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THE SUP METRIC ON INFINITE PRODUCTS

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We study the properties of the sup metric on infinite products $Z = \prod_{\alpha\in\Lambda} X$. (If d is a bounded metric on X then ρ, defined by $\rho((x_\alpha), (y_\alpha)) = \sup_\alpha d(x_\alpha, y_\alpha)$, is the sup metric on Z.) In particular, we prove that if X is an AR(metric) or a topological group then so is Z.

Of all the equivalent metrics in Euclidean spaces, the sup metric is the one which can immediately be extended to infinite products of real lines. Nonetheless, it appears that no one has investigated the properties of this metric; since we could not think of any good reason for this, we decided to have a go at it and, in our opinion, the results which follow show that this metric is useful. Many pertinent questions remain open.

We conclude this study by introducing the *sup uniformity* in infinite products of uniform spaces; however, we know very little about its properties.

1. THE SUP METRIC

In this section we develop the general results which explain the behaviour of the sup metric on infinite products, as well as some of the extension properties and linear properties of this metric.

Let (Y, d) be a bounded metric space. Let $Z = Z(Y, d, \Lambda) = \prod_{\alpha\in\Lambda} Y$ be a cartesian product of Λ (assume Λ is a cardinal number) copies of Y and define a function $\rho\colon Z \times Z \to \mathbb{R}$ by $\rho((x_\alpha), (y_\alpha)) = \sup_{\alpha\in\Lambda} d(x_\alpha, y_\alpha)$.

LEMMA 1.1. *(Z, ρ) is a metric space.*

PROOF: Straightforward. ∎

REMARK. Clearly, one can define the sup metric for infinite products $\prod_{\alpha\in\Lambda} X_\alpha$ of *any* metric spaces (X_α, d_α), as long as all the metrics d_α are bounded by a common constant γ; however, there is no need for this since we can consider $\prod_{\alpha\in\Lambda} X_\alpha$ as a subspace of

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$Z(Y, d, \Lambda)$, where Y is the disjoint topological union $\bigvee_{\alpha\in\Lambda} X_\alpha$ of the spaces X_α with the metric d defined by the d_α and $d(x, y) = \gamma$ for $x \in X_\alpha$, $y \in X_\beta$ with $\alpha \neq \beta$.

Lemma 1.5 proves that bounded equivalent metrics on X generate equivalent metrics on Z.

Lemma 1.2. *For each $(x_\alpha) \in Z$ and $\varepsilon > 0$, $B((x_\alpha), \varepsilon) = \bigcup_{\delta<\varepsilon} \prod_{\alpha\in\Lambda} B(x_\alpha, \delta)$.*

Proof: It is clear that $\bigcup_{\delta<\varepsilon} \prod_{\alpha\in\Lambda} B(x_\alpha, \delta) \subset B((x_\alpha), \varepsilon)$, since $(y_\alpha) \in \prod_{\alpha\in\Lambda} B(x_\alpha, \delta)$ implies that $\rho((x_\alpha), (y_\alpha)) \leqslant \delta$. Conversely, if $\rho((x_\alpha), (y_\alpha)) = \delta < \varepsilon$ then $(y_\alpha) \in \prod_{\alpha\in\Lambda} B(x_\alpha, \delta')$, for any $\delta < \delta' < \varepsilon$. ∎

Lemma 1.3. *For each ν, the projection map $p_\nu \colon Z \to Y_\nu$ is an open continuous function.*

Proof: This is immediate from Lemma 1.2, since $p_\nu(B((x_\alpha), \varepsilon)) = \bigcup_{\delta<\varepsilon} B(x_\nu, \delta) = B(x_\nu, \varepsilon)$. ∎

Lemma 1.4. *A function $g \colon T \to Z$ is continuous if and only if $\{p_\alpha \circ g\}_{\alpha\in\Lambda}$ is equicontinuous.*

Proof: The "if" part: Pick $w \in T$ and $\varepsilon > 0$. Pick a neighbourhood N_w of w such that $p_\alpha(g(N_w)) \subset B(p_\alpha(g(w)), \varepsilon/2)$, for each $\alpha \in \Lambda$. Therefore, $g(N_w) \subset \prod_{\alpha\in\Lambda} B(p_\alpha(g(w)), \varepsilon/2) \subset B(g(w), \varepsilon)$. This proves that g is continuous.

