

## MICROLOCAL EULER CLASSES AND HOCHSCHILD HOMOLOGY

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(Received 4 December 2012; revised 22 May 2013; accepted 22 May 2013;  
first published online 18 July 2013)

*Abstract* We define the notion of a trace kernel on a manifold  $M$ . Roughly speaking, it is a sheaf on  $M \times M$  for which the formalism of Hochschild homology applies. We associate a microlocal Euler class with such a kernel, a cohomology class with values in the relative dualizing complex of the cotangent bundle  $T^*M$  over  $M$ , and we prove that this class is functorial with respect to the composition of kernels.

This generalizes, unifies and simplifies various results from (relative) index theorems for constructible sheaves,  $\mathcal{D}$ -modules and elliptic pairs.

*Keywords:* sheaves; D-modules; microlocal sheaf theory; Euler classes

2010 *Mathematics subject classification:* Primary 14F05; 35A27

### 1. Introduction

Our constructions mainly concern real manifolds, but in order to introduce the subject we first consider a complex manifold  $(X, \mathcal{O}_X)$ . Denote by  $\omega_X^{\text{hol}}$  the dualizing complex in the category of  $\mathcal{O}_X$ -modules, that is,  $\omega_X^{\text{hol}} = \Omega_X[d_X]$ , where  $d_X$  is the complex dimension of  $X$  and  $\Omega_X$  is the sheaf of holomorphic forms of degree  $d_X$ . Denote by  $\mathcal{O}_{\Delta_X}$  and  $\omega_{\Delta_X}^{\text{hol}}$  the direct images of  $\mathcal{O}_X$  and  $\omega_X^{\text{hol}}$  respectively under the diagonal embedding  $\delta: X \hookrightarrow X \times X$ . It is well-known (see in particular [3, 4]) that the Hochschild homology of  $\mathcal{O}_X$  may be defined by using the isomorphism

$$\delta_* \mathcal{H} \mathcal{H}(\mathcal{O}_X) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times X}}(\mathcal{O}_{\Delta_X}, \omega_{\Delta_X}^{\text{hol}}). \quad (1.1)$$

Moreover, if  $\mathcal{F}$  is a coherent  $\mathcal{O}_X$ -module and  $\mathbf{D}_{\mathcal{O}} \mathcal{F} := \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^{\text{hol}})$  denotes its dual, there are natural morphisms

$$\mathcal{O}_{\Delta_X} \rightarrow \mathcal{F} \boxtimes \mathbf{D}_{\mathcal{O}} \mathcal{F} \rightarrow \omega_{\Delta_X}^{\text{hol}} \quad (1.2)$$

This work was partially supported by Grant-in-Aid for Scientific Research (B) 22340005, from the Japan Society for the Promotion of Science.

whose composition defines the Hochschild class of  $\mathcal{F}$ :

$$\mathrm{hh}_{\mathcal{O}}(\mathcal{F}) \in H_{\mathrm{Supp}(\mathcal{F})}^0(X; \mathcal{H}\mathcal{H}(\mathcal{O}_X)).$$

These constructions have been extended when replacing  $\mathcal{O}_X$  with a so-called DQ-algebroid stack  $\mathcal{A}_X$  in [15] (DQ stands for “deformation quantization”). One of the main results of this reference is that Hochschild classes are functorial with respect to the composition of kernels, a kind of (relative) index theorem for coherent DQ-modules.

On the other hand, the notion of Lagrangian cycles of constructible sheaves on real analytic manifolds has been introduced by the first-named author (see [9]) in order to prove an index theorem for such sheaves, after they first appeared in the complex case (see [8, 19]). We refer the reader to [13, Chapter 9] for a systematic study of Lagrangian cycles and for historical comments. Let us briefly recall the construction.

Consider a real analytic manifold  $M$  and let  $\mathbf{k}$  be a unital commutative ring with finite global dimension. Denote by  $\omega_M$  the (topological) dualizing complex of  $M$ , that is,  $\omega_M = \mathrm{or}_M[\dim M]$  where  $\mathrm{or}_M$  is the orientation sheaf of  $M$  and  $\dim M$  is the dimension. Finally, denote by  $\pi_M: T^*M \rightarrow M$  the cotangent bundle of  $M$ . Let  $\Lambda$  be a conic subanalytic Lagrangian subset of  $T^*M$ . The group of Lagrangian cycles supported by  $\Lambda$  is given by  $H_{\Lambda}^0(T^*M; \pi_M^{-1}\omega_M)$ . Denote by  $\mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbf{k}_M)$  the bounded derived category of  $\mathbb{R}$ -constructible sheaves on  $M$ . With an object  $F$  of this category, one associates a Lagrangian cycle supported by  $\mathrm{SS}(F)$ , the microsupport of  $F$ . This cycle is called the characteristic cycle, or the Lagrangian cycle or else the *microlocal Euler class* of  $F$  and is denoted here by  $\mu\mathrm{eu}_M(F)$ .

In fact, it is possible to treat the microlocal Euler classes of  $\mathbb{R}$ -constructible sheaves on real manifolds like Hochschild classes of coherent sheaves on complex manifolds. Denote as above by  $\mathbf{k}_{\Delta_M}$  and  $\omega_{\Delta_M}$  the direct image of  $\mathbf{k}_M$  and  $\omega_M$  under the diagonal embedding  $\delta_M: M \hookrightarrow M \times M$ . Then we have an isomorphism

$$H_{\Lambda}^0(T^*M; \pi_M^{-1}\omega_M) \simeq H_{\Lambda}^0(T^*M; \mu\mathrm{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})), \tag{1.3}$$

where  $\mu\mathrm{hom}$  is the microlocalization of the functor  $\mathrm{R}\mathcal{H}om$ . Then  $\mu\mathrm{eu}_M(F)$  is obtained as follows. Denote by  $\mathbf{D}_M F := \mathrm{R}\mathcal{H}om(F, \omega_M)$  the dual of  $F$ . There are natural morphisms

$$\mathbf{k}_{\Delta_M} \rightarrow F \boxtimes \mathbf{D}_M F \rightarrow \omega_{\Delta_M}, \tag{1.4}$$

whose composition gives the microlocal Euler class of  $F$ .

In this paper, we construct the microlocal Euler class for a wide class of sheaves, including of course the constructible sheaves but also the sheaves of holomorphic solutions of coherent  $\mathcal{D}$ -modules and, more generally, of elliptic pairs in the sense of [23]. To treat such situations, we are led to introduce the notion of a trace kernel.

On a real manifold  $M$  (say of class  $C^\infty$ ), a *trace kernel* is the data of a triplet  $(K, u, v)$  where  $K$  is an object of the derived category of sheaves  $\mathbf{D}^b(\mathbf{k}_{M \times M})$  and  $u, v$  are morphisms

$$u: \mathbf{k}_{\Delta_M} \rightarrow K, \quad v: K \rightarrow \omega_{\Delta_M}. \tag{1.5}$$

One then naturally defines the microlocal Euler class  $\mu eu_M(K, u, v)$  of such a kernel, an element of  $H^0_\Lambda(T^*M; \mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}))$  where  $\Lambda = \text{SS}(K) \cap T^*_{\Delta_M}(M \times M)$ . By (1.4), a constructible sheaf gives rise to a trace kernel.

If  $X$  is a complex manifold and  $\mathcal{M}$  is a coherent  $\mathcal{D}_X$ -module, we construct natural morphisms (over the base ring  $\mathbf{k} = \mathbb{C}$ )

$$\mathbb{C}_{\Delta_X} \rightarrow \Omega_{X \times X} \overset{L}{\otimes}_{\mathcal{D}_{X \times X}} (\mathcal{M} \boxtimes D_D \mathcal{M}) \rightarrow \omega_{\Delta_X}, \tag{1.6}$$

where  $D_D \mathcal{M}$  denotes the dual of  $\mathcal{M}$  as a  $\mathcal{D}$ -module. In other words, one naturally associates a trace kernel on  $X$  with a coherent  $\mathcal{D}_X$ -module. Moreover, we prove that under suitable microlocal conditions, the tensor product of two trace kernels is again a trace kernel, and it follows that one can associate a trace kernel with an elliptic pair.

We study trace kernels and their microlocal Euler classes, showing that some proofs of [15] can be easily adapted to this situation. One of our main results is the functoriality of the microlocal Euler classes: the microlocal Euler class of the composition  $K_1 \circ K_2$  of two trace kernels is the composition of the microlocal Euler classes of  $K_1$  and  $K_2$  (see Theorem 6.3 for a precise statement). Another essential result is that the composition of classes coincides with the composition for  $\pi_M^{-1} \omega_M$  constructed in [13] via the isomorphism between  $\mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})$  and  $\pi_M^{-1} \omega_M$ .

As an application, we recover in a single proof the classical results on the index theorem for constructible sheaves (see [13, §9.5]) as well as the index theorem for elliptic pairs of [23], that is, sheaves of generalized holomorphic solutions of coherent  $\mathcal{D}$ -modules. We also briefly explain how to adapt trace kernels to the formalism of the Lefschetz trace formula.

We call here  $\mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})$  the *microlocal homology of  $M$* , and this paper shows that, in some sense, the microlocal homology of real manifolds plays the same role as the Hochschild homology of complex manifolds.

To conclude this introduction, let us make a general remark. The category  $D_{\mathbb{R}\text{-c}}^b(\mathbf{k}_M)$  of constructible sheaves on a compact real analytic manifold  $M$  is “proper” in the sense of Kontsevich (that is, Ext finite) but it does not admit a Serre functor (in the sense of Bondal and Kapranov) and it is not clear whether it is smooth (again in the sense of Kontsevich). However this category naturally appears in mirror symmetry (see [5]) and it would be a natural aim to try to understand its Hochschild homology in the sense of [17, 16]. We do not know how to compute it, but the above construction, with the use of  $\mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})$ , provides an alternative approach to the Hochschild homology of this category. This result is not totally surprising if one recalls the formula (see [13, Proposition 8.4.14])

$$D_{T^*M}(\mu hom(F, G)) \simeq \mu hom(G, F) \otimes \pi_M^{-1} \omega_M.$$

Hence, in some sense,  $\pi_M^{-1} \omega_M$  plays the role of a microlocal Serre functor. Note that thanks to Nadler and Zaslow [18], we have that the category  $D_{\mathbb{R}\text{-c}}^b(\mathbf{k}_M)$  is equivalent to the Fukaya category of the symplectic manifold  $T^*M$ , and this is another argument for treating sheaves from a microlocal point of view.

2. A short review on sheaves

Throughout this paper, a manifold means a real manifold of class  $C^\infty$ . We shall mainly follow the notation of [13] and use some of the main notions introduced there, in particular that of microsupport and the functor  $\mu hom$ .

Let  $M$  be a manifold. We denote by  $\pi_M : T^*M \rightarrow M$  its cotangent bundle. For a submanifold  $N$  of  $M$ , we denote by  $T_N^*M$  the conormal bundle to  $N$ . In particular,  $T_M^*M$  denotes the zero-section. We set  $\dot{T}^*M := T^*M \setminus T_M^*M$  and we denote by  $\dot{\pi}_M$  the restriction of  $\pi_M$  to  $\dot{T}^*M$ . If there is no risk of confusion, we write simply  $\pi$  and  $\dot{\pi}$  instead of  $\pi_M$  and  $\dot{\pi}_M$ . One denotes by  $a : T^*M \rightarrow T^*M$  the antipodal map,  $(x; \xi) \mapsto (x; -\xi)$ , and for a subset  $S$  of  $T^*M$ , one denotes by  $S^a$  its image under this map. A set  $A \subset T^*M$  is conic if it is invariant under the action of  $\mathbb{R}^+$  on  $T^*M$ .

Let  $f : M \rightarrow N$  be a morphism of manifolds. With  $f$  one associates as usual the maps

$$\begin{array}{ccccc}
 T^*M & \xleftarrow{f_d} & M \times_N T^*N & \xrightarrow{f_\pi} & T^*N \\
 & \searrow \pi_M & \downarrow \pi & & \downarrow \pi_N \\
 & & M & \xrightarrow{f} & N.
 \end{array} \tag{2.1}$$

(Note that in the above citation the map  $f_d$  is denoted by  ${}^t f'^{-1}$ .)

Let  $\Lambda$  be a closed conic subset of  $T^*N$ . One says that  $f$  is *non-characteristic* for  $\Lambda$  if the map  $f_d$  is proper on  $f_\pi^{-1}\Lambda$  or, equivalently,  $f_\pi^{-1}\Lambda \cap f_d^{-1}(T_M^*M) \subset M \times_N T_N^*N$ .

Let  $\mathbf{k}$  be a commutative unital ring with finite global homological dimension. One denotes by  $\mathbf{k}_M$  the constant sheaf on  $M$  with stalk  $\mathbf{k}$  and by  $D^b(\mathbf{k}_M)$  the bounded derived category of sheaves of  $\mathbf{k}$ -modules on  $M$ . When  $M$  is a real analytic manifold, one denotes by  $D_{\mathbb{R}\text{-c}}^b(\mathbf{k}_M)$  the full triangulated subcategory of  $D^b(\mathbf{k}_M)$  consisting of  $\mathbb{R}$ -constructible objects.

One denotes by  $\omega_M$  the dualizing complex on  $M$  and by  $\omega_M^{\otimes -1}$  its dual, that is,  $\omega_M^{\otimes -1} = R\mathcal{H}om(\omega_M, \mathbf{k}_M)$ . More generally, for a morphism  $f : M \rightarrow N$ , one denotes by  $\omega_{M/N} := f^! \mathbf{k}_N \simeq \omega_M \otimes f^{-1}(\omega_N^{\otimes -1})$  the relative dualizing complex. Recall that  $\omega_M \simeq \text{or}_M[\dim M]$  where  $\text{or}_M$  is the orientation sheaf and  $\dim M$  is the dimension of  $M$ . Also recall the natural morphism of functors

$$\omega_{M/N} \otimes f^{-1} \rightarrow f^! \tag{2.2}$$

We have the duality functors

$$D'_M F = R\mathcal{H}om(F, \mathbf{k}_M), \quad D_M F = R\mathcal{H}om(F, \omega_M).$$

For  $F \in D^b(\mathbf{k}_M)$ , one denotes by  $\text{Supp}(F)$  the support of  $F$  and by  $\text{SS}(F)$  its microsupport, a closed  $\mathbb{R}^+$ -conic co-isotropic subset of  $T^*M$ . For a morphism  $f : M \rightarrow N$  and  $G \in D^b(\mathbf{k}_N)$ , one says that  $f$  is non-characteristic for  $G$  if  $f$  is non-characteristic for  $\text{SS}(G)$ .

