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Generating functions on covering groups

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Abstract

In this paper we prove a conjecture relating the Whittaker function of a certain generating function with the Whittaker function of the theta representation $\Theta_n^{(n)}$. This enables us to establish that a certain global integral is factorizable and hence deduce the meromorphic continuation of the standard partial L function $L^S(s,\pi^{(n)})$. In fact we prove that this partial L function has at most a simple pole at $s=1$. Here, $\pi^{(n)}$ is a genuine irreducible cuspidal representation of the group $\mathrm{GL}_r^{(n)}(\mathbf{A})$.

1. Introduction

Let $\mathbf{A}$ denote the ring of adeles of a global field F. Assume that F contains a full set of nth roots of unity. Let $\mathrm{GL}_r^{(n)}(\mathbf{A})$ denote the n-fold metaplectic cover of the group $\mathrm{GL}_r(\mathbf{A})$ as constructed in [KP84]. Let $\pi^{(n)}$ denote a genuine irreducible cuspidal representation of $\mathrm{GL}_r^{(n)}(\mathbf{A})$. To this representation one can attach the partial standard L function, denoted by $L^S(s,\pi^{(n)})$.

The first to consider convolutions of cuspidal representations with theta representations were Bump and Hoffstein in [BH86, BH87]. In these papers they considered the global construction involving the cubic theta representation and established that this construction represents an L function. It was their idea that in order to study the properties of the above L functions, one needs to start with the well-known Rankin–Selberg convolution of two cuspidal representations of $\mathrm{GL}_r(\mathbf{A})$ and $\mathrm{GL}_n(\mathbf{A})$, and adjust this construction to the covering groups.

In detail, let $\pi^{(n)}$ be as defined above. Let $\Theta_n^{(n)}$ denote the global theta representation of the group $\mathrm{GL}_n^{(n)}(\mathbf{A})$. The latter representation was constructed in [KP84]. Assume that $r>n$. Then the proposed construction is given by

$$\int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A})}\int_{V_{r,n}(F)\backslash V_{r,n}(\mathbf{A})}\phi\left(v\begin{pmatrix} g & \\ & I_{r-n}\end{pmatrix}\right)\overline{\theta(g)}\psi_{V_{r,n}}(v)|\det g|^{s-(r-n)/2}\,dv\,dg. \tag{1}$$

Here ϕ is a vector in the space of $\pi^{(n)}$, and θ is a vector in the space of $\Theta_n^{(n)}$. The group $V_{r,n}$ and the character $\psi_{V_{r,n}}$ are defined in § 4. This was the starting point of [BF99]. In fact in [BF99] the authors concentrated on the case when $r<n$, but up to some modifications as explained in [BF99, § 2], the idea is the same.

A straightforward unfolding implies that for $\mathrm{Re}(s)$ large, integral (1) is equal to

$$\int_{V_n(\mathbf{A})\backslash \mathrm{GL}_n(\mathbf{A})} W_\phi\begin{pmatrix} g & \\ & I_{r-n}\end{pmatrix}\overline{W_\theta(g)}|\det g|^{s-(r-n)/2}\,dg. \tag{2}$$

Here W_ϕ denotes the Whittaker coefficient of ϕ, and we define W_θ similarly.

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Since in general the Whittaker coefficient W_ϕ is not factorizable, it is not obvious that the above integral represents an Euler product. To show that it does, one needs to apply the method referred to as the 'new way' which was developed in [PR88]. See [BF99] for a discussion of and references for this method.

In our context, to deduce that integral (2) is Eulerian one proceeds in two steps. Let F denote a local non-archimedean field containing a full set of nth roots of unity. Let $\pi^{(n)}$ denote a local unramified representation, and let $L(s,\pi^{(n)})$ denote the corresponding local L function. This function is defined in equation (4). The first step is to find a generating function for $L(s,\pi^{(n)})$. In [BF99, p. 5], such a function was introduced and was denoted by $\widetilde{\Delta}_s$. We give its definition in § 2 right after equation (4).