The "only if" part: Pick $w \in T$ and $\varepsilon > 0$. Pick a neighbourhood N_w of w such that $g(N_w) \subset B(g(w), \varepsilon/2)$. Since $B(g(w), \varepsilon/2) \subset \prod_{\alpha\in\Lambda} B(p_\alpha(g(w)), \varepsilon)$ we then get that $p_\alpha(g(N_w)) \subset B(p_\alpha(g(w)), \varepsilon)$, for each $\alpha \in \Lambda$. This completes the proof. ∎

Lemma 1.5. *If d_1 and d_2 are bounded equivalent metrics on X then they generate equivalent metrics ρ_1 and ρ_2 on $Z(X, d_1, \Lambda) = Z(X, d_2, \Lambda)$.*

Proof: Let $h \colon (X, d_1) \to (X, d_2)$ be a homeomorphism. Define $g \colon Z(X, d_1, \Lambda) \to Z(X, d_2, \Lambda)$ by $p_\alpha g((x_\alpha)) = h(x_\alpha)$; hence $p_\alpha g^{-1}((x_\alpha)) = h^{-1}(x_\alpha)$. It follows immediately from Lemma 1.4 that g and g^{-1} are continuous, which completes the proof. ∎

Proposition 1.6. *If Y is a closed ball of a Banach space then (Z, ρ) is an AR(metrisable).*

Proof: Let (X, μ) be a metric space, A a closed subset of X and $f \colon A \to Z$ a continuous function. Without loss of generality, assume that Y is the unit ball centred at the origin of a Banach space.

Note that, by Lemma 1.4, $\{p_\alpha \circ g\}_{\alpha\in\Lambda}$ is an equicontinuous family of functions from A to Y. Consequently, by Theorem 1 in the Appendix, $\{\varphi(p_\alpha \circ g)\}_{\alpha\in\Lambda}$ is an equicontinuous family of functions from X to Y.

Finally, let us define a function $\overline{g}: X \to Z$ by $p_\alpha \overline{g}(x) = \varphi(p_\alpha \circ g)(x)$, for each $x \in X$. Lemma 1.4 immediately implies that $\overline{g}$ is continuous. Since it is clear that $\overline{g}|A = g$, we have proved that Z is an AR(metrisable), which completes the proof. $\square$

THEOREM 1.7. *Let (Y, d) be an AR(metric), where d is a bounded metric. Then $Z(Y, d, \Lambda)$ is an AR(metric).*

PROOF: Embed Y in a closed ball of $C(Y)$, the space of bounded, real-valued functions with the sup norm (and metric $\widehat{d}$). Let $\widehat{Y}$ be a closed ball in $C(Y)$ which contains Y and let $r: \widehat{Y} \to Y$ be a continuous retraction. Then the map $\widehat{r}: Z\left(\widehat{Y}, \widehat{d}, \Lambda\right) \to Z(Y, d, \Lambda)$, defined by $\widehat{r}((\widehat{y}_\alpha)) = (r(\widehat{y}_\alpha))$, is a continuous retraction (see Lemma 1.4). Therefore, by Proposition 1.6, $Z(Y, d, \Lambda)$ is an AR(metric). $\square$

QUESTION. Does $Z(Y, d, \Lambda)$ inherit connectedness, equiconnectedness, ... , from (Y, d)?

For convenience, let $\delta(X) = \inf\{\text{card } D \mid \overline{D} = X\}$. (Recall that the density $d(X)$ of X is defined by $d(X) = \aleph_0 \delta(X)$.)