We shall use systematically the functor  $\mu hom$ , a variant of Sato's microlocalization functor. Recall that for a closed submanifold  $N$  of  $M$ , there is a functor  $\mu_N : D^b(\mathbf{k}_M) \rightarrow D^b(\mathbf{k}_{T_N^*M})$  constructed by Sato (see [22]) and for  $F_1, F_2 \in D^b(\mathbf{k}_M)$ , one defines in [13] the

functor

$$\begin{aligned} \mu hom: D^b(\mathbf{k}_M)^{op} \times D^b(\mathbf{k}_M) &\rightarrow D^b(\mathbf{k}_{T^*M}), \\ \mu hom(F_1, F_2) &:= \mu_{\Delta} R\mathcal{H}om(q_2^{-1}F_1, q_1^!F_2) \end{aligned}$$

where  $q_1$  and  $q_2$  are the first and second projections defined on  $M \times M$  and  $\Delta$  is the diagonal. This sheaf is supported by  $T^*_{\Delta}(M \times M)$  that we identify with  $T^*M$  via the first projection  $T^*(M \times M) \simeq T^*M \times T^*M \rightarrow T^*M$ . Note that

$$\text{Supp}(\mu hom(F_1, F_2)) \subset \text{SS}(F_1) \cap \text{SS}(F_2) \tag{2.3}$$

and we have Sato’s distinguished triangle, functorial in  $F_1$  and  $F_2$ :

$$R\pi_! \mu hom(F_1, F_2) \rightarrow R\pi_* \mu hom(F_1, F_2) \rightarrow R\dot{\pi}_*(\mu hom(F_1, F_2)|_{\dot{T}^*M}) \xrightarrow{+1}. \tag{2.4}$$

Moreover, we have the isomorphism

$$R\pi_* \mu hom(F_1, F_2) \simeq R\mathcal{H}om(F_1, F_2), \tag{2.5}$$

and, assuming that  $M$  is real analytic and  $F_1$  is  $\mathbb{R}$ -constructible, the isomorphism

$$R\pi_! \mu hom(F_1, F_2) \simeq D'_M F_1 \overset{L}{\otimes} F_2. \tag{2.6}$$

In particular, assuming that  $F_1$  is  $\mathbb{R}$ -constructible and  $\text{SS}(F_1) \cap \text{SS}(F_2) \subset T^*_M M$ , we have the natural isomorphism (see [13, Corollary 6.4.3])

$$D'_M F_1 \overset{L}{\otimes} F_2 \xrightarrow{\sim} R\mathcal{H}om(F_1, F_2). \tag{2.7}$$

As recalled in the Introduction, assuming that  $M$  is real analytic and the sheaves are constructible, we have the formula (see [13, Proposition 8.4.14])

$$D_{T^*M}(\mu hom(F_1, F_2)) \simeq \mu hom(F_2, F_1) \otimes \pi_M^{-1} \omega_M \quad \text{for } F_1, F_2 \in D^b_{\mathbb{R}\text{-c}}(\mathbf{k}_M). \tag{2.8}$$

### 3. Compositions of kernels

**Notation 3.1.** (i) For a manifold  $M$ , let  $\delta_M: M \rightarrow M \times M$  denote the diagonal embedding, and  $\Delta_M$  the diagonal set of  $M \times M$ .

- (ii) Let  $M_i$  ( $i = 1, 2, 3$ ) be manifolds. For short, we write  $M_{ij} := M_i \times M_j$  ( $1 \leq i, j \leq 3$ ),  $M_{123} = M_1 \times M_2 \times M_3$ ,  $M_{1223} = M_1 \times M_2 \times M_2 \times M_3$ , etc.
- (iii) We will often write for short  $\mathbf{k}_i$  instead of  $\mathbf{k}_{M_i}$  and  $\mathbf{k}_{\Delta_i}$  instead of  $\mathbf{k}_{\Delta_{M_i}}$ , and similarly with  $\omega_{M_i}$ , etc., and with the index  $i$  replaced with several indices  $ij$ , etc.
- (iv) We denote by  $\pi_i, \pi_{ij}$ , etc. the projection  $T^*M_i \rightarrow M_i, T^*M_{ij} \rightarrow M_{ij}$ , etc.
- (v) We denote by  $q_i$  the projection  $M_{ij} \rightarrow M_i$  or the projection  $M_{123} \rightarrow M_i$  and by  $q_{ij}$  the projection  $M_{123} \rightarrow M_{ij}$ . Similarly, we denote by  $p_i$  the projection  $T^*M_{ij} \rightarrow T^*M_i$  or the projection  $T^*M_{123} \rightarrow T^*M_i$  and by  $p_{ij}$  the projection  $T^*M_{123} \rightarrow T^*M_{ij}$ .

(vi) We also need to introduce the maps  $p_j^a$  or  $p_{ij}^a$ , the composition of  $p_j$  or  $p_{ij}$  and the antipodal map on  $T^*M_j$ . For example,

$$p_{12^a}((x_1, x_2, x_3; \xi_1, \xi_2, \xi_3)) = (x_1, x_2; \xi_1, -\xi_2).$$

(vii) We let  $\delta_2: M_{123} \rightarrow M_{1223}$  be the natural diagonal embedding.

We consider the operation of composition of kernels:

$$\begin{aligned} \circ_2: D^b(\mathbf{k}_{M_{12}}) \times D^b(\mathbf{k}_{M_{23}}) &\rightarrow D^b(\mathbf{k}_{M_{13}}) \\ (K_1, K_2) &\mapsto K_1 \circ_2 K_2 := Rq_{13!}(q_{12}^{-1}K_1 \overset{L}{\otimes} q_{23}^{-1}K_2) \\ &\simeq Rq_{13!}\delta_2^{-1}(K_1 \overset{L}{\boxtimes} K_2). \end{aligned} \tag{3.1}$$

We will use a variant of  $\circ$ :

$$\begin{aligned} *_2: D^b(\mathbf{k}_{M_{12}}) \times D^b(\mathbf{k}_{M_{23}}) &\rightarrow D^b(\mathbf{k}_{M_{13}}) \\ (K_1, K_2) &\mapsto K_1 *_2 K_2 := Rq_{13*}(q_2^{-1}\omega_2 \otimes \delta_2^!(K_1 \overset{L}{\boxtimes} K_2)). \end{aligned} \tag{3.2}$$

We also have  $\omega_{M_{123}/M_{1223}} \simeq q_2^{-1}\omega_{M_2}^{\otimes -1}$  and we deduce from (2.2) a morphism  $\delta_2^{-1} \rightarrow q_2^{-1}\omega_{M_2} \otimes \delta_2^!$ . Using the morphism  $Rp_{13!} \rightarrow Rp_{13*}$  we obtain a natural morphism for  $K_1 \in D^b(\mathbf{k}_{M_{12}})$  and  $K_2 \in D^b(\mathbf{k}_{M_{23}})$ :

$$K_1 \circ K_2 \rightarrow K_1 * K_2. \tag{3.3}$$

It is an isomorphism if  $p_{12^a}^{-1}SS(K_1) \cap p_{23^a}^{-1}SS(K_2) \rightarrow T^*M_{13}$  is proper.

We define the composition of kernels on cotangent bundles (see [13, Proposition 4.4.11]):

$$\begin{aligned} \overset{a}{\circ}_2: D^b(\mathbf{k}_{T^*M_{12}}) \times D^b(\mathbf{k}_{T^*M_{23}}) &\rightarrow D^b(\mathbf{k}_{T^*M_{13}}) \\ (K_1, K_2) &\mapsto K_1 \overset{a}{\circ}_2 K_2 := Rp_{13!}(p_{12^a}^{-1}K_1 \overset{L}{\otimes} p_{23}^{-1}K_2) \\ &\simeq Rp_{13^a!}(p_{12^a}^{-1}K_1 \overset{L}{\otimes} p_{23^a}^{-1}K_2). \end{aligned} \tag{3.4}$$

We also define the corresponding operations for subsets of cotangent bundles. Let  $A \subset T^*M_{12}$  and  $B \subset T^*M_{23}$ . We set

$$\begin{aligned} A \overset{a}{\times}_2 B &= p_{12^a}^{-1}(A) \cap p_{23}^{-1}(B), \\ A \overset{a}{\circ}_2 B &= p_{13}(A \overset{a}{\times}_2 B) \\ &= \left\{ (x_1, x_3; \xi_1, \xi_3) \in T^*M_{13}; \text{ there exists } (x_2; \xi_2) \in T^*M_2 \right. \\ &\quad \left. \text{such that } (x_1, x_2; \xi_1, -\xi_2) \in A, (x_2, x_3; \xi_2, \xi_3) \in B \right\}. \end{aligned} \tag{3.5}$$

We have the following result which slightly strengthens Proposition 4.4.11 of [13] in which the composition  $*$  is not used.

**Proposition 3.2.** For  $G_1, F_1 \in \mathbb{D}^b(\mathbf{k}_{M_{12}})$  and  $G_2, F_2 \in \mathbb{D}^b(\mathbf{k}_{M_{23}})$  there exists a canonical morphism (whose construction is similar to that of [13, Proposition 4.4.11]):

$$\mu\text{hom}(G_1, F_1) \overset{a}{\underset{2}{\circ}} \mu\text{hom}(G_2, F_2) \rightarrow \mu\text{hom}(G_1 * G_2, F_1 \overset{2}{\circ} F_2).$$

**Proof.** In Proposition 4.4.8(i) of the earlier citation, one may replace  $F_2 \overset{L}{\boxtimes}_S G_2$  with  $j^!(F_2 \overset{L}{\boxtimes} G_2) \otimes \omega_{X \times_S Y / X \times Y}^{\otimes -1}$ . Then the proof goes exactly like that of Proposition 4.4.11 in the earlier citation.  $\square$

Let  $\Lambda_{ij} \subset T^*M_{ij}$  ( $i = 1, 2, j = i + 1$ ) be closed conic subsets and consider the condition

$$\text{the projection } p_{13}: \Lambda_{12} \overset{a}{\times} \Lambda_{23} \longrightarrow T^*M_{13} \text{ is proper.} \tag{3.6}$$

We set

$$\Lambda_{13} = \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23}. \tag{3.7}$$

**Corollary 3.3.** Assume that  $\Lambda_{ij}$  ( $i = 1, 2, j = i + 1$ ) satisfy (3.6). We have a composition morphism

$$R\Gamma_{\Lambda_{12}} \mu\text{hom}(G_1, F_1) \overset{a}{\underset{2}{\circ}} R\Gamma_{\Lambda_{23}} \mu\text{hom}(G_2, F_2) \rightarrow R\Gamma_{\Lambda_{13}} \mu\text{hom}(G_1 * G_2, F_1 \overset{2}{\circ} F_2).$$

**Convention 3.4.** In (3.1), we have introduced the composition  $\overset{2}{\circ}$  of kernels  $K_1 \in \mathbb{D}^b(\mathbf{k}_{M_{12}})$  and  $K_2 \in \mathbb{D}^b(\mathbf{k}_{M_{23}})$ . However we shall also use the notation  $M_{22} = M_2 \times M_2$  and consider for example kernels  $L_1 \in \mathbb{D}^b(\mathbf{k}_{M_{122}})$  and  $L_2 \in \mathbb{D}^b(\mathbf{k}_{M_{223}})$ . Then when writing  $L_1 \overset{2}{\circ} L_2$  we mean that the composition is taken with respect to the last variable of  $M_{22}$  for  $L_1$  and the first variable for  $L_2$ . In other words, set  $M_4 = M_2$  and consider  $L_1$  and  $L_2$  as objects of  $\mathbb{D}^b(\mathbf{k}_{M_{142}})$  and  $\mathbb{D}^b(\mathbf{k}_{M_{243}})$  respectively, in which case the composition  $L_1 \overset{2}{\circ} L_2$  is unambiguously defined.

### 4. Microlocal homology

Let  $M$  be a real manifold. Recall that  $\delta_M: M \hookrightarrow M \times M$  denotes the diagonal embedding. We shall identify  $M$  with the diagonal  $\Delta_M$  of  $M \times M$  and we sometimes write  $\Delta$  instead of  $\Delta_M$  if there is no risk of confusion. We shall identify  $T^*M$  with  $T^*_\Delta(M \times M)$  via the map

$$\delta_{T^*M}^a: T^*M \hookrightarrow T^*(M \times M), \quad (x; \xi) \mapsto (x, x; \xi, -\xi).$$

We denote by  $\mathbf{k}_{\Delta_M}$ ,  $\omega_{\Delta_M}$  and  $\omega_{\Delta_M}^{\otimes -1}$  the direct image under  $\delta_M$  of  $\mathbf{k}_M$ ,  $\omega_M$  and  $\omega_M^{\otimes -1} := R\mathcal{H}om(\omega_M, \mathbf{k}_M)$ , respectively.

The next definition is inspired by that of Hochschild homology on complex manifolds (see the Introduction).

**Definition 4.1.** Let  $\Lambda$  be a closed conic subset of  $T^*M$ . We set

$$\begin{aligned} \mathcal{MH}_\Lambda(\mathbf{k}_M) &:= R\Gamma_\Lambda(\delta_{T^*M}^a)^{-1} \mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}), \\ \text{MH}_\Lambda(\mathbf{k}_M) &:= R\Gamma(T^*M; \mathcal{MH}_\Lambda(\mathbf{k}_M)), \\ \text{MH}_\Lambda^k(\mathbf{k}_M) &:= H^k(\text{MH}_\Lambda(\mathbf{k}_M)) = H^k(T^*M; \mathcal{MH}_\Lambda(\mathbf{k}_M)). \end{aligned} \tag{4.1}$$

We call  $\mathcal{MH}_\Lambda(\mathbf{k}_M)$  the *microlocal homology* of  $M$  with support in  $\Lambda$ .

We also write  $\mathcal{MH}(\mathbf{k}_M)$  instead of  $\mathcal{MH}_{T^*M}(\mathbf{k}_M)$ .

**Remark 4.2.** (i) We have  $\mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \simeq (\delta_{T^*M}^a)_* \pi_M^{-1} \omega_M$ . In particular, we have  $\text{MH}_\Lambda(\mathbf{k}_M) \simeq R\Gamma_\Lambda(T^*M; \pi_M^{-1} \omega_M)$  and  $\text{MH}(\mathbf{k}_M) \simeq R\Gamma(M; \omega_M)$ . Assuming that  $M$  is real analytic and  $\Lambda$  is a closed conic subanalytic Lagrangian subset of  $T^*M$ , we recover the space of Lagrangian cycles with support in  $\Lambda$  as defined in [13, § 9.3].

(ii) The support of  $\mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})$  is  $T^*_{\Delta_M}(M \times M)$ . Hence, we have  $R\Gamma_{\delta_{T^*M}^a \Lambda} \mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \simeq (\delta_{T^*M}^a)_* \mathcal{MH}_\Lambda(\mathbf{k}_M)$ .

(iii) If  $M$  is real analytic and  $\Lambda$  is a Lagrangian subanalytic closed conic subset, then we have  $H^k(\mathcal{MH}_\Lambda(\mathbf{k}_M)) = 0$  for  $k < 0$  (see [13, Proposition 9.2.2]).

In the sequel, we denote by  $\Delta_i$  (resp.  $\Delta_{ij}$ ) the diagonal subset  $\Delta_{M_i} \subset M_{ii}$  (resp.  $\Delta_{M_{ij}} \subset M_{ijj}$ ).

