beta_q^2 m_q^*}{2a_q} . \tag{19}$$

It would seem natural to put $\beta_p^2 = k^2(m_0 + m_p)$ such that, in view of (10), (15), and (16), F_0 would absorb the entire $1/r_q$ part of the potential. The mean motion n_p would then be related to a_p precisely as for two-body motion, viz.

$$n_p = \beta_p a_p^{-3/2} . \tag{20}$$

However, to satisfy the periodicity requirements of the variation orbits, it is necessary to take

$$\beta_p^2 (1 - \mu_p) = k^2 (m_0 + m_p) = k^2 \frac{m_0 m_p}{m_p^*} , \tag{21}$$

where μ_p (de Sitter, 1918) is a small constant to be determined together with the elements of the variation orbits. The perturbing Hamiltonian, $F_1 = F - F_0$, then becomes

$$F_1 = -\sum_{q=1}^{n} \mu_q \beta_q^2 \frac{m_q^*}{r_q} + \sum_{q=2}^{n} \sum_{r=1}^{q-1} \left[\frac{k^2 m_q m_r}{r_{qr}} - \frac{1}{m_0} (X_q X_r + Y_q Y_r + Z_q Z_r) \right] . \tag{22}$$

It will be seen that $F/m_p^* = (F_0 + F_1)/m_p^*$ can be regarded as the p^{th} body's Hamiltonian (we can neglect all terms not dependent on the elements of this body) to which corresponds a set of canonical variables; e.g., the Delaunay set $L_p = \beta_p\sqrt{a_p}$, $G_p = L_p\sqrt{1 - e_p^2}$, $H_p = G_p \cos I_p$, ℓ_p, ω_p, and Ω_p. Here, e_p is the eccentricity, I_p the inclination, ℓ_p the mean anomaly, ω_p the argument of the pericenter, and Ω_p the longitude of the ascending node. It follows that the combined, modified Delaunay set,

$$\left\{\begin{matrix} L_p = m_p^*\beta_p\sqrt{a_p} & , & G_p = L_p\sqrt{1-e_p^2} & , & H_p = G_p \cos I_p \\ \ell_p & , & \omega_p & , & \Omega_p \end{matrix}\right\} p = 1, 2, \ldots, n \tag{23}$$

obeys the canonical equations,

$$\left.\begin{matrix} \dfrac{dL_p}{dt} = \dfrac{\partial F}{\partial \ell_p} & , & \dfrac{dG_p}{dt} = \dfrac{\partial F}{\partial \omega_p} & , & \dfrac{dH_p}{dt} = \dfrac{\partial F}{\partial \Omega_p} \\ \dfrac{d\ell_p}{dt} = -\dfrac{\partial F}{\partial L_p} & , & \dfrac{d\omega_p}{dt} = -\dfrac{\partial F}{\partial G_p} & , & \dfrac{d\Omega_p}{dt} = -\dfrac{\partial F}{\partial H_p} \end{matrix}\right\} p = 1, 2, \ldots, n \quad , \tag{24}$$

with the common Hamiltonian

$$F = F_0 + F_1 \quad , \tag{25}$$

given by (19) and (22).

By the principle of variation of arbitrary constants, the elliptic formulas relating the Delaunay variables to the positions and velocities for unperturbed motion also hold for perturbed motion. However, in the latter case, we have to make an important distinction. Whereas the position x_p in the perturbed intermediary must be equal to the true relative position, say x_{pt},

$$x_{pt} = x_p \quad , \qquad p = 1, 2, \ldots, n \quad , \tag{26}$$

the same will not be true of the velocities $\dot{x}_p$ and $\dot{x}_{pt}$. This can be seen as follows. From (13), in which $\dot{x}_p$ and $\dot{x}_q$ must now be replaced by $\dot{x}_{pt}$ and $\dot{x}_{qt}$, we have that $\dot{x}_{qt} = \dot{x}_{pt} - X_p/m_p + X_q/m_q$. If this expression is substituted back into (13), there results

$$\frac{M}{m_p} X_p = M\dot{x}_{pt} - (M - m_0)\left(\dot{x}_{pt} - \frac{X_p}{m_p}\right) - \sum_{q=1}^{n} X_q \quad ,$$

or, by means of (17),

$$\dot{x}_{pt} = \frac{X_p}{m_p} + \frac{1}{m_0}\sum_{q=1}^{n} X_q = \dot{x}_p + \sum_{\substack{q=1 \\ q\neq p}}^{n} \frac{m_q^*}{m_q}\dot{x}_q \quad , \qquad p = 1, 2, \ldots, n \ . \tag{27}$$

The intermediaries are therefore not osculating orbits, but this matters very little. The only difference from the use of osculating orbits is that, at the very end, if we wish to compute the true relative velocities (which are usually not needed anyway), they must be obtained from (27). The velocity-dependent indirect term under the double summation sign in (22) can be expanded, as we shall see, perhaps even more easily than can the corresponding indirect term in de Sitter's semicanonical formulation, and the direct terms are identical, apart from an extra mass factor in (22). Note that, while the present formulation is closely related to de Sitter's formulation, the two do not lead to identical intermediaries.

