BULL. AUSTRAL. MATH. SOC. 16A15, 16A99
VOL. 40 (1989) [363-364]

ON A PROBLEM OF SZÁSZ

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Dedicated to Professor Miyuki Yamada on his 60th birthday.

Let R be a ring with centre Z. In this note, we prove the following: If the additive group Z^+ of Z has finite group-theoretic index in R^+, then R has an ideal I contained in Z such that R/I is a finite ring. This is a solution of a problem posed by F.A. Szász.

Throughout this note, R denotes a ring with centre Z. We write R^+ for the additive group of R, and $C(R)$ for the commutator ideal of R.

In Problem 84 of [**1**], F.A. Szász asks: In which rings R has the additive group Z^+ of the centre Z, a finite group-theoretic index in R^+? We show that such a ring R has an ideal I contained in Z such that R/I is a finite ring.

We begin with the following

PROPOSITION 1. *If Z^+ has finite index in R^+, then $C(R)$ is finite.*

PROOF: Let n be the index of Z^+ in R^+ and $\{r_1 = 0, r_2, \ldots, r_n\}$ a complete set of coset representatives of Z^+ in R^+. Since $[r_i + z, r_j + z'] = [r_i, r_j]$, $C(R)$ is additively generated by $r[r_i, r_j]s$ where r, $s \in R$. But $r[r_i, r_j] = (r_k + z)[r_i, r_j] = r_k[r_i, r_j] + [r_i z, r_j]$ and $[r_i z, r_j] = [r_l + z', r_j] = [r_l, r_j]$. A similar result holds for $[r_i, r_j]s$ and so $C(R)$ is additively generated by the finite set $\{r_k[r_i, r_j], [r_i, r_j]r_l, r_k[r_i, r_j]r_l \mid 2 \leqslant i, j, k \leqslant n\}$. Also each $[r_i, r_j]$ has finite additive order, otherwise there exist distinct integers n and n' such that $(n - n')[r_i, r_j] \neq 0$ but $nr_i + Z^+ = n'r_i + Z^+$. However the latter equation yields $(n - n')r_i \in Z$, so $0 = [(n - n')r_i, r_j] = (n - n')[r_i, r_j]$, a contradiction. Hence, as an abelian group, $C(R)$ has a finite set of generators of finite order, and so is finite. ∎

PROPOSITION 2. *Assume that $C(R)$ is finite. Then there exists a finite nilpotent ideal N of R such that R/N is the direct sum of a finite semisimple ring and a commutative ring.*

PROOF: Let N be the Jacobian radical of $C(R)$. It is easily seen that N is a finite nilpotent ideal of R. Let $R' = R/N$. Since $C(R')$ is the canonical homorphic image

Received 9 November, 1988

of $C(R)$ in R/N, $C(R')$ is a finite semisimple ring. Hence $C(R')$ has an identity e. Since $C(R')$ is an ideal of R', e is a central idempotent of R'. Hence $R' = C(R') \oplus S$, where $S = \{r - re \mid r \in R\}$. Clearly, S is a commutative ring. ∎

As an immediate consequence of Propositions 1 and 2, we have

COROLLARY 1. *Let R be a semiprime ring with centre Z. Then the following statements are equivalent:*

(1) *Z^+ has finite index in R^+;*
(2) *$C(R)$ is finite;*
(3) *R is the direct sum of a finite ring and a commutative ring.*

The following example shows that the statements (1) and (2) are not equivalent in general.

Example. Let $\mathbb{Z}$ denote the ring of integers. Let R be the set of all matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with a, $c \in \mathbb{Z}$ and $b \in \mathbb{Z}/2\mathbb{Z}$. In R, define addition and multiplication as in ordinary matrices. Then R is a ring. The ideal $I = \begin{pmatrix} 2\mathbb{Z} & 0 \\ 0 & 2\mathbb{Z} \end{pmatrix}$ is contained in the centre of Z and R, and R/I is a finite ring. Hence Z^+ has finite index in R^+. However R does not satisify (3) in Corollary 1.

Now we come to our main theorem.

THEOREM 1. *Let R be a ring with centre Z. Then the following statements are equivalent:*

(1) *The additive group Z^+ of Z has finite index in R^+;*
(2) *R has an ideal I contained in Z such that R/I is a finite ring.*

PROOF: It suffices to prove the implication (1) $\Rightarrow$ (2). By Proposition 1, $C(R)$ is a finite ideal. Let $f\colon Z \to \mathrm{End}(C(R))$ be the ring homomorphism defined by $f(z)(r) = rz$ for all $z \in Z$ and $r \in C(R)$, and let $I = \mathrm{Ker}\, f$. Then Z/I is a finite ring. Let $r_R(C(R))$ denote the right annihilator of $C(R)$ in R. Then, $I = r_R(C(R)) \cap Z$. Let $a \in I$ and $x \in R$. Then, for any y in R, we get $[ax, y] = a[x, y] = 0$. Hence, $ax \in Z$, and so $ax \in r_R(C(R)) \cap Z = I$. This proves that I is an ideal of R. Since Z/I is finite and since Z^+ has finite index in R^+, R/I is a finite ring. This completes the proof. ∎

REFERENCES

[1] F.A. Szász, *Radicals of Rings* (John Wiley and Sons, Chichester, New York, Brisbane, Toronto, 1981).

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