**Lemma 4.3.** We have natural morphisms:

$$\begin{aligned} \text{(i)} \quad & \omega_{\Delta_{12}} \circlearrowleft_{22} (\mathbf{k}_{\Delta_2} \overset{\text{L}}{\boxtimes} \omega_{\Delta_3}) \rightarrow \omega_{\Delta_{13}}, \\ \text{(ii)} \quad & \mathbf{k}_{\Delta_{13}} \rightarrow \mathbf{k}_{\Delta_{12}} * (\omega_{\Delta_2}^{\otimes -1} \overset{\text{L}}{\boxtimes} \mathbf{k}_{\Delta_3}). \end{aligned}$$

**Proof.** Denote by  $\delta_{22}$  the diagonal embedding  $M_{112233} \hookrightarrow M_{11222233}$ .

(i) We have the morphisms

$$\begin{aligned} \omega_{\Delta_{12}} \circlearrowleft_{22} (\mathbf{k}_{\Delta_2} \overset{\text{L}}{\boxtimes} \omega_{\Delta_3}) &= Rq_{1133!} \delta_{22}^{-1} (\omega_{\Delta_{12}} \overset{\text{L}}{\boxtimes} \mathbf{k}_{\Delta_2} \overset{\text{L}}{\boxtimes} \omega_{\Delta_3}) \\ &\simeq Rq_{1133!} \omega_{\Delta_{123}} \\ &\rightarrow \omega_{\Delta_{13}}. \end{aligned}$$

(ii) The isomorphism

$$\delta_{22}^! (\mathbf{k}_{\Delta_2} \boxtimes \omega_{\Delta_2}) \simeq \mathbf{k}_{\Delta_2}$$

gives rise to the isomorphisms

$$\begin{aligned} \mathbf{k}_{\Delta_{12}} * (\omega_{\Delta_2}^{\otimes -1} \overset{\text{L}}{\boxtimes} \mathbf{k}_{\Delta_3}) &= Rq_{1133*} (q_{1133}^{-1} \omega_{22} \otimes \delta_{22}^! (\mathbf{k}_{\Delta_{12}} \overset{\text{L}}{\boxtimes} \omega_{\Delta_2}^{\otimes -1} \overset{\text{L}}{\boxtimes} \mathbf{k}_{\Delta_3})) \\ &\simeq Rq_{1133*} \delta_{22}^! (\mathbf{k}_{\Delta_1} \overset{\text{L}}{\boxtimes} \omega_{\Delta_2} \overset{\text{L}}{\boxtimes} \mathbf{k}_{\Delta_{23}}) \\ &\simeq Rq_{1133*} \mathbf{k}_{\Delta_{123}} \end{aligned}$$



and the result follows by adjunction from the morphism

$$q_{1133}^{-1} \mathbf{k}_{\Delta_{13}} \simeq \mathbf{k}_{\Delta_1} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{22} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{\Delta_3} \rightarrow \mathbf{k}_{\Delta_1} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{\Delta_2} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{\Delta_3} = \mathbf{k}_{\Delta_{123}}. \quad \square$$

**Proposition 4.4.** *Let  $M_i$  ( $i = 1, 2, 3$ ) be manifolds. We have a natural composition morphism (whose construction will be given in the course of the proof):*

$$\mu\text{hom}(\mathbf{k}_{\Delta_{12}}, \omega_{\Delta_{12}}) \overset{a}{\underset{2}{\circ}} \mu\text{hom}(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) \rightarrow \mu\text{hom}(\mathbf{k}_{\Delta_{13}}, \omega_{\Delta_{13}}). \quad (4.2)$$

In particular, let  $\Lambda_{ij}$  be a closed conic subset of  $T^*M_{ij}$  ( $ij = 12, 13, 23$ ). If  $\Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23} \subset \Lambda_{13}$ , then we have a morphism

$$\mathcal{MH}_{\Lambda_{12}}(\mathbf{k}_{12}) \overset{a}{\underset{2}{\circ}} \mathcal{MH}_{\Lambda_{23}}(\mathbf{k}_{23}) \rightarrow \mathcal{MH}_{\Lambda_{13}}(\mathbf{k}_{13}). \quad (4.3)$$

**Proof.** Consider the morphism (see Proposition 3.2 and Convention 3.4)

$$\begin{aligned} \mu\text{hom}(\omega_{\Delta_2}^{\otimes -1}, \omega_{\Delta_2}^{\otimes -1}) \overset{a}{\underset{2}{\circ}} \mu\text{hom}(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) &\rightarrow \mu\text{hom}(\omega_{\Delta_2}^{\otimes -1} *_2 \mathbf{k}_{\Delta_{23}}, \omega_{\Delta_2}^{\otimes -1} \overset{\circ}{\underset{2}{\circ}} \omega_{\Delta_{23}}) \\ &\simeq \mu\text{hom}(\omega_{\Delta_2}^{\otimes -1} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathbb{L}}{\boxtimes} \omega_{\Delta_3}). \end{aligned}$$

It induces an isomorphism

$$\mu\text{hom}(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) \simeq \mu\text{hom}(\omega_{\Delta_2}^{\otimes -1} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathbb{L}}{\boxtimes} \omega_{\Delta_3}). \quad (4.4)$$

Note that this isomorphism is also obtained from

$$\begin{aligned} \mu\text{hom}(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) &\simeq \mu\text{hom}((\omega_2^{\otimes -1} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{233}) \overset{\mathbb{L}}{\otimes} \mathbf{k}_{\Delta_{23}}, (\omega_2^{\otimes -1} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{233}) \overset{\mathbb{L}}{\otimes} \omega_{\Delta_{23}}) \\ &\simeq \mu\text{hom}(\omega_{\Delta_2}^{\otimes -1} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathbb{L}}{\boxtimes} \omega_{\Delta_3}). \end{aligned}$$

Applying Proposition 3.2, we get a morphism:

$$\begin{aligned} \mu\text{hom}(\mathbf{k}_{\Delta_{12}}, \omega_{\Delta_{12}}) \overset{a}{\underset{2}{\circ}} \mu\text{hom}(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) \\ \rightarrow \mu\text{hom}(\mathbf{k}_{\Delta_{12}} *_2 (\omega_{\Delta_2}^{\otimes -1} \overset{\mathbb{L}}{\boxtimes} \mathbf{k}_{\Delta_3}), \omega_{\Delta_{12}} \overset{\circ}{\underset{2}{\circ}} (\mathbf{k}_{\Delta_2} \overset{\mathbb{L}}{\boxtimes} \omega_{\Delta_3})). \end{aligned} \quad (4.5)$$

It remains to apply Lemma 4.3. □

**Corollary 4.5.** *Let  $\Lambda_{ij}$  ( $i = 1, 2, j = i + 1$ ) satisfying (3.6) and let  $\Lambda_{13} = \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23}$ . The composition of kernels in (4.3) induces a morphism*

$$\overset{a}{\underset{2}{\circ}} : \text{MH}_{\Lambda_{12}}(\mathbf{k}_{12}) \overset{\mathbb{L}}{\otimes} \text{MH}_{\Lambda_{23}}(\mathbf{k}_{23}) \rightarrow \text{MH}_{\Lambda_{13}}(\mathbf{k}_{13}). \quad (4.6)$$

In particular, each  $\lambda \in \text{MH}_{\Lambda_{12}}^0(\mathbf{k}_{12})$  defines a morphism

$$\lambda \overset{a}{\underset{2}{\circ}} : \text{MH}_{\Lambda_{23}}(\mathbf{k}_{23}) \rightarrow \text{MH}_{\Lambda_{13}}(\mathbf{k}_{13}). \quad (4.7)$$

**Proof.** These morphisms follow from (4.3). The second assertion follows from the isomorphism  $H^0(X) \simeq \text{Hom}_{\mathbb{D}^b(\mathbf{k})}(\mathbf{k}, X)$  in the category  $\mathbb{D}^b(\mathbf{k})$ . □

**Theorem 4.6.** (i) *We have the isomorphisms*

$$\begin{aligned} \mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) &\simeq (\delta_{T^*M}^a)_* \pi_M^{-1} \mathbf{R}\mathcal{H}om(\mathbf{k}_M, \omega_M) \\ &\simeq (\delta_{T^*M}^a)_* \pi_M^{-1} \omega_M. \end{aligned}$$

(ii) *We have a commutative diagram*

$$\begin{array}{ccc} \mu\text{hom}(\mathbf{k}_{\Delta_{12}}, \omega_{\Delta_{12}}) \overset{a}{\underset{22}{\circ}} \mu\text{hom}(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) &\longrightarrow & \mu\text{hom}(\mathbf{k}_{\Delta_{13}}, \omega_{\Delta_{13}}) \\ \downarrow \wr & & \downarrow \wr \\ (\delta_{T^*M_{13}}^a)_* (\pi_{M_{12}}^{-1} \omega_{M_{12}} \overset{a}{\underset{2}{\circ}} \pi_{M_{23}}^{-1} \omega_{M_{23}}) &\longrightarrow & (\delta_{T^*M_{13}}^a)_* \pi_{M_{13}}^{-1} \omega_{M_{13}}. \end{array} \tag{4.8}$$

Here the top horizontal arrow of (4.8) is given in Proposition 4.4, and the bottom horizontal arrow is induced by

$$\begin{aligned} p_{12}^{-1} \pi_{M_{12}}^{-1} \omega_{M_{12}} \overset{L}{\otimes} p_{23}^{-1} \pi_{M_{23}}^{-1} \omega_{M_{23}} &\simeq \pi_{M_1}^{-1} \omega_{M_1} \overset{L}{\boxtimes} \pi_{M_2}^{-1} (\omega_{M_2} \overset{L}{\otimes} \omega_{M_2}) \overset{L}{\boxtimes} \pi_{M_3}^{-1} \omega_{M_3}, \\ \pi_{M_2}^{-1} (\omega_{M_2} \overset{L}{\otimes} \omega_{M_2}) &\simeq \omega_{T^*M_2}, \\ \mathbf{R}p_{13}(\pi_{M_1}^{-1} \omega_{M_1} \overset{L}{\boxtimes} \omega_{T^*M_2} \overset{L}{\boxtimes} \pi_{M_3}^{-1} \omega_{M_3}) &\longrightarrow \pi_{M_1}^{-1} \omega_{M_1} \overset{L}{\boxtimes} \pi_{M_3}^{-1} \omega_{M_3}. \end{aligned}$$

**Proof.** (i) is obvious.

(ii)-(a) By [13, Proposition 4.4.8], we have natural morphisms for  $(i, j) = (1, 2)$  or  $(i, j) = (2, 3)$ :

$$\mu\text{hom}(\mathbf{k}_{\Delta_i}, \omega_{\Delta_i}) \overset{L}{\boxtimes} \mu\text{hom}(\mathbf{k}_{\Delta_j}, \omega_{\Delta_j}) \rightarrow \mu\text{hom}(\mathbf{k}_{\Delta_{ij}}, \omega_{\Delta_{ij}})$$

and it follows from (i) that these morphisms are isomorphisms. These isomorphisms give rise to the isomorphism

$$\begin{aligned} \mu\text{hom}(\mathbf{k}_{\Delta_{12}}, \omega_{\Delta_{12}}) \overset{a}{\underset{22}{\circ}} \mu\text{hom}(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) \\ \simeq \mu\text{hom}(\mathbf{k}_{\Delta_1}, \omega_{\Delta_1}) \overset{L}{\boxtimes} (\mu\text{hom}(\mathbf{k}_{\Delta_2}, \omega_{\Delta_2}) \overset{a}{\underset{22}{\circ}} \mu\text{hom}(\mathbf{k}_{\Delta_2}, \omega_{\Delta_2})) \overset{L}{\boxtimes} \mu\text{hom}(\mathbf{k}_{\Delta_3}, \omega_{\Delta_3}). \end{aligned}$$

Similarly, we have an isomorphism

$$\pi_{M_{12}}^{-1} \omega_{M_{12}} \overset{a}{\underset{2}{\circ}} \pi_{M_{23}}^{-1} \omega_{M_{23}} \simeq \pi_{M_1}^{-1} \omega_{M_1} \boxtimes (\pi_{M_2}^{-1} \omega_{M_2} \overset{a}{\underset{2}{\circ}} \pi_{M_2}^{-1} \omega_{M_2}) \boxtimes \pi_{M_3}^{-1} \omega_{M_3}.$$

Hence, we are reduced to the case where  $M_1 = M_3 = \text{pt}$ , which we shall assume now.

(ii)-(b) We change our notation and set

$$\begin{aligned}
 M &:= M_2, & Y &:= M \times M, \\
 \delta_M &: M \hookrightarrow Y \text{ the diagonal embedding, } \Delta_M = \delta_M(M), \\
 j &: Y \hookrightarrow Y \times Y \text{ the diagonal embedding, } \Delta_Y = j(Y), \\
 \delta_{T^*M}^a &: T^*M \hookrightarrow T^*Y, (x; \xi) \mapsto (x, x; \xi, -\xi), \\
 \delta_{T^*Y}^a &: T^*Y \hookrightarrow T^*Y \times T^*Y, \\
 p &: T^*Y \rightarrow \text{pt the projection,} \\
 a_Y &: Y \rightarrow \text{pt the projection.}
 \end{aligned}$$

With this new notation, the composition  $\overset{a}{\underset{22}{\circ}}_{T^*Y}$  will be denoted by  $\overset{a}{\circ}_{T^*Y}$ .