To take advantage of the fact that the eccentricities and inclinations are very small, we shall introduce the modified Delaunay set

$$\left\{\begin{array}{lll} G_p = m_p^*\beta_p\sqrt{a_p(1-e_p^2)} \ , & M_p = m_p^*\beta_p\sqrt{a_p}(1-\sqrt{1-e_p^2}) \ , & N_p = G_p(\cos I_p - 1) \\ \lambda_p = \ell_p + \omega_p + \Omega_p \ , & \ell_p \ , & \Omega_p \end{array}\right\} \tag{28}$$

but since e_p and I_p may pass through zero, it is convenient also to make use of the following set, in which the conjugate pairs (M_p, ℓ_p) and (N_p, Ω_p) are replaced by the associated Poincaré variables (p_p, q_p) and (u_p, v_p) given by

$$\left\{\begin{array}{lll} G_p \ , & p_p = \sqrt{2M_p}\cos \ell_p \ , & u_p = \sqrt{-2N_p}\cos \Omega_p \\ \lambda_p \ , & q_p = \sqrt{2M_p}\sin \ell_p \ , & v_p = -\sqrt{-2N_p}\sin \Omega_p \end{array}\right\} . \tag{29}$$

Finally, closely related to the set (28) is the Hill set,

$$\left\{\begin{array}{lll} G_p \ , & \dot{r}_p \ , & N_p \\ f_p + \omega_p + \Omega_p \ , & r_p \ , & \Omega_p \end{array}\right\} , \tag{30}$$

where f_p is the true anomaly. The Hill variables may prove useful for obtaining the perturbations in r_p and $\dot{r}_p$ which are of interest if range and range-rate observations of the satellites become available. The canonical equations for these three sets of variables can be written down immediately from (24) by replacing the conjugate pairs of variables there by the appropriate new ones. The sets (28) and (29) are particularly convenient if we expand the disturbing function in powers of the auxiliary eccentricities ϵ_p and the auxiliary inclinations γ_p, defined by,

$$\left\{\begin{array}{l} \epsilon_p = \sqrt{2M_p/G_p} = \sqrt{2(1-e_p^2)^{-1/2} - 2} = e_p + 0(e_p^3) \ , \\ \gamma_p = \sqrt{-2N_p/G_p} = 2\sin\frac{I}{2}p = I_p + 0(I_p^3) \ , \end{array}\right\} \tag{31}$$

rather than, as usual, in powers of e_p and I_p (or $\sin I_p$), whose derivatives with respect to G_p, M_p, N_p, and the Poincaré variables are rather cumbersome. de Sitter pointed out the advantages of the set (28) over the noncanonical Kepler variables, which he introduced only to be able to make easy use of the existing expansions of the disturbing function in powers of e_p and I_p. The new expansion proposed here is greatly facilitated today by utilizing an algebra program on an electronic computer. In place of (28), Marsden (1964) adopted the canonical set

$$\left.\begin{matrix} L_p = m_p^*\beta_p\sqrt{a_p} \quad , & M_p = L_p\left(\sqrt{1-e_p^2}-1\right) \quad , & N_p = L_p\sqrt{1-e_p^2}(\cos I_p - 1) \\ \lambda_p = \ell_p + \omega_p + \Omega_p \quad , & \bar{\omega}_p = \omega_p + \Omega_p \quad , & \Omega_p \end{matrix}\right\} , \tag{32}$$

which has the advantage that it contains only one "fast" variable, λ_p. However, corresponding to (31), we now have

$$\begin{aligned} \epsilon_p &= \sqrt{-2M_p/L_p} = \sqrt{2 - 2(1-e_p^2)^{1/2}} \quad , \\ \gamma_p &= \sqrt{-2N_p/L_p} = (1-e_p^2)^{1/4}\, 2 \sin\frac{I}{2}p \quad , \end{aligned} \tag{33}$$

and e_p in the last equation introduces a considerable complication, since sin $I_p/2$ occurs quite naturally by itself in the disturbing function. We note that with the von Zeipel method used by Marsden, it is difficult to handle more than one fast variable at once, but this is not so with Hori's method.