THEOREM 1.8. *If (X, d) is a metric space with $\delta(X) = \Lambda$ then X is isometric to a subspace of $Z(\mathbb{R}, \Lambda)$.*

PROOF: Without loss of generality, let us assume that $\Lambda \geqslant \aleph_0$. Let D be a dense subspace of X such that $\text{card}\, D = \Lambda$. Let $\mathcal{B} = \{B(x, r) \mid x \in D \text{ and } r \text{ is a rational number}\}$; we clearly get that $\mathcal{B}$ is a basis for the topology associated with d such that $\text{card}\, \mathcal{B} = \Lambda$. For each $B \in \mathcal{B}$, let $f_B: X \to \mathbb{R}$ be the function defined by $f_B(x) = d(x, X - B)$. Note that $\{f_B \mid B \in \mathcal{B}\}$ is equicontinuous. (Note that, if $w \in X$, $\varepsilon > 0$ and $d(w, x) < \varepsilon$ then $|f_B(w) - f_B(x)| = |d(w, X - B) - d(x, X - B)| \leqslant d(w, x) < \varepsilon$, because $d(a, X - B) \leqslant d(a, b) + d(b, X - B)$ for any $a, b \in X$.) Therefore, by letting $h: X \to Z(\mathbb{R}, \mathcal{B})$ be defined by $p_B(h(x)) = f_B(x)$, we immediately get from Lemma 1.4 that h is continuous. Indeed, h is an isometry: Note that, for any distinct $x, y \in X$, there exists a sequence $\{x_n\} \subset D$ and a sequence $\{r_n\}$ of rational numbers such that

(i) $d(x, x_n) < (1/n) d(x, y)$,
(ii) $x \in B(x_n, r_n), y \notin B(x_n, r_n)$,
(iii) $\lim\limits_n r_n = d(x, y)$.

Letting $B_n = B(x_n, r_n)$, we then get that $\lim\limits_n |f_{B_n}(x) - f_{B_n}(y)| = \lim\limits_n f_{B_n}(x) = d(x, y)$. This shows that $\rho(h(x), h(y)) \geqslant d(x, y)$. Since the preceding parenthetical argument also shows that $d(x, y) \geqslant \rho(h(x), h(y))$, we then get that h is an isometry. This completes the proof. $\square$

COROLLARY 1.9. *If (X, d) is a complete metric space with $\delta(X) = \Lambda$ then X is isometric to a closed subspace of $Z(\mathbb{R}, \Lambda)$.*

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Unfortunately, Theorem 1.8 is *not good enough*! Let us recall that any separable metric space is a subspace of the space s which is the countable product of real lines with the generalised Euclidean metric (that is, $\prod_{n\in w} \mathbb{R}$ with the metric $\rho((x_n),(y_n)) = \sum_n 2^{-n} \min\{1, |x_n - y_n|\}$). The space s is *separable* and its topology is the Tychonoff product topology (recall that uncountable Tychonoff products of nondegenerate spaces are not metrisable). Theorem 1.8 fails to duplicate the preceding result in one important aspect: If $\delta(X) = \Lambda \geqslant \aleph_0$, then $\delta(Z(X, d, \Lambda)) = 2^\Lambda$. (From Theorem 1.1(c) of [3] we get that there exists a natural number n and a subset $D = \{x_\alpha \mid \alpha \in \Lambda\}$ of X such that $\operatorname{card} D = \Lambda$ and $\{B(x_\alpha, 1/n) \mid \alpha \in \Lambda\}$ is a pairwise disjoint collection of open balls of X with the *same* radius $1/n$. It follows that $\{B((x_\alpha), 1/n) \mid (x_\alpha) \in \prod_{\alpha\in\Lambda} D\}$ is a pairwise disjoint collection of open balls of $Z(X, d, \Lambda)$ whose cardinality is $\Lambda^\Lambda = 2^\Lambda$. Again, by Theorem 1.1(a) of [3], $\delta(Z) = 2^\Lambda$.) The preceding observation raises the obvious question.

QUESTION. If $\Lambda \geqslant \aleph_0$, is there a "well-behaved" metric ν on $Z(\mathbb{R}, \Lambda)$ such that $\delta(Z) = \Lambda$? (By "well-behaved" we hope that ν should be linear.)