Consider the diagram (4.9) similar to Diagram (4.4.15) of [13]:

$$\begin{array}{ccccc}
 T^*M \times T^*M & \xrightarrow{i} & T^*Y \times T^*Y & \xleftarrow{\delta_{T^*Y}^a} & T^*Y \\
 & & \uparrow j_\pi & & \uparrow \sim p_1 \\
 & & T^*Y \times_Y T^*Y & \xleftarrow{\tilde{s}} & T^*_{\Delta_Y}(Y \times Y) \\
 & & \downarrow j_d & \square & \downarrow \pi_Y \\
 T^*Y & \xleftarrow{s} & Y & \xrightarrow{a_Y} & \text{pt.}
 \end{array} \tag{4.9}$$

Here,  $i$  is the canonical embedding induced by  $\delta_{T^*M}^a$ ,  $p_1$  is induced by the first projection  $T^*Y \times T^*Y \rightarrow T^*Y$ ,  $s: Y \hookrightarrow T^*Y$  is the zero-section embedding and  $\tilde{s}$  is the natural embedding. Note that the square labeled by  $\square$  is Cartesian. We have

$$\begin{aligned}
 Rp_! \circ (\delta_{T^*Y}^a)^{-1} &\simeq Ra_{Y!} \circ R\pi_{Y!} \circ p_1^{-1} \circ (\delta_{T^*Y}^a)^{-1} \\
 &\simeq Ra_{Y!} \circ R\pi_{Y!} \circ \tilde{s}^{-1} \circ j_\pi^{-1} \\
 &\simeq Ra_{Y!} \circ s^{-1} \circ Rj_{d!} \circ j_\pi^{-1}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &\mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \overset{a}{\circ}_{T^*Y} \mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \\
 &\simeq Rp_!(\delta_{T^*Y}^a)^{-1} (\mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \overset{L}{\boxtimes} \mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})) \\
 &\simeq Ra_{Y!} s^{-1} Rj_{d!} j_\pi^{-1} \mu\text{hom}(\mathbf{k}_{\Delta_M} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{L}{\boxtimes} \omega_{\Delta_M}).
 \end{aligned}$$

Hence, by adjunction, giving a morphism

$$\mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \overset{a}{\circ}_{T^*Y} \mu\text{hom}(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \rightarrow \mathbf{k}$$

is equivalent to giving a morphism in  $D^b(\mathbf{k}_Y)$

$$s^{-1} Rj_{d!} j_\pi^{-1} \mu\text{hom}(\mathbf{k}_{\Delta_M} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{L}{\boxtimes} \omega_{\Delta_M}) \rightarrow a_Y^! \mathbf{k}_{\text{pt}}. \tag{4.10}$$

Note that the left hand side of (4.10) is supported on  $\Delta_M$ . Hence in order to give a morphism (4.10), it is necessary and sufficient to give a morphism in  $D^b(\mathbf{k}_M)$

$$\delta_M^{-1} s^{-1} Rj_{d!} j_\pi^{-1} \mu hom(\mathbf{k}_{\Delta_M} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{L}{\boxtimes} \omega_{\Delta_M}) \rightarrow \delta_M^! a_Y^! \mathbf{k}_{pt}. \tag{4.11}$$

Hence, it is enough to check the commutativity of the upper square in the following diagram in  $D^b(\mathbf{k}_M)$ :

$$\begin{array}{ccc} \delta_M^{-1} s^{-1} Rj_{d!} j_\pi^{-1} \mu hom(\mathbf{k}_{\Delta_M} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{L}{\boxtimes} \omega_{\Delta_M}) & \longrightarrow & \delta_M^! a_Y^! \mathbf{k}_{pt} \\ \downarrow \sim & & \downarrow \text{id} \\ \delta_M^{-1} s^{-1} Rj_{d!} j_\pi^{-1} i_* (\pi_M^{-1} \omega_M \overset{L}{\boxtimes} \pi_M^{-1} \omega_M) & \longrightarrow & \delta_M^! a_Y^! \mathbf{k}_{pt} \\ \downarrow \sim & \xrightarrow{\text{id}} & \downarrow \sim \\ \omega_M & \longrightarrow & \omega_M. \end{array} \tag{4.12}$$

The top horizontal arrow is constructed from a chain of morphisms (see [13, § 4.4]):

$$\begin{aligned} & Rj_{d!} j_\pi^{-1} \mu hom(\mathbf{k}_{\Delta_M} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{L}{\boxtimes} \omega_{\Delta_M}) \\ & \rightarrow \mu hom(j^! (\mathbf{k}_{\Delta_M} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_M}) \overset{L}{\otimes} \omega_Y, j^{-1} (\omega_{\Delta_M} \overset{L}{\boxtimes} \omega_{\Delta_M})) \\ & \simeq \mu hom(\omega_{\Delta_M}, \omega_{\Delta_M} \overset{L}{\otimes} \omega_{\Delta_M}) \simeq (\delta_{T^*M}^a)_* \pi_M^{-1} \omega_M \end{aligned}$$

and

$$\delta_M^{-1} s^{-1} Rj_{d!} j_\pi^{-1} \mu hom(\mathbf{k}_{\Delta_M} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{L}{\boxtimes} \omega_{\Delta_M}) \rightarrow \delta_M^{-1} s^{-1} (\delta_{T^*M}^a)_* \pi_M^{-1} \omega_M \simeq \omega_M. \tag{4.13}$$

Hence, the commutativity of the diagram (4.12) is reduced to the commutativity of the diagram below:

$$\begin{array}{ccc} \delta_M^{-1} s^{-1} Rj_{d!} j_\pi^{-1} \mu hom(\mathbf{k}_{\Delta_M} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{L}{\boxtimes} \omega_{\Delta_M}) & & \\ \downarrow & \searrow \lambda & \\ \delta_M^{-1} s^{-1} Rj_{d!} j_\pi^{-1} i_* (\pi_M^{-1} \omega_M \overset{L}{\boxtimes} \pi_M^{-1} \omega_M) & \xrightarrow{\sim} & \omega_M \end{array} \tag{4.14}$$

where the morphism  $\lambda$  is given by the morphisms in (4.13). All terms of (4.14) are concentrated at the degree  $-\dim M$ . Hence the commutativity of (4.14) is a local problem in  $M$  and we can assume that  $M$  is a Euclidean space. We can check directly in this case. □

**Remark 4.7.** Theorem 4.6 may be applied as follows. Let  $\Lambda_{ij}$  be a closed conic subset of  $T^*M_{ij}$  ( $i = 1, 2, j = i + 1$ ). Assume (3.6), that is, the projection  $p_{13}: \Lambda_{12} \overset{a}{\times} \Lambda_{23} \rightarrow T^*M_{13}$

is proper, and set  $\Lambda_{13} = \Lambda_{12} \overset{a}{\circ} \Lambda_{23}$ . Let  $\lambda_{ij} \in \mathbb{M}\mathbb{H}_{\Lambda_{ij}}^0(\mathbf{k}_{M_{ij}}) \simeq H_{\Lambda_{ij}}^0(T^*M_{ij}; \pi^{-1}\omega_{ij})$ . Then

$$\lambda_{12} \overset{a}{\circ} \lambda_{23} = \int_{T^*M_2} \lambda_{12} \cup \lambda_{23} \tag{4.15}$$

where the right hand side is obtained as follows. Set  $\Lambda := \Lambda_{12} \overset{a}{\times} \Lambda_{23}$  and consider the morphisms

$$\begin{aligned} &H_{\Lambda_{12}}^0(T^*M_{12}; \pi^{-1}\omega_{12}) \times H_{\Lambda_{23}}^0(T^*M_{23}; \pi^{-1}\omega_{23}) \\ &\rightarrow H_{\Lambda}^0(T^*M_{123}; \pi^{-1}\omega_1 \boxtimes \omega_{T^*M_2} \boxtimes \pi^{-1}\omega_3) \\ &\rightarrow H_{\Lambda_{13}}^0(T^*M_{13}; \pi^{-1}\omega_{13}). \end{aligned}$$

The first morphism is the cup product and the second one is the integration morphism with respect to  $T^*M_2$ .

### 5. Microlocal Euler classes of trace kernels

In this section, we often write  $\Delta$  instead of  $\Delta_M$ .

**Definition 5.1.** A trace kernel  $(K, u, v)$  on  $M$  is the data of  $K \in \mathbf{D}^b(\mathbf{k}_{M \times M})$  together with morphisms

$$\mathbf{k}_{\Delta} \xrightarrow{u} K \quad \text{and} \quad K \xrightarrow{v} \omega_{\Delta}. \tag{5.1}$$

In the sequel, as long as there is no risk of confusion, we simply write  $K$  instead of  $(K, u, v)$ .

For a trace kernel  $K$  as above, we set

$$\text{SS}_{\Delta}(K) := \text{SS}(K) \cap T_{\Delta}^*(M \times M) = (\delta_{T^*M}^a)^{-1} \text{SS}(K). \tag{5.2}$$

(Recall that one often identifies  $T^*M$  and  $T_{\Delta}^*(M \times M)$  through  $\delta_{T^*M}^a: T^*M \hookrightarrow T^*M \times T^*M$ .)

**Definition 5.2.** Let  $(K, u, v)$  be a trace kernel.

- (a) The morphism  $u$  defines an element  $\tilde{u}$  in  $H_{\text{SS}_{\Delta}(K)}^0(T^*M; \mu\text{hom}(\mathbf{k}_{\Delta}, K))$  and the microlocal Euler class  $\mu\text{eu}_M(K)$  of  $K$  is the image of  $\tilde{u}$  under the morphism  $\mu\text{hom}(\mathbf{k}_{\Delta}, K) \rightarrow \mu\text{hom}(\mathbf{k}_{\Delta}, \omega_{\Delta})$  associated with the morphism  $v$ .
- (b) Let  $\Lambda$  be a closed conic subset of  $T^*M$  containing  $\text{SS}_{\Delta}(K)$ . One denotes by  $\mu\text{eu}_{\Lambda}(K)$  the image of  $\tilde{u}$  in  $H_{\Lambda}^0(T^*M; \mu\text{hom}(\mathbf{k}_{\Delta}, \omega_{\Delta}))$ .

Hence,

$$\mu\text{eu}_{\Lambda}(K) \in \mathbb{M}\mathbb{H}_{\Lambda}^0(\mathbf{k}_M) \simeq H_{\Lambda}^0(T^*M; \pi^{-1}\omega_M). \tag{5.3}$$

Let  $\tilde{v}$  be the element of  $H_{\text{SS}_{\Delta}(K)}^0(T^*M; \mu\text{hom}(K, \omega_{\Delta}))$  induced by  $v$ . Then the microlocal Euler class  $\mu\text{eu}_M(K)$  of  $K$  coincides with the image of  $\tilde{v}$  under the morphism  $\mu\text{hom}(K, \omega_{\Delta_M}) \rightarrow \mu\text{hom}(\mathbf{k}_{\Delta}, \omega_{\Delta})$  associated with the morphism  $u$ , which can be easily

seen from the following commutative diagram:

$$\begin{CD}
 (\delta_{T^*M}^a)^{-1} \mu\text{hom}(K, K) @>v>> (\delta_{T^*M}^a)^{-1} \mu\text{hom}(K, \omega_\Delta) \\
 @VVuV @VVuV \\
 (\delta_{T^*M}^a)^{-1} \mu\text{hom}(\mathbf{k}_\Delta, K) @>v>> (\delta_{T^*M}^a)^{-1} \mu\text{hom}(\mathbf{k}_\Delta, \omega_\Delta).
 \end{CD}$$

One denotes by  $\text{eu}(K)$  the restriction of  $\mu\text{eu}(K)$  to the zero-section  $M$  of  $T^*M$  and calls it the *Euler class* of  $K$ . Hence

$$\text{eu}_M(K) \in H_{\text{Supp}(K) \cap \Delta}^0(M; \omega_M). \tag{5.4}$$

It is nothing but the class induced by the composition  $\mathbf{k}_{\Delta_M} \rightarrow K \rightarrow \omega_{\Delta_M}$ .

We say that  $L \in \mathbf{D}^b(\mathbf{k}_M)$  is *invertible* if  $L$  is locally isomorphic to  $\mathbf{k}_M[d]$  for some  $d \in \mathbb{Z}$ . Then,  $L^{\otimes -1} := \mathbf{R}\mathcal{H}om(L, \mathbf{k}_M)$  is also invertible and  $L \overset{\mathbf{L}}{\boxtimes} L^{\otimes -1} \simeq \mathbf{k}_M$ .

**Proposition 5.3.** *Let  $L$  be an invertible object in  $\mathbf{D}^b(\mathbf{k}_M)$  and  $K$  a trace kernel. Then  $K \overset{\mathbf{L}}{\boxtimes} (L \overset{\mathbf{L}}{\boxtimes} L^{\otimes -1})$  is a trace kernel and  $\mu\text{eu}(K \overset{\mathbf{L}}{\boxtimes} (L \overset{\mathbf{L}}{\boxtimes} L^{\otimes -1})) = \mu\text{eu}(K)$ .*

**Proof.**  $L \overset{\mathbf{L}}{\boxtimes} L^{\otimes -1}$  is canonically isomorphic to  $\mathbf{k}_{M \times M}$  on a neighborhood of the diagonal set  $\Delta_M$  of  $M \times M$ . □

**Remark 5.4.** Of course, we could also have defined a trace kernel as a sequence of morphisms

$$\omega_{\Delta_M}^{\otimes -1} \rightarrow \tilde{K} \rightarrow \mathbf{k}_{\Delta_M}. \tag{5.5}$$

When treating sheaves, the two definitions would give the same microlocal Euler class on taking  $K = \tilde{K} \otimes (\mathbf{k}_M \overset{\mathbf{L}}{\boxtimes} \omega_M)$ . However, when working with  $\mathcal{O}$ -modules or with DQ-modules as in [15], the two constructions give different classes. Note that we have chosen an analogue of (5.5) in [15].

**Trace kernels for constructible sheaves**

Let us denote by  $\mathbf{D}_{cc}^b(\mathbf{k}_M)$  the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_M)$  consisting of cohomologically constructible sheaves (see [13, § 3.4]).

**Lemma 5.5.** *Let  $F \in \mathbf{D}_{cc}^b(\mathbf{k}_M)$ . There are natural morphisms in  $\mathbf{D}_{cc}^b(\mathbf{k}_{M \times M})$ :*

$$\mathbf{k}_{\Delta_M} \rightarrow F \overset{\mathbf{L}}{\boxtimes} D_M F, \tag{5.6}$$

$$F \overset{\mathbf{L}}{\boxtimes} D_M F \rightarrow \omega_{\Delta_M}. \tag{5.7}$$

In other words, an object  $F \in \mathbf{D}_{cc}^b(\mathbf{k}_M)$  defines naturally a trace kernel on  $M$ .

**Proof.** (i) We have

$$\mathbf{k}_M \rightarrow \mathbf{R}\mathcal{H}om(F, F) \simeq \delta^!(F \overset{\mathbf{L}}{\boxtimes} D_M F).$$

Hence, the result follows by adjunction.

(ii) The morphism (5.7) may be deduced from (5.6) by duality, or by adjunction from the morphism

$$\delta^{-1}(F \boxtimes^{\mathbb{L}} D_M F) \rightarrow \omega_M. \quad \square$$

**Notation 5.6.** We shall denote by  $\text{TK}(F)$  the trace kernel associated with  $F \in \mathbf{D}_{\text{cc}}^b(\mathbf{k}_M)$ , that is the data of  $F \boxtimes^{\mathbb{L}} D_M F$  and the morphisms (5.6), (5.7). Note that we always have  $\text{SS}_{\Delta}(\text{TK}(F)) \subset \text{SS}(F)$  and the equality holds if  $M$  is real analytic and  $F$  is  $\mathbb{R}$ -constructible.

We have the chain of morphisms

$$\begin{aligned} \mu\text{hom}(F, F) &\simeq (\delta_{T^*M}^a)^{-1} \mu\text{hom}(\mathbf{k}_{\Delta}, F \boxtimes^{\mathbb{L}} DF) \\ &\rightarrow (\delta_{T^*M}^a)^{-1} \mu\text{hom}(\mathbf{k}_{\Delta}, \omega_{\Delta}). \end{aligned}$$

We deduce the map

$$H_{\text{SS}(F)}^0(T^*M; \mu\text{hom}(F, F)) \rightarrow \mathbb{MH}_{\text{SS}(F)}^0(\mathbf{k}_M). \quad (5.8)$$

**Definition 5.7.** Let  $F \in \mathbf{D}_{\text{cc}}^b(\mathbf{k}_M)$ . The image of  $\text{id}_F$  under the map (5.8) is called the microlocal Euler class of  $F$  and is denoted by  $\mu\text{eu}_M(F)$ .

Clearly, one has

$$\mu\text{eu}_M(F) = \mu\text{eu}_M(\text{TK}(F)). \quad (5.9)$$

Assume that  $M$  is real analytic and denote by  $\mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbf{k}_M)$  the full triangulated subcategory of  $\mathbf{D}^b(\mathbf{k}_M)$  consisting of  $\mathbb{R}$ -constructible complexes. Of course,  $\mathbb{R}$ -constructible complexes are cohomologically constructible. In [13, §9.4] the microlocal Euler class of an object  $F \in \mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbf{k}_M)$  is constructed as above and this class is also called the characteristic cycle, or else, the Lagrangian cycle, of  $F$ .