In the remainder of this section, we introduce a very convenient formulation due to Marsden. Since Jupiter's (m_0's) oblateness has a pronounced effect on the Galilean satellites (m_1, m_2, m_3, m_4), it is necessary to add to F_1 the potential

$$F_{1J} = -k^2 m_0 \sum_{q=1}^{4} \left[J_2 R_0^2 \frac{m_q}{r_q^3} P_2(\sin\phi_q) + J_4 R_0^4 \frac{m_q}{r_q^5} P_4(\sin\phi_q) \right] \quad , \tag{34}$$

where R_0 is the equatorial radius and J_2 and J_4 are the dynamical form factors of Jupiter, ϕ_q is the latitude of m_q on Jupiter's equator, and P_2 and P_4 are the second and fourth Legendre polynomials. Note that in this potential, unlike in the corresponding one in de Sitter's formulation, there is no interaction between the oblateness and the indirect part of F_1. There is a converse effect of the satellites on the motion of Jupiter's equator. This can be described by allowing the index p to take on also the value zero in the expressions (28) for N_p and Ω_p, defining Ω_0 and I_0 to be, respectively, the longitude of the ascending node and inclination of Jupiter's equator on the fixed reference plane, e_0 to be zero, and

$$G_0 = Cn_0 = \text{constant} \tag{35}$$

to be the angular momentum of Jupiter about its polar axis, n_0 being the angular velocity of rotation and C the moment of inertia about this axis. With these definitions, putting

$$\alpha = \sin I_0 \sin\Omega_0 \quad , \quad \beta = -\sin I_0 \cos\Omega_0 \quad , \quad \gamma = \cos I_0 \quad , \tag{36}$$

sin ϕ_q in (34) can be written

$$\sin\phi_q = \frac{1}{r_q}(\alpha x_q + \beta y_q + \gamma z_q) \quad . \tag{37}$$

Finally, if we define m_5 to be the Sun, its attraction can be included by taking $n = 5$ in (19) and (22). It is sufficient to adopt a fixed ellipse for the Sun's motion about Jupiter, such that for $p = 5$ in (32) all the variables are constants, except for λ_5, which is a linear function of time. (The use of the set (28) would lead to two time-dependent variables, ℓ_5 and λ_5.) It follows that we can remove the time from the Hamiltonian by including the canonical pair (L_5, λ_5) (provided that, as far as the Sun is concerned, F_1 is regarded as a function of L_5, λ_5, $\bar{\omega}_5$, and Ω_5). Our problem, then, has altogether 14 degrees of freedom – three for each of the satellites, one for Jupiter's equator, and one for the Sun. The attractions of the remaining planets could, of course, be included in the same way, but according to de Sitter, even the perturbations by Saturn are entirely negligible to the order of accuracy that he aimed for, i.e., 10^{-6} radians in the longitudes of the satellites.

3. EXPANSION OF THE DISTURBING FUNCTION

Following de Sitter, we shall take Jupiter's equator at a certain epoch, e.g. 1950.0, as our reference plane. This choice makes the inclinations of the satellite orbits less than about a tenth of that of the Sun's orbit, $I_5 \approx 3°$. For simplicity, we shall take m_0 as the unit of mass and $(n_2 - n_3)^{-1} \approx 1.1222$ ephemeris days as the unit of time. We also put $k = 1$ (k will denote a dummy index in what follows), which leads to a unit of length of about 0.0070854 a.u., being very close to the mean distance of the third satellite.

The approximate values (de Sitter, 1918) of the various small parameters are given in Table I, where

Table I. Collection of small parameters.

p	m_p	J_{2p}	$-J_{4p}$	e_p	I_p
1	4×10^{-5}	$\frac{2}{5} \times 10^{-3}$	$\frac{1}{2} \times 10^{-6}$	4×10^{-3}	6×10^{-4}
2	$\frac{5}{2} \times 10^{-5}$	$\frac{1}{6} \times 10^{-3}$	$\frac{2}{3} \times 10^{-7}$	9×10^{-3}	8×10^{-3}
3	8×10^{-5}	$\frac{2}{3} \times 10^{-4}$	1×10^{-8}	2×10^{-3}	3×10^{-3}
4	$\frac{9}{2} \times 10^{-5}$	2×10^{-5}	1×10^{-9}	$\frac{3}{4} \times 10^{-2}$	4×10^{-3}

$$J_{ip} = J_i R_0^i / \sigma_p^i \quad ; \quad i = 2, 4 \quad ; \quad p = 1, 2, 3, 4 \quad , \tag{38}$$

with

$$\sigma_p = a_p(1 - e_p^2) = \frac{\nu_p G_p^2}{m_p^2} \quad , \quad \nu_p = (1 - \mu_p)(1 + m_p) \quad , \tag{39}$$

which relations follow from (15), (21), and (28). The values for e_p include both the free and the forced eccentricities, and we note that the inclinations enter the disturbing function only in the combination I_pI_q (p, q = 1, 2, 3, 4). If we regard e_p and I_p as quantities of the first order, then m_p and J_{2p} are roughly of the second order, and J_{4p}, of the third order. According to de Sitter, it is necessary to develop the principal terms of $F_1 + F_{1J}$, as given by (22) and (34), to the eighth order (sixth order in de Sitter's case, since our disturbing function contains an extra mass factor) to achieve an accuracy of 10^{-6} in the longitudes. Hence, we must include terms of the order m_p times

$$m_pe_p^4\ ,\quad m_pe_p^2I_p^2\ ,\quad J_{2p}e_p^4\ ,\quad J_{2p}e_p^2I_p^2\ ,\quad J_{4p}e_p^3\ ,\quad J_{4p}e_pI_p^2\ . \tag{40}$$

Since $m_5/r_5^2 \approx 2.1 \times 10^{-3}$ is comparable to m_p/r_p^2 (p = 1, 2, 3, 4), the terms in (22) that involve the Sun and one satellite and those that involve pairs of satellites will give rise to perturbations of roughly the same order, but the former terms can be expanded much more easily on account of the smallness of $1/r_5 \approx 1.4 \times 10^{-3}$.