THEOREM 1.10. *If (G, d) is a topological group, with a bounded and translation-invariant metric d, then $Z(G, d, \Lambda)$ is a topological group.*

PROOF: (Note that any translation-invariant metric d' yields an equivalent metric d which is translation-invariant and bounded: Simply, let $d(x, y) = d'(x, y)$ if $d'(x, y) \leqslant 1$, and $d(x, y) = 1$ otherwise.) Clearly, the inverse map $(x_\alpha) \to (x_\alpha^{-1})$ is continuous. In order to show that the coordinatewise multiplication map $((x_\alpha), (y_\alpha)) \to (x_\alpha y_\alpha)$ is continuous, pick $\varepsilon > 0$ and $\delta > 0$ such that $B(1, \delta)B(1, \delta) \subset B(1, \varepsilon/2)$. Since d is translation-invariant, we then get that $B(x_\alpha, \delta)B(y_\alpha, \delta) \subset B(x_\alpha y_\alpha, \varepsilon/2)$, for each $\alpha \in \Lambda$. (Note that $d(x_\alpha, z_\alpha) < \delta$, $\left(d(y_\alpha, w_\alpha) < \delta \Leftrightarrow d(1, w_\alpha y_\alpha^{-1}) < \delta\right) \Rightarrow d(1, x_\alpha^{-1} z_\alpha w_\alpha y_\alpha^{-1}) < \varepsilon/2 \Leftrightarrow d(x_\alpha, z_\alpha w_\alpha y_\alpha^{-1}) < \varepsilon/2 \Leftrightarrow d(x_\alpha y_\alpha, z_\alpha w_\alpha) < \varepsilon/2$.) Hence, $B((x_\alpha), \delta)B((y_\alpha), \delta) \subset B((x_\alpha y_\alpha), \varepsilon)$. This completes the proof. ∎

REMARK. It is noteworthy that the preceding proof depends on the two-sided invariance of the metric d. It may be interesting to know whether $Z(G, \Lambda)$ will still be a topological group if the metric d is only left-invariant or only right-invariant or neither.

THEOREM 1.11. *If (L, d) is a linear metric space, where d is bounded and translation invariant, then $Z(L, d, \Lambda)$ is a linear metric space.*

PROOF: (It is noteworthy that every linear metric space does have a bounded and translation-invariant metric, by a theorem of Kakutani — see Theorem 1.1.1 of [2].) By Theorem 1.10, we only need prove that the scalar map $\mathbb{R} \times Z \to Z$, defined by $(r, (\ell_\alpha)) \to (r\ell_\alpha)$ is continuous, and this is straightforward. ∎

2. The sup uniformity

Let $(Y, \mathcal{U})$ be a uniform space. Using the pseudometric approach to uniformities, let $\mathcal{G} = \{d_\alpha \mid \alpha \in \Gamma$ and each d_α is a bounded pseudometric on $X\}$ be a gauge for $\mathcal{U}$. Since each d_α generates a pseudometric ρ_α on $Z = Z(X, \Lambda) = \prod_{\alpha\in\Lambda} Y$, we get a family $\mathcal{G}^* = \{\rho_\alpha \mid \alpha \in \Gamma\}$ of pseudometrics on Z. It turns out that $\mathcal{G}^*$ actually generates a base for a uniformity for Z: Simply note that if $\rho_\alpha, \rho_\beta \in \mathcal{U}^*$ then $\sup(\rho_\alpha, \rho_\beta) \in \mathcal{U}^*$, since $\sup(d_\alpha, d_\beta) \in \mathcal{U}$ and it generates $\sup(\rho_\alpha, \rho_\beta)$.

Appendix

We will re-examine the Dugundji Extension Theorem in order to obtain a new result which is necessary for the preceding work.

First of all, let us recall that, for any metric space (X, d) and closed subset A of X, the proofs of Lemma 2.1 and Theorem 4.1 of [1] imply that there exists a locally finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ of $X - A$, $x_\alpha \in U_\alpha$ and $a_\alpha \in A$, and a partition of unity $\{p_\alpha\}_{\alpha\in\Lambda}$ subordinated to $\mathcal{U}$ such that

(i) $d(x_\alpha, a_\alpha) < 2d(x_\alpha, A)$;

(ii) for each $a \in A$ and $\delta > 0$ there exists $\delta'(a, \delta) = \delta' > 0$ such that $U_\alpha \cap B(a, \delta') \neq \emptyset$ implies that $U_\alpha \subset B(a, \delta/3)$ and $d(a, a_\alpha) < \delta$;