**Remark 5.8.** Let  $(K, u, v)$  be a trace kernel on  $M$ . Let  $\delta: M \rightarrow M \times M$  be the diagonal embedding. Then  $u$  and  $v$  decompose as

$$\mathbf{k}_{\Delta_M} \rightarrow \delta_* \delta^! K \rightarrow K \rightarrow \delta_* \delta^{-1} K \rightarrow \omega_{\Delta_M}.$$

Hence  $\delta_* \delta^! K$  and  $\delta_* \delta^{-1} K$  are also trace kernels. We have evidently

$$\mu\text{eu}_M(\delta_* \delta^! K) = \mu\text{eu}_M(\delta_* \delta^{-1} K) = \mu\text{eu}_M(K) \quad \text{as elements in } \mathbb{MH}_{T^*M}^0(\mathbf{k}_M).$$

### Trace kernels over one point

Let us consider the particular case where  $M$  is a single point,  $M = \text{pt}$ , and let us identify a sheaf over  $\text{pt}$  with a  $\mathbf{k}$ -module. In this situation, a trace kernel  $(K, u, v)$  is the data of  $K \in \mathbf{D}^b(\mathbf{k})$  together with linear maps

$$\mathbf{k} \xrightarrow{u} K \xrightarrow{v} \mathbf{k}.$$

The (microlocal) Euler class  $\text{eu}_{\text{pt}}(K)$  of this kernel is the image of  $1 \in \mathbf{k}$  under  $v \circ u$ .

Assume now that  $\mathbf{k}$  is a field and denote by  $D_f^b(\mathbf{k})$  the full triangulated subcategory of  $D^b(\mathbf{k})$  consisting of objects with finite-dimensional cohomologies. Let  $V \in D_f^b(\mathbf{k})$  and set  $V^* = \text{RHom}(V, \mathbf{k})$ . Let  $K = \text{TK}(V) = V \otimes V^*$ , and let  $v$  be the trace morphism and  $u$  its dual. Then

- (a)  $\text{eu}_{\text{pt}}(V \otimes V^*) = \text{tr}(\text{id}_V)$ , the trace of the identity of  $V$ .
- (b) If  $\mathbf{k}$  has characteristic 0, then (5.10)  
 $\text{eu}_{\text{pt}}(V \otimes V^*) = \chi(V)$ , the Euler–Poincaré index of  $V$ .

**Trace kernels for  $\mathcal{D}$ -modules**

In this subsection, we denote by  $X$  a complex manifold of complex dimension  $d_X$  and the base ring  $\mathbf{k}$  is the field  $\mathbb{C}$ . We denote by  $\mathcal{O}_X$  the structure sheaf and by  $\Omega_X$  the sheaf of holomorphic forms of maximal degree. We still denote by  $\omega_X$  the topological dualizing complex and recall the isomorphism  $\omega_X \simeq \mathbb{C}_X [2d_X]$ .

One denotes by  $\mathcal{D}_X$  the sheaf of  $\mathbb{C}_X$ -algebras of (finite-order) holomorphic differential operators on  $X$  and we refer the reader to [11] for a detailed exposition of the theory of  $\mathcal{D}$ -modules. We denote by  $\text{Mod}(\mathcal{D}_X)$  the category of left  $\mathcal{D}_X$ -modules and by  $D^b(\mathcal{D}_X)$  its bounded derived category. We also denote by  $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$  the abelian category of coherent  $\mathcal{D}_X$ -modules and by  $D_{\text{coh}}^b(\mathcal{D}_X)$  the full triangulated subcategory of  $D^b(\mathcal{D}_X)$  consisting of objects with coherent cohomologies.

We denote by  $D_{\mathcal{D}}: D^b(\mathcal{D}_X)^{\text{op}} \rightarrow D^b(\mathcal{D}_X)$  the duality functor for left  $\mathcal{D}$ -modules:

$$D_{\mathcal{D}} \mathcal{M} := \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1} [d_X].$$

We denote by  $\cdot \boxtimes \cdot$  the external product for  $\mathcal{D}$ -modules:

$$\mathcal{M} \boxtimes \mathcal{N} := \mathcal{D}_{X \times X} \otimes_{\mathcal{D}_X \boxtimes \mathcal{D}_X} (\mathcal{M} \boxtimes^L \mathcal{N}).$$

Let  $\Delta$  be the diagonal of  $X \times X$ . The left  $\mathcal{D}_{X \times X}$ -module  $H_{[\Delta]}^{d_X}(\mathcal{O}_{X \times X})$  (the algebraic cohomology with support in  $\Delta$ ) is denoted as usual by  $\mathcal{B}_{\Delta}$ . Note that

$$D_{\mathcal{D}} \mathcal{B}_{\Delta} \simeq \mathcal{B}_{\Delta}.$$

One should be aware that here, the dual is taken over  $X \times X$ . We also introduce

$$\mathcal{B}_{\Delta}^{\vee} := \mathcal{B}_{\Delta} [2d_X].$$

For  $\mathcal{M} \in D_{\text{coh}}^b(\mathcal{D}_X)$ , we have the isomorphism

$$\text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{M}) \simeq \text{RHom}_{\mathcal{D}_{X \times X}}(\mathcal{B}_{\Delta}, \mathcal{M} \boxtimes D_{\mathcal{D}} \mathcal{M}) [d_X].$$

We deduce the morphism in  $D^b(\mathcal{D}_{X \times X})$

$$\mathcal{B}_{\Delta} \rightarrow \mathcal{M} \boxtimes D_{\mathcal{D}} \mathcal{M} [d_X] \tag{5.11}$$

and by duality, the morphism in  $D^b(\mathcal{D}_{X \times X})$

$$\mathcal{M} \boxtimes D_{\mathcal{D}} \mathcal{M} [d_X] \rightarrow \mathcal{B}_{\Delta}^{\vee}. \tag{5.12}$$



Denote by  $\mathcal{E}_X$  the sheaf on  $T^*X$  of microdifferential operators of [22]. For a coherent  $\mathcal{D}_X$ -module  $\mathcal{M}$  set

$$\mathcal{M}^E := \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$$

and recall that, denoting by  $\text{char}(\mathcal{M})$  the characteristic variety of  $\mathcal{M}$ , we have  $\text{char}(\mathcal{M}) = \text{Supp}(\mathcal{M}^E)$ . One also sets

$$\mathcal{C}_\Delta := \mathcal{B}_\Delta^E, \quad \mathcal{C}_\Delta^\vee := (\mathcal{B}_\Delta^\vee)^E.$$

We denote by  $D_\mathcal{E} : D^b(\mathcal{E}_X)^{\text{op}} \rightarrow D^b(\mathcal{E}_X)$  the duality functor for left  $\mathcal{E}$ -modules:

$$D_\mathcal{E}\mathcal{M} := R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_X) \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega_X^{\otimes -1} [d_X]$$

and we denote by  $\boxtimes$  the external product for  $\mathcal{E}$ -modules:

$$\mathcal{M} \boxtimes \mathcal{N} := \mathcal{E}_{X \times X} \otimes_{\mathcal{E}_X \boxtimes \mathcal{E}_X} (\mathcal{M} \overset{L}{\boxtimes} \mathcal{N}).$$

The morphisms (5.11) and (5.12) give rise to the morphisms

$$\mathcal{C}_\Delta \rightarrow \mathcal{M}^E \boxtimes D_\mathcal{E}\mathcal{M}^E [d_X] \rightarrow \mathcal{C}_\Delta^\vee. \tag{5.13}$$

Let  $\Lambda$  be a closed conic subset of  $T^*X$ . One sets

$$\begin{aligned} \mathcal{H}\mathcal{H}(\mathcal{E}_X) &= (\delta_{T^*X}^a)^{-1} R\mathcal{H}om_{\mathcal{E}_{X \times X}}(\mathcal{C}_\Delta, \mathcal{C}_\Delta^\vee), \\ \mathbb{H}\mathbb{H}_\Lambda(\mathcal{E}_X) &= R\Gamma_\Lambda(T^*X; \mathcal{H}\mathcal{H}(\mathcal{E}_X)), \\ \mathbb{H}\mathbb{H}_\Lambda^k(\mathcal{E}_X) &= H^k(\mathbb{H}\mathbb{H}_\Lambda(\mathcal{E}_X)) = H^k_\Lambda(T^*X; \mathcal{H}\mathcal{H}(\mathcal{E}_X)). \end{aligned}$$

We call  $\mathbb{H}\mathbb{H}_\Lambda(\mathcal{E}_X)$ , the *Hochschild homology* of  $\mathcal{E}_X$  with support in  $\Lambda$ .

The morphisms in (5.13) define a class

$$\text{hh}_\mathcal{E}(\mathcal{M}) \in \mathbb{H}\mathbb{H}_{\text{char}(\mathcal{M})}^0(\mathcal{E}_X) \tag{5.14}$$

that we call the *Hochschild class* of  $\mathcal{M}$ .

Let  $S$  be a closed subset of  $X$ . By restricting the above construction to the zero-section  $X$  of  $T^*X$ , we obtain the Hochschild homology of  $\mathcal{D}_X$ :

$$\begin{aligned} \mathcal{H}\mathcal{H}(\mathcal{D}_X) &= (\delta_X)^{-1} R\mathcal{H}om_{\mathcal{D}_{X \times X}}(\mathcal{B}_\Delta, \mathcal{B}_\Delta^\vee) \simeq \mathcal{H}\mathcal{H}(\mathcal{E}_X)|_X, \\ \mathbb{H}\mathbb{H}_S(\mathcal{D}_X) &= R\Gamma_S(X; \mathcal{H}\mathcal{H}(\mathcal{D}_X)), \\ \mathbb{H}\mathbb{H}_S^k(\mathcal{D}_X) &= H^k(\mathbb{H}\mathbb{H}_S(\mathcal{D}_X)) = H^k_S(X; \mathcal{H}\mathcal{H}(\mathcal{D}_X)). \end{aligned}$$

Then, for  $\mathcal{M} \in D_{\text{coh}}^b(\mathcal{D}_X)$  one obtains

$$\text{hh}_\mathcal{D}(\mathcal{M}) := \text{hh}_\mathcal{E}(\mathcal{M})|_X \in \mathbb{H}\mathbb{H}_{\text{Supp}(\mathcal{M})}^0(\mathcal{D}_X).$$

We shall make a link between the Hochschild class of  $\mathcal{M}$  and the microlocal Euler class of a trace kernel attached to the sheaves of holomorphic solutions of  $\mathcal{M}$ . We need a lemma.

**Lemma 5.9.** *For  $\mathcal{N}_1$  and  $\mathcal{N}_2$  in  $D_{\text{coh}}^b(\mathcal{D}_X)$ , there exists a natural morphism*

$$R\mathcal{H}om_\mathcal{E}(\mathcal{N}_1^E, \mathcal{N}_2^E) \rightarrow \mu\text{hom}(\Omega_X \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{N}_1, \Omega_X \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{N}_2). \tag{5.15}$$

Moreover, this morphism is compatible with the composition

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{E}}(\mathcal{N}_1^E, \mathcal{N}_2^E) \otimes \mathrm{R}\mathcal{H}om_{\mathcal{E}}(\mathcal{N}_2^E, \mathcal{N}_3^E) &\rightarrow \mathrm{R}\mathcal{H}om_{\mathcal{E}}(\mathcal{N}_1^E, \mathcal{N}_3^E), \\ \mu\mathrm{hom}(F_1, F_2) \otimes \mu\mathrm{hom}(F_2, F_3) &\rightarrow \mu\mathrm{hom}(F_1, F_3). \end{aligned}$$

**Proof.** We have the natural morphism in  $\mathrm{D}^b(\pi^{-1}\mathcal{D}_X \otimes \pi^{-1}\mathcal{D}_X^{\mathrm{op}})$  (see [12, Proposition 10.6.2])

$$\mathcal{E}_X \rightarrow \mu\mathrm{hom}(\Omega_X, \Omega_X).$$

This gives rise to the morphisms

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{N}_1, \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{N}_2) \\ \rightarrow \mathrm{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{N}_1, \mu\mathrm{hom}(\Omega_X, \Omega_X)) \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{N}_2 \\ \simeq \mu\mathrm{hom}(\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N}_1, \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N}_2). \end{aligned} \quad \square$$

We have

$$\begin{aligned} \Omega_{X \times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X \times X}} \mathcal{B}_{\Delta} &\simeq \mathbb{C}_{\Delta}, \\ \Omega_{X \times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X \times X}} \mathcal{B}_{\Delta}^{\vee} &\simeq \omega_{\Delta}. \end{aligned}$$

Applying Lemma 5.9, one deduces the morphisms

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{E}_{X \times X}}(\mathcal{C}_{\Delta}, \mathcal{C}_{\Delta}^{\vee}) &\rightarrow \mu\mathrm{hom}(\Omega_{X \times X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X \times X}} \mathcal{B}_{\Delta}, \Omega_{X \times X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X \times X}} \mathcal{B}_{\Delta}^{\vee}) \\ &\simeq \mu\mathrm{hom}(\mathbb{C}_{\Delta}, \omega_{\Delta}). \end{aligned}$$

An easy calculation shows that the first arrow is also an isomorphism. Therefore, we get the isomorphism

$$\mathcal{H}\mathcal{H}(\mathcal{E}_X) \xrightarrow{\sim} \mathcal{M}\mathcal{H}(\mathbb{C}_X). \tag{5.16}$$

Recall that the Hochschild homology of  $\mathcal{E}_X$  has already been calculated in [2].

Applying the functor  $\Omega_{X \times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X \times X}} \cdot$  to (5.11) and (5.12) we get the morphisms

$$\mathbb{C}_{\Delta} \rightarrow \Omega_{X \times X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X \times X}} (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}) \rightarrow \omega_{\Delta}. \tag{5.17}$$

**Notation 5.10.** For  $\mathcal{M} \in \mathrm{D}_{\mathrm{coh}}^b(\mathcal{D}_X)$ , we denote by  $\mathrm{TK}(\mathcal{M})$  the trace kernel given by (5.17).

Since  $\mathrm{char}(\mathcal{M}) = \mathrm{SS}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))$  by [13, Theorem 11.3.3], we get that  $\mu\mathrm{eu}_{\mathcal{M}}(\mathrm{TK}(\mathcal{M}))$  is supported by  $\mathrm{char}(\mathcal{M})$ , the characteristic variety of  $\mathcal{M}$ .

**Proposition 5.11.** *After identifying  $\mathcal{H}\mathcal{H}(\mathcal{E}_X)$  and  $\mathcal{M}\mathcal{H}(\mathbb{C}_X)$  through the isomorphism (5.16), we have  $\mathrm{hh}_{\mathcal{E}}(\mathcal{M}) = \mu\mathrm{eu}_X(\mathrm{TK}(\mathcal{M}))$  in  $\mathrm{HH}_{\mathrm{char}(\mathcal{M})}^0(\mathbb{C}_X)$ .*

**Proof.** This follows from Lemma 5.9 applied to (5.13). □

Note that the class  $\mu\mathrm{eu}_X(\mathrm{TK}(\mathcal{M}))$  coincides with the microlocal Euler class of  $\mathcal{M}$  already introduced by Schapira and Schneiders in [23].