In the subsequent derivations, we need consider only one pair of bodies. To ease the notation, we shall drop the subscripts for the body numbers and attach primes to the symbols relating to the outer body, i.e., $a < a'$; the distance between the bodies will be denoted by Δ. For the expansion of the indirect part of F_1, we have the following formulas for elliptic motion,

$$\begin{aligned}
\dot{x} &= \beta\sigma^{-1/2}\,[-P_1\ \sin f + \bar{P_1}(e + \cos f)]\ ,\\
\dot{y} &= \beta\sigma^{-1/2}\,[-P_2\ \sin f + \bar{P_2}(e + \cos f)]\ ,\\
\dot{z} &= \beta\sigma^{-1/2}\,[-P_3\ \sin f + \bar{P_3}(e + \cos f)]\ ,
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
P_1 &= c^2\ \cos\bar{\varpi} + s^2\ \cos(\bar{\varpi} - 2\Omega)\ ,\\
P_2 &= c^2\ \sin\varpi - s^2\ \sin(\varpi - 2\Omega)\ ,\\
P_3 &= 2sc\ \sin(\bar{\omega} - \Omega)\ ,
\end{aligned} \tag{42}$$

and $\bar{P_k}$ (k = 1, 2, 3) can be obtained from the expressions for P_k by first replacing "cos" by "sin" and "sin" by "cos" and then changing the signs of the arguments. In (42), we have introduced the abbreviations

$$s = \sin\frac{I}{2} = \frac{\gamma}{2}\ , \qquad c = \cos\frac{I}{2}\ . \tag{43}$$

In Cayley's tables (Cayley, 1861), we find the following expansions to the fourth order in e,

$$\begin{aligned}
\sin f = \sum_{i=1}^{\infty} a_i \sin i\ell &= \left(1 - \frac{7}{8}e^2 + \frac{17}{192}e^4\right)\sin\ell + \left(e - \frac{7}{6}e^3\right)\sin 2\ell\\
&+ \left(\frac{9}{8}e^2 - \frac{207}{128}e^4\right)\sin 3\ell + \frac{4}{3}e^3\ \sin 4\ell + \frac{625}{384}e^4\ \sin 5\ell + 0(e^5)\ ,
\end{aligned}$$

$$e + \cos f = \sum_{i=1}^{\infty} b_i \cos i\ell = \left(1 - \frac{9}{8}e^2 + \frac{25}{192}e^4\right)\cos\ell + \left(e - \frac{4}{3}e^3\right)\cos 2\ell$$
$$+ \left(\frac{9}{8}e^2 - \frac{225}{128}e^4\right)\cos 3\ell + \frac{4}{3}e^3 \cos 4\ell + \frac{625}{384}e^4 \cos 5\ell + 0(e^5) \quad , \tag{44}$$

where the powers of e may easily be replaced by powers of ϵ by means of (31). By using (15), (17), (21), (39), and (41), the indirect term inside the double summation sign in (22) may now be written

$$-(XX' + YY' + ZZ') = mm'(\nu\nu'\sigma\sigma')^{-1\ 2} \cdot \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{3} \frac{1}{2}$$
$$\times [(P_k P'_k a_i a'_j - P_k^- P_k'^- b_i b'_j) \cos (i\ell + j\ell')$$
$$+ (P_k P_k'^- a_i b'_j + P'_k P_k^- a'_j b_i) \sin (i\ell + j\ell')] \quad , \tag{45}$$

where it is understood that $a_i = -a_{-i}$ (and therefore $a_0 = 0$), $b_i = b_{-i}$, and $b_0 = 0$. Now,

$$\sum_{k=1}^{3} P_k P'_k = c^2 c'^2 \cos (\varpi - \varpi') + 2ss'cc' \cos (\varpi - \varpi' - \Omega + \Omega')$$
$$+ s^2 s'^2 \cos (\varpi - \varpi' - 2\Omega + 2\Omega') + s^2 c'^2 \cos (\varpi + \varpi' - 2\Omega)$$
$$+ s'^2 c^2 \cos (\varpi + \varpi' - 2\Omega') - 2ss'cc' \cos (\varpi + \varpi' - \Omega - \Omega') \ , \tag{46}$$

and by changing the signs of the last three terms, this expression turns into that for $\sum P_k^- P_k'^-$, while $\sum P'_k P_k^- = \left(\sum P_k P'_k\right)^-$, from which $\sum P_k P_k'^-$ obtains, of course, by interchanging the primed and unprimed quantities. Substituting these results in (45), we find after a considerable amount of calculation,