The second step is to use this function to compute a local integral which is obtained from integral (2), and to show that this local integral is independent of the choice of the local Whittaker function of the representation $\pi^{(n)}$. This is done as follows. First, one proves identity (5). This is done in [BF99, Proposition 1.1]. Here $\omega_{\pi^{(n)}}$, denoted by σ in [BF99], is the unique spherical function attached to $\pi^{(n)}$. Then we apply the argument of [BF99, Proposition 2.1] to deduce the identity

$$L(s,\pi^{(n)}) = \int_{V_r\backslash \mathrm{GL}_r} W_{\pi^{(n)}}(h) \int_{V_r} \widetilde{\Delta}_s(vh)\psi_{V_r}(v)\,dv\,dh. \tag{3}$$

Here $W_{\pi^{(n)}}(h)$ is any local unramified Whittaker function associated with the representation $\pi^{(n)}$, normalized to equal one at the identity. The group V_r is the maximal unipotent subgroup of GL_r, and ψ_{V_r} is the Whittaker character defined on V_r. See equation (15).

Indeed, to obtain (3) from (5), we use the uniqueness of $\omega_{\pi^{(n)}}$. Thus, from this uniqueness we have $\int_{K_r} W_{\pi^{(n)}}(kh)\,dk = \omega_{\pi^{(n)}}(h)W_{\pi^{(n)}}(e)$ where $W_{\pi^{(n)}}$ is any Whittaker function such that $W_{\pi^{(n)}}(e) \neq 0$. Here K_r is the maximal compact subgroup of GL_r, and if $|n|_F = 1$ it can be viewed as a subgroup of $\mathrm{GL}_r^{(n)}$. See § 2. Assuming that $W_{\pi^{(n)}}(e) = 1$, we plug this into integral (5), change variables in h, and we obtain integral (3).

Thus we are reduced to the computation of the Whittaker function of $\widetilde{\Delta}_s$ and relating this computation to the local corresponding integral of (2). This is done in [BF99] for the cases $r = 2, 3$ and arbitrary n. The general case is conjectured in [BF99, Conjecture 1.2]. This conjecture is stated in our notation as identities (20) when $r < n$, and (28) when $r \geqslant n$.

In this paper we prove these two identities, and hence prove [BF99, Conjecture 1.2]. To do this we give a different realization for the function $\widetilde{\Delta}_s$. This realization makes the proof of the stated conjecture relatively simple. The new realization is described in § 2 and is given by a certain unique functional defined on the local theta representation $\Theta_{nr}^{(n)}$. This last representation is defined on the group $\mathrm{GL}_{nr}^{(n)}$. We then use this functional to define a function on the group $\mathrm{GL}_{nr}^{(n)}$, which we denote by $W_{nr}^{(n)}(h)$. Here $h \in \mathrm{GL}_{nr}^{(n)}$. Restricting to the group $\mathrm{GL}_r^{(n)}$, we obtain a function on that group which we use to give the new expression for $\widetilde{\Delta}_s$. Thus our result contains two parts. The first is to prove that the function $W_{nr}^{(n)}(h)$ restricted to $\mathrm{GL}_r^{(n)}$ is indeed the generating function for the standard L function. This we do in Proposition 2. The second, and the main result of this paper, is to obtain the desired expression for the Whittaker function of the generating function. This we do in Theorem 2, which is [BF99, Conjecture 1.2]. In both cases the computations are quite straightforward and are done by a repeated application of Lemma 1 and Corollary 1 stated and proved in § 2.1.

As mentioned above, the global result and some of the computations done in [BF99] assume that $r < n$. This is just a technical point. The authors of [BF99] were well aware that their

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construction works for all r and n. To complete their result, in the last section we give some details in the other two cases, that is, when $r > n$ and $r = n$.

To summarize, combining [BF99] with our result, we have the following theorem.