### 6. Operations on microlocal Euler classes I

In this section, we shall adapt to trace kernels the constructions of [15, Chapter 4 §3] and we shall show that under natural microlocal conditions of properness, the microlocal Euler class of the composition of two kernels is the composition of the classes.

We use Notation 3.1 and we consider a trace kernel  $(K, u, v)$  on  $M_{12}$ .

**Lemma 6.1.** *Let  $K$  be a trace kernel on  $M_{12}$ . There are natural morphisms in  $D^b(\mathbf{k}_{M_{11}})$ :*

$$\mathbf{k}_{\Delta_{13}} \rightarrow K *_2 (\omega_{\Delta_2}^{\otimes -1} \boxtimes^L \mathbf{k}_{\Delta_3}), \tag{6.1}$$

$$K \circ_2 (\mathbf{k}_{\Delta_2} \boxtimes^L \omega_{\Delta_3}) \rightarrow \omega_{\Delta_{13}}. \tag{6.2}$$

**Proof.** (i) By Lemma 4.3(ii) we have a morphism  $\mathbf{k}_{\Delta_{13}} \rightarrow \mathbf{k}_{\Delta_{12}} *_2 (\omega_{\Delta_2}^{\otimes -1} \boxtimes^L \mathbf{k}_{\Delta_3})$ . By composing this morphism with  $\mathbf{k}_{\Delta_{12}} \rightarrow K$ , we get (6.1).

(ii) By Lemma 4.3(i) we have a morphism  $\omega_{\Delta_{12}} \circ_2 (\mathbf{k}_{\Delta_2} \boxtimes^L \omega_{\Delta_3}) \rightarrow \omega_{\Delta_{13}}$ . By composing this morphism with  $K \rightarrow \omega_{\Delta_{12}}$  we get (6.2). □

Let  $K$  be a trace kernel on  $M_{12}$  with microsupport  $SS(K)$  contained in a closed conic subset  $\Lambda_{1122}$  of  $T^*M_{1122}$  and let  $\Lambda_{23}$  a closed conic subset of  $T^*M_{23}$ . We assume

$$\Lambda_{1122} \overset{a}{\times}_{22} \delta_{T^*M_{23}}^a \Lambda_{23} \text{ is proper over } T^*M_{1133}. \tag{6.3}$$

We set

$$\begin{cases} \Lambda_{12} := \Lambda_{1122} \cap T_{\Delta_{12}}^* M_{1122}, \\ \Lambda_{1133} := \Lambda_{1122} \overset{a}{\circ}_{22} \delta_{T^*M_{23}}^a \Lambda_{23}, \\ \Lambda_{13} := \Lambda_{1133} \cap T_{\Delta_{13}}^* M_{1133} = \Lambda_{12} \overset{a}{\circ}_{22} \Lambda_{23}. \end{cases} \tag{6.4}$$

We define a map

$$\Phi_K : \mathbb{M}\mathbb{H}_{\Lambda_{23}}(\mathbf{k}_{23}) \longrightarrow \mathbb{M}\mathbb{H}_{\Lambda_{13}}(\mathbf{k}_{13}) \tag{6.5}$$

by the sequence of morphisms

$$\begin{aligned} \mathbb{M}\mathbb{H}_{\Lambda_{23}}(\mathbf{k}_{23}) &\simeq R\Gamma_{\delta_{T^*M_{23}}^a \Lambda_{23}}(T^*M_{2233}; \mu\text{hom}(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}})) \\ &\simeq R\Gamma_{\delta_{T^*M_{23}}^a \Lambda_{23}}(T^*M_{2233}; \mu\text{hom}(\omega_{\Delta_2}^{\otimes -1} \boxtimes^L \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \boxtimes^L \omega_{\Delta_3})) \\ &\rightarrow R\Gamma_{\Lambda_{1133}}(T^*M_{1133}; \mu\text{hom}(K, K) \overset{a}{\circ}_{22} \mu\text{hom}(\omega_{\Delta_2}^{\otimes -1} \boxtimes^L \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \boxtimes^L \omega_{\Delta_3})) \\ &\rightarrow R\Gamma_{\Lambda_{1133}}(T^*M_{1133}; \mu\text{hom}(K *_2 (\omega_{\Delta_2}^{\otimes -1} \boxtimes^L \mathbf{k}_{\Delta_3}), K \circ_2 (\mathbf{k}_{\Delta_2} \boxtimes^L \omega_{\Delta_3}))) \\ &\rightarrow \Gamma(T^*M_{1133}; \mu\text{hom}(\mathbf{k}_{\Delta_{13}}, \omega_{\Delta_{13}})) \simeq \mathbb{M}\mathbb{H}_{\Lambda_{13}}(\mathbf{k}_{13}). \end{aligned}$$

Here the first arrow is given by  $\text{id}_K$ , the second is given by Proposition 3.2, and the last arrow is induced by the morphisms in Lemma 6.1.

The next result is similar to [15, Theorem 4.3.5].

**Proposition 6.2.** *Let  $\Lambda_{1122} \subset T^*M_{1122}$  and  $\Lambda_{23} \subset T^*M_{23}$  be closed conic subsets satisfying (6.3) and recall the notation (6.4). Let  $K$  be a trace kernel on  $M_{12}$  with microsupport contained in  $\Lambda_{1122}$ . Then the map  $\Phi_K$  in (6.5) is the map  $\mu eu_{M_{12}}(K) \overset{a}{\circ} \overset{L}{\circ} \overset{L}{\circ} \overset{L}{\circ}$  given by Corollary 4.5.*

**Proof.** By using the morphism  $\mathbf{k}_{\Delta_{12}} \rightarrow K$ , we find the commutative diagram below:

$$\begin{CD} R\Gamma_{\Lambda_{23}}(T^*M_{2233}; \mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}})) @>>> R\Gamma_{\Lambda_{13}}(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} * \mathbf{k}_{\Delta_{23}}, \mathbf{k}_{\Delta_{12}} \overset{L}{\circ} \omega_{\Delta_{23}})) \\ @VVV @VVV \\ R\Gamma_{\Lambda_{1133}}(T^*M_{1133}; \mu hom(K * \mathbf{k}_{\Delta_{23}}, K \overset{L}{\circ} \omega_{\Delta_{23}})) @>>> R\Gamma_{\Lambda_{13}}(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} * \mathbf{k}_{\Delta_{23}}, K \overset{L}{\circ} \omega_{\Delta_{23}})). \end{CD}$$

By using the morphism  $K \rightarrow \omega_{\Delta_{12}}$ , we get the commutative diagram

$$\begin{CD} R\Gamma_{\Lambda_{23}}(T^*M_{2233}; \mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}})) @>>> R\Gamma_{\Lambda_{13}}(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} * \mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{12}} \overset{L}{\circ} \omega_{\Delta_{23}})) \\ @V \searrow VV @AA \nearrow A \\ R\Gamma_{\Lambda_{1133}}(T^*M_{1133}; \mu hom(K * \mathbf{k}_{\Delta_{23}}, K \overset{L}{\circ} \omega_{\Delta_{23}})) \end{CD} \tag{6.6}$$

Recall the morphisms in Lemma 4.3:

$$\omega_{\Delta_{12}} \overset{L}{\circ} \overset{L}{\circ} (\mathbf{k}_{\Delta_2} \overset{L}{\boxtimes} \omega_{\Delta_3}) \rightarrow \omega_{\Delta_{13}}, \quad \mathbf{k}_{\Delta_{13}} \rightarrow \mathbf{k}_{\Delta_{12}} * (\omega_{\Delta_2}^{\otimes -1} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_3}). \tag{6.7}$$

We get the morphisms

$$\begin{aligned} w: & R\Gamma_{\delta_{T^*M_{13}}^a} \Lambda_{13}(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} * \mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{12}} \overset{L}{\circ} \omega_{\Delta_{23}})) \\ & \simeq R\Gamma_{\delta_{T^*M_{13}}^a} \Lambda_{13}(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} * (\omega_{\Delta_2}^{\otimes -1} \overset{L}{\boxtimes} \mathbf{k}_{\Delta_3}), \omega_{\Delta_{12}} \overset{L}{\circ} \overset{L}{\circ} (\mathbf{k}_{\Delta_2} \overset{L}{\boxtimes} \omega_{\Delta_3}))) \\ & \rightarrow R\Gamma_{\delta_{T^*M_{13}}^a} \Lambda_{13}(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{13}}, \omega_{\Delta_{13}})). \end{aligned}$$

By its construction, the morphism  $\mu eu_{M_{12}}(K) \circ$  is obtained as the composition with the map  $w$  of the top row of the diagram (6.6). Since the composition with  $w$  of the two other arrows is the morphism  $\Phi_K$ , the proof is complete.  $\square$

The next result is similar to [15, Theorem 4.3.6].

Let  $i = 1, 2, j = i + 1$  and let  $\Lambda_{ijj}$  be a closed conic subset of  $T^*M_{ijj}$ . Assume that

$$\Lambda_{1122} \overset{a}{\times} \overset{L}{\circ} \Lambda_{2233} \text{ is proper over } T^*M_{1133}. \tag{6.8}$$

Set  $\Lambda_{1133} = \Lambda_{1122} \overset{a}{\circ} \overset{L}{\circ} \Lambda_{2233}$  and  $\Lambda_{ij} = \Lambda_{ijj} \cap T_{\Delta_j}^* M_{ijj}$ .

**Theorem 6.3.** *Let  $K_{ij}$  be a trace kernel on  $M_{ij}$  with  $SS(K_{ij}) \subset \Lambda_{ijj}$ . Assume (6.8), set  $\tilde{K}_{23} = \omega_{\Delta_2}^{\otimes -1} \overset{L}{\circ} K_{23} \simeq (\omega_{\Delta_2}^{\otimes -1} \overset{L}{\boxtimes} \mathbf{k}_{233}) \overset{L}{\otimes} K$  and set  $K_{13} = K_{12} \overset{L}{\circ} \tilde{K}_{23}$ . Then*

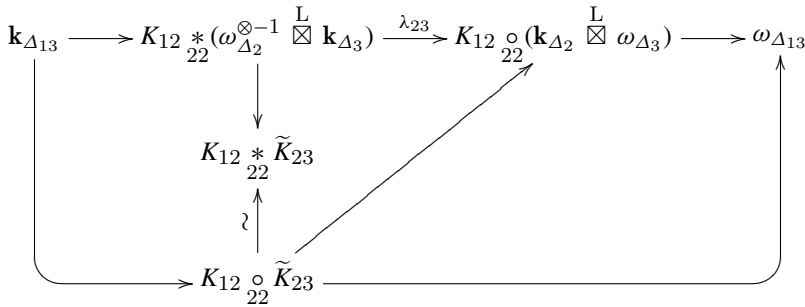
- (a)  $K_{13}$  is a trace kernel on  $M_{13}$ ,
- (b)  $\mu eu_{M_{13}}(K_{13}) = \mu eu_{M_{12}}(K_{12}) \overset{a}{\circ} \mu eu_{M_{23}}(K_{23})$  as elements of  $\mathbb{M}\mathbb{H}^0_{\Delta_{13}}(\mathbf{k}_{13})$ .
- (c) In particular, we have  $\Phi_{K_{12}} \circ \Phi_{K_{23}} \simeq \Phi_{K_{13}}$ .

**Proof.** (a) The trace kernel  $K_{23}$  defines morphisms

$$\omega_{\Delta_2}^{\otimes -1} \boxtimes^L \mathbf{k}_{\Delta_3} \rightarrow \tilde{K}_{23} \rightarrow \mathbf{k}_{\Delta_2} \boxtimes^L \omega_{\Delta_3}.$$

Assuming (6.8) and using (6.1) and (6.2), we get that  $K_{13} = K_{12} \overset{\circ}{\circ}_{22} \tilde{K}_{23}$  is a trace kernel on  $M_{13}$ .

(b) We get a commutative diagram in which we set  $\lambda_{23} = \mu eu_{M_{23}}(K_{23}) \in \mathbb{M}\mathbb{H}^0(\mathbf{k}_{23}) \simeq \text{Hom}(\omega_{\Delta_2}^{\otimes -1} \boxtimes^L \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \boxtimes^L \omega_{\Delta_3})$ :



The composition of the arrows at the bottom is  $\mu eu_{M_{13}}(K_{13})$  and the composition of the arrows at the top is  $\Phi_{K_{12}}(\mu eu_{M_{23}}(K_{23}))$ . Hence, the assertion follows from the commutativity of the diagram by Proposition 6.2.

(c) follows from (b) and Proposition 6.2. □

### 7. Operations on microlocal Euler classes II

We shall combine Theorems 4.6 and 6.3 and make more explicit the operations on microlocal Euler classes for direct or inverse images. In particular, applying our results to the case of constructible sheaves, we shall recover the results of [13, Chapter IX § 5].

Let  $M$  be a manifold and let  $\iota: N \hookrightarrow M$  be a closed embedding of a smooth submanifold  $N$ . If there is no risk of confusion, we shall still denote by  $\mathbf{k}_N$  and  $\omega_N$  the sheaves  $\iota_* \mathbf{k}_N$  and  $\iota_* \omega_N$  on  $M$ . Then  $\mathbf{k}_N$  is cohomologically constructible and moreover

$$D_M \mathbf{k}_N = R\mathcal{H}om(\mathbf{k}_N, \omega_M) \simeq \omega_N.$$

Hence,  $\text{TK}(\mathbf{k}_N) = \mathbf{k}_N \boxtimes^L \omega_N$  is a trace kernel on  $M$ .