$$-(XX' + YY' + ZZ') = mm'(\nu\nu'\sigma\sigma')^{-1/2} \sum_{i,j=-\infty}^{\infty} \frac{1}{4}(a_i + b_i)$$
$$\times \Big[(a'_j - b'_j)\{c^2 c'^2 \cos (i\ell + j\ell' + \varpi - \varpi') + 2ss'cc' \cos (i\ell + j\ell' + \varpi - \varpi' - \Omega + \Omega')$$
$$+ s^2 s'^2 \cos (i\ell + j\ell' + \varpi - \varpi' - 2\Omega + 2\Omega')\} + (a'_j + b'_j)\{s^2 c'^2 \cos (i\ell + j\ell'$$
$$+ \varpi + \varpi' - 2\Omega) + s'^2 c^2 \cos (i\ell + j\ell' + \varpi + \varpi' - 2\Omega') - 2ss'cc' \cos (i\ell + j\ell'$$
$$+ \varpi + \varpi' - \Omega - \Omega')\}\Big] \tag{47}$$

As a partial check, we observe that the d'Alembert rule is obeyed since the lowest powers of e, e', s, and s' occuring in the coefficients are the same as the respective multiples of ℓ, ℓ', Ω, and Ω'. As was to be expected, the indirect term has no secular part since there are not terms with i = j = 0. We shall later show that the indirect term can be conveniently combined with the direct term whose expansion we take up next.

We have that

$$\Delta^{-1} = (r^2 + r'^2 - 2rr' \cos \psi)^{-1/2} \quad , \tag{48}$$

where ψ is the angle between the radius vectors r and r′. We note that the expression (46) is the dot-product of the unit vectors directed along the apses of the two orbits, so that we may obtain $\cos \psi$ from the same expression merely by replacing ϖ and ϖ' by the true longitudes $v = \varpi + f$ and $v' = \varpi' + f'$, respectively:

$$\cos \psi = \left(\sum_{k=1}^{3} P_k P'_k \right) \begin{matrix} \varpi \rightarrow v \\ \varpi' \rightarrow v' \end{matrix} \tag{49}$$

Since we are forced to treat the motions of all the bodies simultaneously, the usual expansions of Δ^{-1} in powers of the mutual inclinations of pairs of bodies cannot be used here. For the expansion in s and s′, we proceed as follows, aided by a novel treatment of the planetary disturbing function by Yuasa and Hori (1975). For this expansion it is sufficient to consider circular orbits, in which case we denote Δ by Δ_0. Then, in view of (46), (48), and (49),

$$\Delta_0^{-1} = \left[a^2 + a'^2 - 2aa' \{\cos (\lambda - \lambda') + \delta\} \right]^{-1/2} \quad , \tag{50}$$

where δ is a quantity of the order ss′ given by

$$\begin{aligned} \delta = {} & (c^2 c'^2 - 1) \cos (\lambda - \lambda') + 2\, ss'cc' \cos (\lambda - \lambda' - \Omega + \Omega') \\ & + s^2 s'^2 \cos (\lambda - \lambda' - 2\Omega + 2\Omega') + s^2 c'^2 \cos (\lambda + \lambda' - 2\Omega) \\ & + s'^2 c^2 \cos (\lambda + \lambda' - 2\Omega') - 2ss'cc' \cos (\lambda + \lambda' - \Omega - \Omega') \quad . \end{aligned} \tag{51}$$

Yuasa and Hori define δ slightly differently such that $\cos (\lambda - \lambda')$ in (50) will contain an additional factor $(cc' - ss')^2$. They found that the convergence of the expansion is thereby improved with the remarkable result that it even holds for intersecting orbits. However, since convergence is no problem in our case because s and s′ are very small, and in order to introduce the familiar Laplace coefficients, we have replaced this factor by unity. By expanding the right-hand side of (50) by means of the binomial theorem, we find

$$\Delta_0^{-1} = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (2\delta)^i \, a'^{-1} b_j^{(i)} \exp [\sqrt{-1}\; j(\lambda - \lambda')] \quad , \tag{52}$$

where the $b_j^{(i)}$'s are the coefficients in the expansion

$$\begin{aligned} \binom{-1/2}{i} (-\alpha)^i \left[1 + \alpha^2 - 2\alpha \cos (\lambda - \lambda')\right]^{-1/2-i} &= b_0^{(i)} + \sum_{j=1}^{\infty} 2b_j^{(i)} \cos (j\lambda - j\lambda') \\ &= \sum_{j=-\infty}^{\infty} b_j^{(i)} \exp [\sqrt{-1}\; j(\lambda - \lambda')] \end{aligned} \tag{53}$$