Theorem 1. *Let $\pi^{(n)}$ denote an irreducible cuspidal representation of the group $\mathrm{GL}_r^{(n)}(\mathbf{A})$. Then the partial L function $L^S(s, \pi^{(n)})$ has a meromorphic continuation to the whole complex plane. When $r \neq n$ this partial L function is holomorphic. When $r = n$ it can have at most a simple pole at $s = 1$.*

As a first remark we mention that in fact we do expect that the partial L function $L^S(s, \pi^{(n)})$ will also be holomorphic in the case when $r = n$. From the global integral given in § 4, see integral (26), we deduce that if this L function has a simple pole at $s = 1$, then $\pi^{(n)}$ will be isomorphic to $\Theta_n^{(n)}$. This we believe cannot happen.

Second, it is worthwhile mentioning that one can extend the above global constructions in two ways. First, one can replace the representation $\Theta_n^{(n)}$ by the representation $\Theta_{n,\chi}^{(n)}$. Here χ is any global character of $\mathrm{GL}_1(\mathbf{A})$ such that $\chi = \chi_1^n$ for some character χ_1 of $\mathrm{GL}_1(\mathbf{A})$. This last representation is defined as a residue of an Eisenstein series, and as $\Theta_n^{(n)}$ it has a unique Whittaker function. A second extension is to replace $\Theta_n^{(n)}$ by a cuspidal theta representation $\Theta_{n,\chi}^{(n)}$ when it exists. Here χ is any global character of $\mathrm{GL}_1(\mathbf{A})$ which is not of the form $\chi = \chi_1^n$. For example, for $n = 2$ such cuspidal representations were constructed in [GP80, Fli80]. For $n = 3$, see [PP84] for the construction of such cuspidal representations. In both of these extensions we expect to get the twisted L function $L^S(s, \pi^{(n)} \otimes \chi^{-1})$. In the first case, when $\Theta_{n,\chi}^{(n)}$ is not cuspidal, we expect this L function to be holomorphic in all cases. However, if $\Theta_{n,\chi}^{(n)}$ is cuspidal, and $r = n$, then we do expect a simple pole at $s = 1$ if $\pi^{(n)} = \Theta_{n,\chi}^{(n)}$.

Finally, we mention that the result proved in this paper simplifies some of the proofs in [FG16]. This is explained in detail in [FG16] before and after equation (1).

2. Generating functions

The main references for this section are [BF99, KP84]. Fix a positive integer $n > 1$. Let F denote a local non-archimedean field which contains a full set of nth roots of unity. Let $\mathrm{GL}_r^{(n)}(F)$ denote the metaplectic n-fold cover of the group $\mathrm{GL}_r(F)$. We realize this group as pairs $\langle g, \epsilon \rangle$, where $g \in \mathrm{GL}_r(F)$ and ϵ is an nth root of unity. We shall assume that $|n|_F = 1$. Let K_r denote the standard maximal compact subgroup of $\mathrm{GL}_r(F)$. Then the group K_r splits under the covering, and can be viewed as a subgroup of $\mathrm{GL}_r^{(n)}(F)$. Henceforth, we shall omit the notation of the field F. For example, we write GL_r for $\mathrm{GL}_r(F)$. Let B_r denote the standard Borel subgroup of GL_r consisting of all upper triangular matrices. Let T_r denote the subgroup of B_r consisting of all diagonal matrices, and let V_r denote the group of all upper unipotent matrices in B_r.

Let $\pi^{(n)}$ denote an unramified representation of $\mathrm{GL}_r^{(n)}$ associated to a character χ of T_r. These representations were defined in [KP84]. In detail, let $T_{r,n}^{(n)}$ denote the center of $T_r^{(n)}$. This defines a genuine character of $T_{r,n}^{(n)}$ which we will denote by χ. Let $T_{r,0}^{(n)}$ denote any maximal abelian subgroup of $T_r^{(n)}$ which contains the group $T_{r,n}^{(n)}$. Choose any extension of the character χ from $T_{r,n}^{(n)}$ to $T_{r,0}^{(n)}$. Extend the character χ trivially to V_r. Then, inducing this extension to $\mathrm{GL}_r^{(n)}$, we obtain the representation $\pi^{(n)} = \mathrm{Ind}_{B_r^{(n)}}^{\mathrm{GL}_r^{(n)}} \chi\delta_{B_r}^{1/2}$. It follows from [KP84] that this representation depends only on the character χ. See also [BF99, p. 5]. In this paper we choose the maximal