Let  $M_i$  be a manifold ( $i = 1, 2$ ), let  $K_i$  be a trace kernel on  $M_i$  and let  $\Lambda_{ii}$  be a closed conic subset of  $T^*M_{ii}$  with  $\text{SS}(K_i) \subset \Lambda_{ii}$ . We set

$$\Lambda_i = \Lambda_{ii} \cap T_{\Delta_i}^* M_{ii}.$$

For a morphism of manifolds  $f : M_1 \rightarrow M_2$ , we denote by  $\Gamma_f$  its graph, a smooth closed submanifold of  $M_{12}$ , and we set for short

$$\Lambda_f := T_{\Gamma_f}^*(M_{12}), \quad \tilde{f} = (f, f) : M_{11} \rightarrow M_{22}.$$

Recall the diagram (2.1)

$$\begin{array}{ccccc} T^*M_1 & \xleftarrow{f_d} & M_1 \times_{M_2} T^*M_2 & \xrightarrow{f_\pi} & T^*M_2 \\ & \searrow^{\pi_{M_1}} & \downarrow \pi & & \downarrow \pi_{M_2} \\ & & M_1 & \xrightarrow{f} & M_2. \end{array}$$

Note that

$$\Lambda_{11} \overset{a}{\underset{11}{\circ}} \Lambda_{\tilde{f}} = \tilde{f}_\pi \tilde{f}_d^{-1} \Lambda_{11}, \quad \Lambda_{\tilde{f}} \overset{a}{\underset{22}{\circ}} \Lambda_{22} = \tilde{f}_d \tilde{f}_\pi^{-1} \Lambda_{22}.$$

In the sequel, we shall identify  $M_{1212}$  with  $M_{1122}$ . We take as kernel the sheaf  $\text{TK}(\mathbf{k}_{\Gamma_f})$ . Then

$$\begin{aligned} \text{TK}(\mathbf{k}_{\Gamma_f}) &= \mathbf{k}_{\Gamma_f} \overset{\text{L}}{\boxtimes} \omega_{\Gamma_f} \simeq \mathbf{k}_{\Gamma_f} \otimes (\mathbf{k}_1 \overset{\text{L}}{\boxtimes} \omega_1 \overset{\text{L}}{\boxtimes} \mathbf{k}_{22}) \\ &\simeq \omega_{\Delta_1} \overset{\circ}{\underset{11}{\circ}} ((\omega_1^{\otimes -1} \overset{\text{L}}{\boxtimes} \omega_1 \overset{\text{L}}{\boxtimes} \mathbf{k}_{22}) \overset{\text{L}}{\otimes} \mathbf{k}_{\Gamma_f}). \end{aligned} \tag{7.1}$$

Moreover, we have (see (5.9))

$$\mu\text{eu}_{M_{12}}(\text{TK}(\mathbf{k}_{\Gamma_f})) = \mu\text{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}).$$

Also note that

$$\text{R}\tilde{f}_! K_1 \simeq K_1 \overset{\circ}{\underset{11}{\circ}} \mathbf{k}_{\Gamma_f}, \quad \tilde{f}^{-1} K_2 \simeq \mathbf{k}_{\Gamma_f} \overset{\circ}{\underset{22}{\circ}} K_2.$$

**External product**

Applying Theorem 4.6 with  $M_2 = \text{pt}$  and  $M_3$  being here  $M_2$ , we get the commutative diagram

$$\begin{array}{ccc} \mathcal{MH}_{\Lambda_1}(\mathbf{k}_{M_1}) \overset{\text{L}}{\boxtimes} \mathcal{MH}_{\Lambda_2}(\mathbf{k}_{M_2}) & \xrightarrow{\circ} & \mathcal{MH}_{\Lambda_1 \times \Lambda_2}(\mathbf{k}_{M_{12}}) \\ \downarrow \sim & & \downarrow \sim \\ \text{R}\Gamma_{\Lambda_1}(\pi_{M_1}^{-1} \omega_{M_1}) \overset{\text{L}}{\boxtimes} \text{R}\Gamma_{\Lambda_2}(\pi_{M_2}^{-1} \omega_{M_2}) & \xrightarrow{\overset{\text{L}}{\boxtimes}} & \text{R}\Gamma_{\Lambda_1 \times \Lambda_2}(\pi_{M_{12}}^{-1} \omega_{M_{12}}) \end{array}$$

and taking the global sections and the zeroth cohomology,

$$\begin{array}{ccc} \text{MH}_{\Lambda_1}^0(\mathbf{k}_{M_1}) \otimes \text{MH}_{\Lambda_2}^0(\mathbf{k}_{M_2}) & \xrightarrow{\circ} & \text{MH}_{\Lambda_1 \times \Lambda_2}^0(\mathbf{k}_{M_{12}}) \\ \downarrow \sim & & \downarrow \sim \\ H_{\Lambda_1}^0(T^*M_1; \pi_{M_1}^{-1} \omega_{M_1}) \otimes H_{\Lambda_2}^0(T^*M_2; \pi_{M_2}^{-1} \omega_{M_2}) & \xrightarrow{\overset{\text{L}}{\boxtimes}} & H_{\Lambda_1 \times \Lambda_2}^0(T^*M_{12}; \pi_{M_{12}}^{-1} \omega_{M_{12}}). \end{array}$$

Applying Theorem 6.3, we obtain

**Proposition 7.1.** *The object  $K_1 \overset{L}{\boxtimes} K_2$  is a trace kernel on  $M_{12}$  and*

$$\mu eu_{M_{12}}(K_1 \overset{L}{\boxtimes} K_2) = \mu eu_{M_1}(K_1) \overset{L}{\boxtimes} \mu eu_{M_2}(K_2).$$

**Direct image**

Let  $f: M_1 \rightarrow M_2$  and  $\Gamma_f$  be as above. Applying Theorem 4.6 with  $M_1 = \text{pt}$  and  $M_2, M_3$  being the current  $M_1, M_2$ , we get the commutative diagram

$$\begin{array}{ccc} \mathcal{MH}(\mathbf{k}_{M_1}) \overset{a}{\underset{1}{\circ}} \mathcal{MH}(\mathbf{k}_{M_{12}}) & \longrightarrow & \mathcal{MH}(\mathbf{k}_{M_2}) \\ \downarrow \sim & & \downarrow \sim \\ \pi_{M_1}^{-1} \omega_{M_1} \overset{a}{\underset{1}{\circ}} \pi_{M_{12}}^{-1} \omega_{M_{12}} & \longrightarrow & \pi_{M_2}^{-1} \omega_{M_2}. \end{array}$$

Now we assume

$$f \text{ is proper on } \Lambda_1 \cap T_{M_1}^* M_1, \text{ or, equivalently, } f_\pi \text{ is proper on } f_d^{-1} \Lambda_1. \tag{7.2}$$

We set

$$f_\mu(\Lambda_1) = \Lambda_1 \circ \Lambda_f = f_\pi(f_d^{-1}(\Lambda_1)).$$

Taking the global sections and the zeroth cohomology of the diagram above, we obtain the commutative diagram

$$\begin{array}{ccc} \mathbb{MH}_{\Lambda_1}^0(\mathbf{k}_{M_1}) & \xrightarrow{\circ \mu eu(\mathbf{k}_{\Gamma_f})} & \mathbb{MH}_{f_\mu \Lambda_1}^0(\mathbf{k}_{M_2}) \\ \downarrow \sim & & \downarrow \sim \\ H_{\Lambda_1}^0(T^* M_1; \pi_{M_1}^{-1} \omega_{M_1}) & \xrightarrow{\circ \mu eu(\mathbf{k}_{\Gamma_f})} & H_{f_\mu \Lambda_1}^0(T^* M_2; \pi_{M_2}^{-1} \omega_{M_2}). \end{array}$$

We have the natural morphism and isomorphisms, already constructed in [13]:

$$\begin{aligned} f_{\pi!} f_d^{-1} \pi_{M_1}^{-1} \omega_{M_1} &\simeq f_{\pi!} \pi^{-1} \omega_{M_1} \simeq \pi_{M_2}^{-1} f_! \omega_{M_1} \\ &\rightarrow \pi_{M_2}^{-1} \omega_{M_2}. \end{aligned}$$

These induce a morphism:

$$f_\mu: R\Gamma_{\Lambda_1}(\pi_{M_1}^{-1} \omega_{M_1}) \rightarrow R\Gamma_{f_\mu \Lambda_1}(\pi_{M_2}^{-1} \omega_{M_2}).$$

**Lemma 7.2.** *Let  $\lambda \in H_{\Lambda_1}^0(T^* M_1; \pi_{M_1}^{-1} \omega_{M_1})$ . Then  $\lambda \circ \mu eu_{M_{12}}(\mathbf{k}_{\Gamma_f}) = f_\mu(\lambda)$ .*

**Proposition 7.3.** *Assume that  $\tilde{f}$  is proper on  $\Lambda_{11} \cap T_{M_{11}}^* M_{11}$ . Then the object  $R\tilde{f}_! K_1$  is a trace kernel on  $M_2$  and*

$$\begin{aligned} \mu eu_{M_2}(R\tilde{f}_! K_1) &= \mu eu_{M_1}(K_1) \overset{a}{\underset{1}{\circ}} \mu eu_{M_{12}}(\mathbf{k}_{\Gamma_f}) \\ &= f_\mu(\mu eu_{M_1}(K_1)). \end{aligned}$$

**Proof.**

Note that  $\mu\text{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}) = \mu\text{eu}_{M_{12}}((\omega_1^{\otimes -1} \overset{\text{L}}{\boxtimes} \omega_1 \overset{\text{L}}{\boxtimes} \mathbf{k}_{22}) \overset{\text{L}}{\otimes} \text{TK}(\mathbf{k}_{\Gamma_f}))$  by Proposition 5.3. We have  $\text{R}\tilde{f}_!K_1 \simeq K_1 \overset{\text{L}}{\circlearrowleft}_{11} (\omega_{\Delta_1}^{\otimes -1} \overset{\text{L}}{\circlearrowleft}_1 ((\omega_1^{\otimes -1} \overset{\text{L}}{\boxtimes} \omega_1 \overset{\text{L}}{\boxtimes} \mathbf{k}_{22}) \overset{\text{L}}{\otimes} \text{TK}(\mathbf{k}_{\Gamma_f})))$ . It remains to apply Theorem 6.3 in which one replaces  $M_1, M_2, M_3$  with  $\text{pt}, M_1, M_2$ , respectively.  $\square$

**Inverse image**

Let  $f: M_1 \rightarrow M_2$  and  $\Gamma_f$  be as above. Applying Theorem 4.6 with  $M_3 = \text{pt}$ , we get the commutative diagram

$$\begin{CD} \mathcal{MH}(\mathbf{k}_{M_{12}}) @>\overset{a}{\circlearrowleft}_2>> \mathcal{MH}(\mathbf{k}_{M_2}) @>>> \mathcal{MH}(\mathbf{k}_{M_1}) \\ @VV\sim V @VV\sim V @VV\sim V \\ \pi_{M_{12}}^{-1}\omega_{M_{12}} @>\overset{a}{\circlearrowleft}_2>> \pi_{M_2}^{-1}\omega_{M_2} @>>> \pi_{M_1}^{-1}\omega_{M_1}. \end{CD}$$

Now we assume

$$f \text{ is non-characteristic for } \Lambda_2, \text{ or, equivalently, } f_d \text{ is proper on } f_\pi^{-1}\Lambda_2. \tag{7.3}$$

We set

$$f^\mu(\Lambda_2) = \Lambda_f \circ \Lambda_1 = f_d(f_\pi^{-1}(\Lambda_2)).$$

Taking the global sections and the zeroth cohomology of the diagram above, we obtain the commutative diagram

$$\begin{CD} \text{MH}_{\Lambda_2}^0(\mathbf{k}_{M_2}) @>\mu\text{eu}(\mathbf{k}_{\Gamma_f})^\circ>> \text{MH}_{f^\mu\Lambda_2}^0(\mathbf{k}_{M_1}) \\ @VV\sim V @VV\sim V \\ H_{\Lambda_2}^0(T^*M_2; \pi_{M_2}^{-1}\omega_{M_2}) @>\mu\text{eu}(\mathbf{k}_{\Gamma_f})^\circ>> H_{f^\mu\Lambda_2}^0(T^*M_1; \pi_{M_1}^{-1}\omega_{M_1}). \end{CD}$$

We have a natural morphism constructed in the proof of [13, Proposition 9.3.2]:

$$f^\mu : f_{d!}f_\pi^{-1}\pi_{M_2}^{-1}\omega_{M_2} \rightarrow \pi_{M_1}^{-1}\omega_{M_1}.$$

Hence, we get a map:

$$f^\mu : \text{R}\Gamma_{\Lambda_2}(\pi_{M_2}^{-1}\omega_{M_2}) \rightarrow \text{R}\Gamma_{f^\mu\Lambda_2}(\pi_{M_1}^{-1}\omega_{M_1}).$$

**Lemma 7.4.** *Let  $\lambda \in H_{\Lambda_1}^0(T^*M_2; \pi_{M_2}^{-1}\omega_{M_2})$ . Then  $\mu\text{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}) \circ \lambda = f^\mu(\lambda)$ .*

**Proposition 7.5.** *Assume that  $\tilde{f}$  is non-characteristic with respect to  $\Lambda_{22}$ . Then the object  $(\mathbf{k}_1 \overset{\text{L}}{\boxtimes} \omega_{M_1/M_2}) \overset{\text{L}}{\otimes} \tilde{f}^{-1}K_2$  is a trace kernel on  $M_1$  and*

$$\begin{aligned} \mu\text{eu}_{M_1}(\omega_{\Delta_1} \overset{\text{L}}{\circlearrowleft}_1 \tilde{f}^{-1}(\omega_{\Delta_2}^{\otimes -1} \overset{\text{L}}{\circlearrowleft}_2 K_2)) &= \mu\text{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}) \overset{a}{\circlearrowleft}_2 \mu\text{eu}_{M_2}(K_2) \\ &= f^\mu(\mu\text{eu}_{M_2}(K_2)). \end{aligned}$$



**Proof.** Applying Theorem 6.3 with  $M_3 = \text{pt}$ , we get that

$$(\mathbf{k}_1 \boxtimes \omega_{M_1/M_2}) \otimes^{\mathbb{L}} \tilde{f}^{-1} K_2 \simeq \text{TK}(\mathbf{k}_f) \circ_{22} (\omega_{\Delta_2}^{\otimes -1} \circ_2 (\omega_2 \boxtimes \omega_2^{\otimes -1}) \otimes^{\mathbb{L}} K_2)$$

is a trace kernel. Since  $\mu\text{eu}_{M_2}((\omega_2 \boxtimes \omega_2^{\otimes -1}) \otimes^{\mathbb{L}} K_2) = \mu\text{eu}_{M_2}(K_2)$  by Proposition 5.3, we obtain the result.  $\square$

**Tensor product**

Consider now the case where  $M_1 = M_2 = M$  and the  $\Lambda_{ii}$  satisfy the transversality condition

$$\Lambda_{11} \cap \Lambda_{22}^a \subset T_{M \times M}^*(M \times M). \tag{7.4}$$

Then by composing the external product with the restriction to the diagonal, we get a convolution map

$$\star: \text{MH}_{\Lambda_1}(\mathbf{k}_M) \times \text{MH}_{\Lambda_2}(\mathbf{k}_M) \rightarrow \text{MH}_{\Lambda_1 + \Lambda_2}(\mathbf{k}_M). \tag{7.5}$$

Applying Propositions 7.1 and 7.5, we get

**Proposition 7.6.** Assume (7.4). Then the object  $K_1 \otimes^{\mathbb{L}} (\mathbf{k}_M \boxtimes \omega_M^{\otimes -1}) \otimes^{\mathbb{L}} K_2$  is a trace kernel on  $M$  and

$$\mu\text{eu}_M(K_1 \otimes^{\mathbb{L}} (\mathbf{k}_M \boxtimes \omega_M^{\otimes -1}) \otimes^{\mathbb{L}} K_2) = \mu\text{eu}_M(K_1) \star \mu\text{eu}_M(K_2).$$

Following [23, II, Corollary 5.6], we shall recall the link between the product  $\star$  and the cup product.

**Proposition 7.7.** Let  $\lambda_i \in H_{\Lambda_i}^0(T^*M_i; \pi_M^{-1}\omega_M)$  ( $i = 1, 2$ ), and assume that  $\Lambda_1 \cap \Lambda_2^a \subset T_M^*M$ . Then

$$(\lambda_1 \star \lambda_2)|_M = \int_{\pi_M} (\lambda_1 \cup \lambda_2) \tag{7.6}$$

as elements of  $H_{\pi(\Lambda_1 \cap \Lambda_2)}^0(M; \omega_M)$ .