(iii) if Y is a locally convex subset of a linear topological space and $f\colon A \to Y$ is a continuous function, then the function $\varphi(f)\colon X \to Y$ defined by

$$\varphi(f)(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \sum_{\alpha\in\Lambda} p_\alpha(x) f(a_\alpha) & \text{if } x \in X - A, \end{cases}$$

is a continuous extension of f

Theorem 1. *Let (X, d) be a metric space, A a closed subset of X and Y a locally convex bounded subset of a Banach space. If $\{f_\gamma\}_{\gamma\in\Gamma}$ is an equicontinuous family of functions $f_\gamma\colon A \to Y$ then $\{\varphi(f_\gamma)\}_{\gamma\in\Gamma}$ is a equicontinuous family of functions $\varphi(f_\gamma)\colon X \to Y$.*

Proof: Without loss of generality, let us assume that $\|Y\| = \sup\{\|y\| \mid y \in Y\} \leqslant 1$. Let us first prove that $\{\varphi(f_\gamma)\}_{\gamma\in\Gamma}$ is equicontinuous at each $a \in A$. Note that $\|f_\gamma(a) - \sum_\alpha p_\alpha(x) f_\gamma(a_\alpha)\| = \|\sum_\alpha p_\alpha(x) f_\gamma(a) - \sum_\alpha p_\alpha(x) f_\gamma(a_\alpha)\| \leqslant \sum_\alpha p_\alpha(x)\, \|f_\gamma(a) - f_\gamma(a_\alpha)\|$. From hypothesis, given $\varepsilon > 0$ there exists $\delta > 0$ such that $f_\gamma(B(a, \delta) \cap A) \subset B(f_\gamma(a), \varepsilon)$ for each $\gamma \in \Gamma$. Consequently, by (ii), $x \in B(a, \delta') \cap U_\alpha$ implies that $d(a, a_\alpha) < \delta$ which implies that $\|\varphi(f_\gamma)(a) - \varphi(f_\gamma)(x)\| \leqslant \sum_\alpha p_\alpha(x)\, \|f_\gamma(a) - f_\gamma(a_\alpha)\| \leqslant \sum_\alpha p_\alpha(x)\varepsilon = \varepsilon$, for each $\gamma \in \Gamma$; that is, $\varphi(f_\gamma)(B(a, \delta')) \subset B(\varphi(f_\gamma)(a), \varepsilon)$, for each $\gamma \in \Gamma$, as required.

Finally, we prove that $\{\varphi(f_\gamma)\}_{\gamma\in\Gamma}$ is equicontinuous at each $w \in X - A$: Pick $B(w, \delta) \subset X - A$ which meets only $U_{\alpha_1}, \dots, U_{\alpha_n}$ in $\mathcal{U}$. Then, for any $x \in B(w, \delta)$, $\|\sum_\alpha p_\alpha(x) f_\gamma(a_\alpha) - \sum_\alpha p_\alpha(w) f_\gamma(a_\alpha)\| \leqslant \sum_\alpha |p_\alpha(x) - p_\alpha(w)| \, \|f_\gamma(a_\alpha)\| \leqslant \sum_\alpha |p_\alpha(x) - p_\alpha(w)| \, \|Y\| \leqslant \sum_\alpha |p_\alpha(x) - p_\alpha(w)| = \sum_{i=1}^{n} \left|p_{\alpha_i}(x) - p_{\alpha_i}(w)\right|$. Consequently, given $\varepsilon > 0$ there exists $B(w, \overline{\delta}) \subset B(w, \delta)$ such that $\|\varphi(f_\gamma)(x) - \varphi(f_\gamma)(w)\| = \sum_{i=1}^{n} \left|p_{\alpha_i}(x) - p_{\alpha_i}(w)\right| < \varepsilon$, for all $x \in B(w, \overline{\delta})$ and $\gamma \in \Gamma$; this implies that $\varphi(f_\gamma)\big(B(w, \overline{\delta})\big) \subset B(\varphi(f_\gamma)(w), \varepsilon)$, for all $\gamma \in \Gamma$. This completes the proof. $\square$

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