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with $\alpha = a/a' < 1$ and where $b_j^{(i)} = b_{-j}^{(i)}$ depends on α only ($b_j^{(i)}$ here would be written

$$\frac{1}{2}\binom{-1/2}{i}(-\alpha)^i \cdot b_{1/2+i}^{(j)}$$

in the usual notation for the Laplace coefficients). In our problem we need consider values of i only up to two, and we may then readily perform the expansion

$$(2\delta)^i = c_{0000} + 2\sum s^{|k_3|} s'^{|k_4|} c^{(i)}_{k_1k_2k_3k_4} \cos(k_1\lambda + k_2\lambda' + k_3\Omega + k_4\Omega')$$
$$= \sum s^{|k_3|} s'^{|k_4|} c^{(i)}_{k_1k_2k_3k_4} \exp\left[\sqrt{-1}(k_1\lambda + k_2\lambda' + k_3\Omega + k_4\Omega')\right] \quad ,(54)$$

where the first expression includes a finite number of terms for all the occurring combinations of the k's with $k_1 \geq 0$. In the last sum we include for each of these terms (except when all the k's are zero) an additional term with coefficient $c^{(i)}_{-k_1,-k_2,-k_3,-k_4} = c^{(i)}_{k_1,k_2,k_3,k_4}$. We finally substitute (54) in (52) to obtain

$$\Delta_0^{-1} = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k} a'^{-1} b_j^{(i)} c_k^{(i)} s^{|k_3|} s'^{|k_4|} \exp\left[\sqrt{-1}\{(j+k_1)\lambda + (-j+k_2)\lambda' + k_3\Omega + k_4\Omega'\}\right] \quad , \tag{55}$$

where k is an abbreviation for k_1, k_2, k_3, and k_4.

The expansion in powers of s and s′ having been completed, we may in the following regard Δ_0^{-1} as a function of only a, a′, λ, and λ′. We see from (50) that Δ^{-1} is the same function of r, r′, v, and v′ and hence may be obtained from (55) if the former variable set is replaced by the latter. This replacement can be done most easily by introducing the complex variables

$$\begin{aligned} w_1 &= \exp(\sqrt{-1}\lambda) \quad , & w_1' &= \exp(\sqrt{-1}\lambda') \quad , \\ w_2 &= \exp(\sqrt{-1}v) \quad , & w_2' &= \exp(\sqrt{-1}v') \quad , \\ w_3 &= \exp(\sqrt{-1}\ell) \quad , & w_3' &= \exp(\sqrt{-1}\ell') \quad , \\ w_4 &= \exp(\sqrt{-1}f) \quad , & w_4' &= \exp(\sqrt{-1}f') \quad , \end{aligned} \tag{56}$$

and the differential operators,

$$D = a\partial/\partial a \ , \quad D' = a'\partial/\partial a' \ , \quad D_1 = w_1\partial/\partial w_1 \ , \quad D_1' = w_1'\partial/\partial w_1' \ . \tag{57}$$

Reference is made to Izsak et al. (1964) for a detailed exposition of the following method. It is easy to show that we may write symbolically

$$\Delta^{-1} = (r/a)^D (r'/a')^{D'} (w_4/w_3)^{D_1} (w_4'/w_3')^{D_1'} \Delta_0^{-1}(a, a', w_1, w_1') \ , \tag{58}$$

where the D's may be treated formally as if they were exponents, and $(r/a)^D$ and $(w_4/w_3)^{D_1}$ are supposed to be expanded in Laurent series in positive and

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negative powers of w_3, and similarly for the primed quantities. The product of these series for the inner body will be of the form

$$(r/a)^D (w_4/w_3)^{D_1} = \sum_{n=-\infty}^{\infty} \sum_{q} \Pi_n^q(D, D_1)\, e^q w_3^n \quad , \tag{59}$$

where q is summed over all values for which $q - |n| = 0, 2, 4, \ldots, \infty$. The corresponding result for the outer body is obtained by adding a prime on all the symbols in (59). The Newcomb operator $\Pi_n^q(D, D_1)$ is a polynomial of degree q in D and D_1.

In performing the operations indicated in (58), we observe that D affects only the factor $a'^{-1}b_j^{(i)}$ in (55) and, for any integer q, $D'^q a'^{-1} b_j^{(i)}(a/a') = (-D-1)^q a'^{-1} b_j^{(i)}(a/a')$. Furthermore, the effect of D_1^q and $D_1'^q$ on Δ_0^{-1} is to multiply each term of Δ_0^{-1} by the factors $(j + k_1)^q$ and $(-j + k_2)^q$, respectively. We have thus succeeded in reducing the four D-operators to the single one, D, and we have the combined result

$$\Pi_{nn'}^{qq'}(D, j, k_1, k_2) = \Pi_n^q(D, j+k_1) \cdot \Pi_{n'}^{q'}(-D-1, -j+k_2) \quad . \tag{60}$$

We note that the only structural difference between this equation and the corresponding equation (26) in Izsak et al. (1964) is that we have two distinct indices k_1 and k_2 where they have only one. This is due to the fact that only one mutual inclination and one common node occur in their expansion which is equivalent to ours with $s' = 0$ and $\Omega = \Omega'$. We are otherwise led to precisely the same Newcomb operators which may be taken from an existing table or generated by means of the very convenient recursion formulas developed by those authors. Izsak and Benima (1963) have also published a computer algorithm for computation of the Laplace coefficients and their Newcomb derivatives.