**Proof.** Denote by  $\delta: \Delta \hookrightarrow M_{12} = M \times M$  the diagonal embedding and let us identify  $M$  with  $\Delta$ . Consider the diagram

$$\begin{CD} T_{\Delta}^*M_{12} @>f>> \Delta \times_{M_{12}} T^*M_{12} \\ @V{\pi}VV @VV{\delta_d}V \\ \Delta @>s>> T^*\Delta \end{CD} \tag{7.7}$$

where  $\pi$  is the projection,  $\delta_d$  is the map associated with  $\delta$ ,  $s$  is the zero-section embedding and  $f$  is the restriction to  $\Delta \times_M T^*M_{12}$  of the embedding  $T_{\Delta}^*M_{12} \hookrightarrow T^*M_{12}$ . Since this diagram is Cartesian, we have

$$s^{-1}\delta_d! \simeq \pi!f^{-1}.$$

Now let  $\lambda_1 \times \lambda_2 \in H^0_{\Lambda_1 \times \Lambda_2}(T^*M_{12}; \pi^{-1}\omega_{M_{12}})$  and denote by  $\lambda_1 \times_M \lambda_2$  its image under the map

$$H^0_{\Lambda_1 \times \Lambda_2}(T^*M_{12}; \pi^{-1}\omega_{M_{12}}) \rightarrow H^0_{\Lambda_1 \times_M \Lambda_2}(\Delta \times_{M_{12}} T^*M_{12}; \pi^{-1}\omega_{M_{12}}).$$

(Here, on the right hand side, we still denote by  $\pi$  the restriction of the projection  $\pi_{M_{12}}$  to  $\Delta \times_{M_{12}} T^*M_{12}$ .) Then

$$\begin{aligned} \int_{\pi} (\lambda_1 \cup \lambda_2) &= \pi_! f^{-1}(\lambda_1 \times_M \lambda_2), \\ (\lambda_1 \star \lambda_2)|_M &= s^{-1} \delta_{d!}(\lambda_1 \times_M \lambda_2). \end{aligned} \quad \square$$

**Corollary 7.8.** *Let  $K_1$  and  $K_2$  be two trace kernels on  $M$  with  $SS(K_i) \subset \Lambda_{ii}$ . Assume (7.4) and assume moreover that  $\text{Supp}(K_1) \cap \text{Supp}(K_2)$  is compact. Then the object  $R\Gamma(M \times M; K_1 \overset{L}{\boxtimes} (\mathbf{k}_M \overset{L}{\boxtimes} \omega_M^{\otimes -1}) \overset{L}{\boxtimes} K_2)$  is a trace kernel on pt and*

$$\text{eu}_{\text{pt}}(R\Gamma(M \times M; K_1 \overset{L}{\boxtimes} (\mathbf{k}_M \overset{L}{\boxtimes} \omega_M^{\otimes -1}) \overset{L}{\boxtimes} K_2)) = \int_{T^*M} \mu\text{eu}(K_1) \cup \mu\text{eu}(K_2).$$

**Remark 7.9.** Let  $M$  be a real analytic manifold and let  $F \in \text{D}^b_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ . Recall that one associates with  $F$  the trace kernel  $\text{TK}(F) = F \overset{L}{\boxtimes} D_M F$  and that  $\mu\text{eu}_M(F) = \mu\text{eu}_M(\text{TK}(F))$ . Assume now that  $f: M_1 \rightarrow M_2$  is a morphism of real analytic manifolds.

Let  $F_1 \in \text{D}^b_{\mathbb{R}\text{-c}}(\mathbf{k}_{M_1})$  and assume that  $f$  is proper on  $\text{Supp}(F_1)$ . Applying Proposition 7.3 and noticing that

$$\text{Rf}_! \widetilde{\text{TK}}(F_1) \simeq \text{TK}(\text{Rf}_! F_1), \tag{7.8}$$

we find that  $\mu\text{eu}(\text{Rf}_! F_1) = f_{\mu}(\mu\text{eu}(F_1))$ . This is nothing but [13, Proposition 9.4.2].

Let  $F_2 \in \text{D}^b_{\mathbb{R}\text{-c}}(\mathbf{k}_{M_2})$  and assume that  $f$  is non-characteristic with respect to  $F_2$ . Applying Proposition 7.5 and noticing that

$$\text{TK}(f^{-1}F_2) \simeq (\mathbf{k}_1 \overset{L}{\boxtimes} \omega_{M_1/M_2}) \overset{L}{\boxtimes} \widetilde{f}^{-1} \text{TK}(F_2),$$

we find that  $\mu\text{eu}(f^{-1}F_2) = f^{\mu}(\mu\text{eu}(F_2))$ . Hence, we recover [13, Proposition 9.4.3].

### 8. Applications: $\mathcal{D}$ -modules and elliptic pairs

We shall, as an application of Theorem 6.3, recover the theorem of [23] on the index of elliptic pairs. In this section,  $X$  is a complex manifold,  $\mathbf{k} = \mathbb{C}$ ,  $\mathcal{M}$  is an object of  $\text{D}^b_{\text{coh}}(\mathcal{D}_X)$  and  $F$  is an object of  $\text{D}^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ .

Recall that we have denoted by  $\text{TK}(F)$  and  $\text{TK}(\mathcal{M})$  (see Notation 5.10) the trace kernels associated with  $F$  and with  $\mathcal{M}$ , respectively:

$$\begin{aligned} \text{TK}(F) &:= F \overset{L}{\boxtimes} D_X F, \\ \text{TK}(\mathcal{M}) &:= \Omega_{X \times X} \overset{L}{\boxtimes} \mathcal{D}_{X \times X}(\mathcal{M} \overset{L}{\boxtimes} D_{\mathcal{D}} \mathcal{M}). \end{aligned}$$

The pair  $(\mathcal{M}, F)$  is called an *elliptic pair* in the earlier citation if  $\text{char}(\mathcal{M}) \cap SS(F) \subset T^*X$ . From now on, we assume that  $(\mathcal{M}, F)$  is an elliptic pair.

It follows from Proposition 7.6 that the tensor product of  $\text{TK}(F)$  and  $\text{TK}(\mathcal{M})$  shifted by  $-2d_X$  is again a trace kernel. We denote it by  $\text{TK}(\mathcal{M}, F)$ . Hence

$$\text{TK}(\mathcal{M}, F) \simeq \Omega_{X \times X} \overset{\text{L}}{\otimes}_{\mathcal{D}_{X \times X}} (\mathcal{M} \boxtimes \mathbb{D}_{\mathcal{D}} \mathcal{M}) \otimes (F \boxtimes \mathbb{D}'_X F). \tag{8.1}$$

Moreover the same statement gives

$$\mu\text{eu}_X(\text{TK}(\mathcal{M}, F)) = \mu\text{eu}_X(\mathcal{M}) \star \mu\text{eu}_X(F). \tag{8.2}$$

We set

$$\text{Sol}(\mathcal{M}, F) := \text{RHom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X), \tag{8.3}$$

$$\text{DR}(\mathcal{M}, F) := \text{R}\Gamma(X; \Omega_X \overset{\text{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M} \otimes F) [d_X]. \tag{8.4}$$

As explained in [23], [13, Theorem 11.3.3] and isomorphism (2.7) provide a generalization of the classical Petrovsky regularity theorem, namely, the natural isomorphisms

$$\text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbb{D}'_X F \otimes \mathcal{O}_X) \xrightarrow{\sim} \text{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X). \tag{8.5}$$

Now assume that  $\text{Supp}(\mathcal{M}) \cap \text{Supp}(F)$  is compact and let us take the global sections of the isomorphism (8.5). We find the isomorphism

$$\text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathbb{D}'_X F \otimes \mathcal{O}_X) \xrightarrow{\sim} \text{RHom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X). \tag{8.6}$$

It is proved in [23] (assuming  $\mathcal{M}$  has a good filtration) that one can represent the left hand side of (8.6) by a complex of topological vector spaces of type DFN and the right hand side of (8.6) by a complex of topological vector spaces of type FN. It follows that the complexes  $\text{Sol}(\mathcal{M}, F)$  and  $\text{DR}(\mathcal{M}, F)$  have finite-dimensional cohomology and are dual to each other. More precisely, denoting by  $(\cdot)^*$  the duality functor in  $\mathbb{D}_f^b(\mathbb{C})$ , we have

$$(\text{Sol}(\mathcal{M}, F))^* \simeq \text{DR}(\mathcal{M}, F).$$

It follows from the finiteness of the cohomology of the complexes  $\text{Sol}(\mathcal{M}, F)$  and  $\text{DR}(\mathcal{M}, F)$  that

$$\text{R}\Gamma(X \times X; \text{TK}(\mathcal{M}, F)) \simeq \text{Sol}(\mathcal{M}, F) \otimes \text{DR}(\mathcal{M}, F).$$

One checks that this isomorphism commutes with the composition of the morphisms  $\mathbb{C} \rightarrow \text{R}\Gamma(X \times X; \text{TK}(\mathcal{M}, F)) \rightarrow \mathbb{C}$  and  $\mathbb{C} \rightarrow \text{Sol}(\mathcal{M}, F) \otimes \text{DR}(\mathcal{M}, F) \rightarrow \mathbb{C}$ , which implies

$$\text{eu}_{\text{pt}}(\text{R}\Gamma(X \times X; \text{TK}(\mathcal{M}, F))) = \chi(\text{Sol}(\mathcal{M}, F)). \tag{8.7}$$

Therefore, one recovers the index formula of the earlier citation:

$$\begin{aligned} \chi(\text{RHom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)) &= \int_X (\mu\text{eu}_X(\mathcal{M}) \star \mu\text{eu}_X(F))|_X \\ &= \int_{T^*X} \mu\text{eu}_X(\mathcal{M}) \cup \mu\text{eu}_X(F). \end{aligned} \tag{8.8}$$

**Remark 8.1.** In general the direct image of an elliptic pair is no longer an elliptic pair. However, it remains a trace kernel.

**Remark 8.2.** As already mentioned in [23], formula (8.8) has many applications, as long as one is able to calculate  $\mu\text{eu}_X(\mathcal{M})$  (see the final remarks below). For example, if  $M$  is a compact real analytic manifold and  $X$  is a complexification of  $M$ , one recovers the Atiyah–Singer theorem by choosing  $F = D'\mathbb{C}_M$ . If  $X$  is a complex compact manifold, one recovers the Riemann–Roch theorem: one takes  $F = \mathbb{C}_X$  and if  $\mathcal{F}$  is a coherent  $\mathcal{O}_X$ -module, one sets  $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}$ .

**9. The Lefschetz fixed point formula**

In this section, we shall briefly show how to adapt the formalism of trace kernels to the Lefschetz trace formula as treated in [13, §9.6]. Here we assume that  $\mathbf{k}$  is a field.

Assume that we are given two maps  $f, g: N \rightarrow M$  of real analytic manifolds, an object  $F \in D_{\mathbb{R}\text{-c}}^b(\mathbf{k}_M)$  and a morphism

$$\varphi: f^{-1}F \rightarrow g^!F. \tag{9.1}$$

Set

$$\begin{aligned} h &= (g, f): N \times N \rightarrow M \times M, \\ S &= \text{Supp}(F), \quad L = h^{-1}(\Delta_M) = \{(x, y) \in N \times N; g(x) = f(y)\}, \\ i &: L \hookrightarrow N \times N, \\ T &= f^{-1}(S) \cap g^{-1}(S). \end{aligned}$$

One makes the following assumption:

$$\text{The set } T \text{ is compact.} \tag{9.2}$$

Then we have the maps

$$R\Gamma(M; F) \rightarrow R\Gamma_{f^{-1}S}(N; f^{-1}F) \xrightarrow{\varphi} R\Gamma_T(N; g^!F) \rightarrow R\Gamma(M; F).$$

The composition gives a map

$$\int \varphi: R\Gamma(M; F) \rightarrow R\Gamma(M; F), \tag{9.3}$$

and this map factorizes through  $R\Gamma_T(N; g^!F)$  which has finite-dimensional cohomologies. Hence, we can define the trace  $\text{tr}(\int \varphi)$ .

We have the chain of morphisms

$$\begin{aligned} \mathbf{k}_N &\rightarrow R\mathcal{H}om(g^!F, g^!F) \\ &\xrightarrow{\varphi} R\mathcal{H}om(f^{-1}F, g^!F) \simeq \delta_N^!(g^!F \boxtimes^L D_N f^{-1}F) \\ &\simeq \delta_N^!(g^!F \boxtimes^L f^!D_M F) \simeq \delta_N^!h^!(F \boxtimes^L D_M F). \end{aligned}$$

We have thus constructed the morphism

$$\mathbf{k}_{\Delta_N} \rightarrow h^!(F \boxtimes^L D_M F).$$

By using the morphism  $F \boxtimes^L D_M F \rightarrow \omega_{\Delta_M}$  and the isomorphism  $h^! \omega_{\Delta_M} \simeq i_* \omega_L$ , we get the morphisms

$$\mathbf{k}_{\Delta_N} \rightarrow h^!(F \boxtimes^L D_M F) \rightarrow i_* \omega_L \tag{9.4}$$

in  $D^b(\mathbf{k}_{N \times N})$ . The support of the composition is contained in  $\delta_N(T) \cap L$ .

**Theorem 9.1** ([13, Proposition 9.6.2]). *The trace  $\text{tr}(\int \varphi)$  coincides with the image of  $1 \in \mathbf{k}$  under the composition of the morphisms*

$$\mathbf{k} \rightarrow R\Gamma(N, \mathbf{k}_N) \rightarrow R\Gamma_c(L, \omega_L) \rightarrow \mathbf{k}.$$

Here the middle arrow is derived from (9.4).

Although (9.4) is not a trace kernel in the sense of Definition 5.1, it should be possible to adapt the previous constructions to the case of  $\mathcal{D}$ -modules and to elliptic pairs, and then to recover a theorem of [7], but we do not develop this point here (see [21] for related results).

**Final remarks**

The microlocal Euler class of constructible sheaves is easy to compute since it is enough to calculate some multiplicities at generic points. We refer the reader to [13] for examples.

On the other hand, there is no direct method for calculating the microlocal Euler class of a coherent  $\mathcal{D}$ -module  $\mathcal{M}$  (except in the holonomic case). In [23], the authors made a precise conjecture relying on  $\mu\text{eu}_X(\mathcal{M})$  and the Chern character of the associated graded module (an  $\mathcal{O}_{T^*X}$ -module), and this conjecture has been proved by Bressler, Nest and Tsygan [1].

Similarly, the Hochschild class of coherent  $\mathcal{O}_X$ -modules is usually calculated through the so-called Hochschild–Kostant–Rosenberg isomorphism, but this isomorphism does not commute with proper direct images, and a precise conjecture (involving the Todd class) has been made by Kashiwara in [10] and this conjecture has recently been proved in the algebraic case by Ramadoss [20] and in the general case by Grivaux [6].

**Acknowledgements.** The second-named author warmly thanks Stéphane Guillermou for helpful discussions.

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