By means of (55) and (58) to (60), we now get

$$\Delta^{-1} = \sum_{i=0}^{\infty} \sum_{\substack{-\infty \\ j,n,n'}}^{\infty} \sum_{k,q,q'} \Pi_{nn'}^{qq'} a'^{-1} b_j^{(i)} c_k^{(i)} e^q e'^{q'} s^{|k_3|} s'^{|k_4|}$$
$$\times \cos\,[n\ell + n'\ell' + (j+k_1)\lambda + (-j+k_2)\lambda' + k_3\Omega + k_4\Omega'] \quad , \tag{61}$$

where $q - |n|$ and $q - |n'|$ take one the values $0, 2, 4, \ldots, \infty$. This expression does not yet have the desired form, since we wish to replace e and e′ above by ϵ and ϵ'. We could, of course, do this by direct substitution by means of (31), but we also wish to replace a and a′ by σ and σ' which involve only the canonical variables G and G′, according to (39). Now, we may replace a by σ and a′ by σ' in (55), (57), and (58) provided that we alter the meaning of α, as it enters through $b_j^{(i)}(\alpha)$, to $\alpha = \sigma/\sigma' < 1$. Furthermore, we may change a into σ and e into ϵ in (59) where $\Pi_n^q(D, D_1)$ will now be a different polynomial but of the same structure as before. This polynomial is a special case of the generalized Newcomb operators devised by Izsak et al. (1964) with $\kappa_I = -1/2$ and $\alpha_I = \sigma$ (where the subscript I for "Izsak" has been added to avoid confusion with the

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meanings of κ and α in the present paper). If we in place of (28) and (31) had adopted (32) and (33), we would have $\kappa_I = 0$ and $\alpha_I = a$. Izsak et al. have derived recursion formulas also for the generalized Newcomb operators.

Before writing down the new version of (61), we shall reformulate (47) to enable a convenient combination of the indirect and direct terms. We notice the similarity of the coefficients in (47) to those in (54) with i = 1. If we put $\omega = \lambda - \ell$ and $\omega' = \lambda' - \ell'$ in (47), it is easy to show that that equation may be written

$$
\begin{aligned}
-(XX' + YY' + ZZ') = mm'(\nu\nu'\sigma\sigma')^{-1/2} \sum_{n,n'=-\infty}^{\infty} \frac{1}{8}\Big[\{(a_{n+1} + b_{n+1})\\
\times (a'_{n'-1} - b'_{n'-1}) \cos(n\ell + n'\ell' + \lambda - \lambda') + (a_{n-1} - b_{n-1})(a'_{n'+1} + b'_{n'+1})\\
\times \cos(n\ell + n'\ell' - \lambda + \lambda')\} + \sum_k s^{|k_s|} s'^{|k_4|} c_k^{(1)} (a_{n+k_1} + k_1 b_{n+k_1})\\
\times (a'_{n'+k_2} + k_2 b'_{n'+k_2}) \cos(n\ell + n'\ell' + k_1\lambda + k_2\lambda' + k_3\Omega + k_4\Omega')\Big] \quad , \qquad (62)
\end{aligned}
$$

where, in view of (51), k_1 and k_2 have only the values ±1. Let p be ±1. Then

$$
\begin{aligned}
a_{n+p} + pb_{n+p} &= p(a_{|n|+1} + b_{|n|+1}) \qquad \text{if } pn \geq 0 \quad , \\
&= -p(a_{|n|-1} - b_{|n|-1}) \qquad \text{if } pn \leq -1 \quad ,
\end{aligned} \qquad (63)
$$

and because of (44) we can write,*

$$
\begin{aligned}
a_{|n|+1} + b_{|n|+1} &= \sum_q \alpha_{q|n|} \epsilon^q \quad , \qquad |n| = 0, 1, \ldots, \infty \quad , \\
a_{|n|-1} - b_{|n|-1} &= \sum_q \beta_{q|n|} \epsilon^q \quad , \qquad |n| = 1, 2, \ldots, \infty \quad , \\
a_{n+p} + pb_{n+p} &= \sum_q a_{qn}^{(p)} \epsilon^q \quad , \qquad n = 0, \pm 1, \ldots, \pm\infty \quad ,
\end{aligned} \qquad (64)
$$

where $q - |n| = 0, 2, 4, \ldots, \infty$ and the coefficients of ϵ^q are pure numbers. If we rewrite (62) by means of the last of equations (64) and add the result to the new version of (61) multiplied by mm', we finally obtain for the term inside the square brackets in (22),

$$
\frac{mm'}{\Delta} - (XX' + YY' + ZZ') = mm' \sum_{i=0}^{\infty} \sum_{j,n,n'}^{\infty}{}_{-\infty} \sum_{k,q,q'} \Big[\Pi_{nn'}^{qq'} \sigma'^{-1} b_j^{(i)}
$$

*The coefficients $a_i + b_i$ and $a_i - b_i$ are also listed in Cayley's tables.

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$$+\frac{1}{8}\alpha_{qn}^{(j+k_1)} \cdot \alpha_{q'n'}^{(-j+k_2)} (\nu\nu'\sigma\sigma')^{-1/2}\Big] c_k^{(i)} s^{|k_3|} s'^{|k_4|} \epsilon^{q} \epsilon'^{q'}$$

$$\times \cos [n\ell + n'\ell' + (j+k_1)\lambda + (-j+k_2)\lambda' + k_3\Omega + k_4\Omega'] \quad , \tag{65}$$

where, for each i, k is summed over a finite number of values of k_1, k_2, k_3, and k_4, and $q - |n|$ and $q' - |n'|$ take on the values 0, 2, 4, ..., ∞, and where $b_j^{(i)}(\alpha) = b_{-j}^{(i)}(\alpha)$, $c_k^{(i)}(s, s') = c_{-k}^{(i)}(s, s')$, and $\Pi_{nn'}^{qq'}(D, j, k_1, k_2) = \Pi_{-n,-n'}^{qq'}(D,-j,-k_1,-k_2)$ are defined by (53), (54), and (60) with

$$\alpha = \sigma/\sigma' \quad , \qquad D = \alpha\partial/\partial\alpha \quad . \tag{66}$$

Furthermore, the α-coefficients are given by

$$\begin{aligned} \alpha_{qn}^{(p)} &= 0 \text{ if } |p| \neq 1 \text{ or } i > 1 \quad , \\ &= p\alpha_{q|n|} \text{ if } pn \geq 0 \text{ and } |p| = 1 \text{ and } i \leq 1 \quad , \\ &= -p\beta_{q|n|} \text{ if } pn \leq -1 \text{ and } |p| = 1 \text{ and } i \leq 1 \quad , \end{aligned} \tag{67}$$

where the only nonzero values of $\alpha_{q|n|}$ and $\beta_{q|n|}$, and with q and $|n|$ below five, are easily found to be, by means of (31), (44), and (64):

$$\begin{aligned} &\alpha_{00} = 2 \;, \quad \alpha_{20} = -2 \;, \quad \alpha_{40} = 55/32 \;, \\ &\alpha_{11} = 2 \;, \quad \alpha_{31} = -13/4 \;, \\ &\alpha_{22} = 9/4 \;, \quad \alpha_{42} = -81/16 \;, \quad \beta_{22} = 1/4 \;, \quad \beta_{42} = -11/48 \\ &\alpha_{33} = 8/3 \;, \quad \beta_{33} = 1/6 \;, \\ &\alpha_{44} = 625/192 \;, \quad \beta_{44} = 9/64 \;. \end{aligned} \tag{68}$$

In view of (40), we need include only the terms with $i \leq 2$ and $q + q' + |k_3| + |k_4| \leq 4$ in (65), and de Sitter's variation orbits depend on only the secular and critical terms which do not involve the inclinations, i.e., $i = k_1 = k_2 = k_3 = k_4 = 0$ and $c_k^{(i)} = c_0^{(0)} = 1$. The secular and critical terms in addition satisfy the respective conditions $n = n' = j = 0$ and $n\ell + n'\ell' + j\lambda - j\lambda' = 0$.

Since $\sigma/r = 1 + e \cos f = 1 - e^2 + \sum_{i=1}^{\infty} eb_i \cos i\ell$, we obtain readily the following expansion for the term under the first summation sign in (22),

$$\mu\beta^2 \frac{m^*}{r} = \frac{\mu m}{\sigma(1-\mu)} \Big[1 - \epsilon^2 + \frac{3}{4}\epsilon^4 + (\epsilon - \frac{3}{2}\epsilon^3) \cos \ell + (\epsilon^2 - \frac{3}{4}\epsilon^4) \cos 2\ell + \frac{9}{8}\epsilon^3 \cos 3\ell + \frac{4}{3}\epsilon^4 \cos 4\ell + 0(\epsilon^5)\Big] \quad , \tag{69}$$

where μ, like m, is of the second order, according to de Sitter.

We do not give here an expansion for the disturbing potential (34) due to Jupiter's oblateness since it may turn out to be simpler to obtain the resulting perturbations from Brouwer's (1959) theory for an artificial satellite.

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NOTE ADDED IN PROOF

After this work had been completed, I became aware that Poincare's canonical relative coordinates had, in fact, been considered for use on the planetary problem by Izsak et al. (1965), who showed how to develop the associated disturbing function. Although their results do not apply here, since the development was made in terms of the mutual inclination, they pointed out that the indirect term in (22) can be written

$$-(XX' + YY' + ZZ') = -mm'(\nu\nu' aa')^{-1/2} \frac{\partial^2}{\partial \ell \partial \ell'} \left(\frac{rr'}{aa'} \cos \psi\right) \quad , \tag{70}$$

and hence can be expanded by means of the Newcomb operators, affording a somewhat simpler derivation than that presented here.

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