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abelian subgroup $T_{r,0}^{(n)}$ to be the following. It is the group generated by $T_{r,n}^{(n)}$ and all elements $\langle t,\epsilon\rangle \in T_r^{(n)}$ such that all diagonal entries of t are units. Then it follows from [KP84] that $T_{r,0}^{(n)}$ is a maximal abelian subgroup of $T_r^{(n)}$. The extension of χ that we choose is the trivial extension.

Assuming that χ is in general position, one can attach to $\pi^{(n)}$ the local L function which is defined as

$$L(s,\pi^{(n)}) = \frac{1}{\prod_{i=1}^r (1-\chi_i^n(p)q^{-s})}. \tag{4}$$

Here p is a generator of the maximal ideal in the ring of integers of F, and $q^{-1}=|p|_F$. Also, s is a complex variable.

In [BF99, formula (1.4)] the function $\widetilde{\Delta}_s(h)$ is defined. This function is a function of $\mathrm{GL}_r^{(n)}$ and is defined as follows. It is an anti-genuine K_r bi-invariant function, and hence it is enough to define $\widetilde{\Delta}_s(h)$ on elements $h=\langle t,1\rangle$ where $t=\mathrm{diag}(p^{n_1},p^{n_2},\ldots,p^{n_r})\in T_r$. On such an element h, $\widetilde{\Delta}_s(h)=0$ unless all n_i are non-negative integers, each one divisible by n. Finally, if all n_i are non-negative integers, and each one of them is divisible by n, then

$$\widetilde{\Delta}_s(\langle t,1\rangle) = |\det(t)|^{s/n+(r-1)/2n}\delta_{B_r}^{(n-1)/2n}(t).$$

Proposition 1.1 in [BF99] states that for $\mathrm{Re}(s)$ large,

$$\int_{\mathrm{GL}_r} \omega_{\pi^{(n)}}(h)\widetilde{\Delta}_s(h)\,dh = L(s,\pi^{(n)}). \tag{5}$$

Here, $\omega_{\pi^{(n)}}$, denoted by σ in [BF99], is the spherical function attached to $\pi^{(n)}$. Thus $\omega_{\pi^{(n)}}$ is a genuine K_r bi-invariant function of $\mathrm{GL}_r^{(n)}$. As is well known, the function $\widetilde{\Delta}_s(h)$ is uniquely determined by [BF99, Proposition 1.1]. This function is referred to as the generating function for the standard L function of the group $\mathrm{GL}_r^{(n)}$.

We will give a different realization of the function $\widetilde{\Delta}_s(h)$. To do that let $\Theta_{nr}^{(n)}$ denote the local unramified theta representation of $\mathrm{GL}_{nr}^{(n)}$ as constructed in [KP84]. This representation is the unramified sub-representation of $\mathrm{Ind}_{B_{nr}^{(n)}}^{\mathrm{GL}_{nr}^{(n)}}\delta_{B_{nr}}^{(n-1)/2n} = \mathrm{Ind}_{B_{nr}^{(n)}}^{\mathrm{GL}_{nr}^{(n)}}\delta_{B_{nr}}^{-1/2n}\delta_{B_{nr}}^{1/2}$. Here, B_{nr} is the Borel subgroup of GL_{nr}. The definition of this induced representation is similar to the definition of the representation $\pi^{(n)}$ given above, replacing the group $\mathrm{GL}_r^{(n)}$ by $\mathrm{GL}_{nr}^{(n)}$, and the character χ by the character $\delta_{B_{nr}}^{-1/2n}$.

This representation is not generic, but it still has a certain unique functional defined on it. To describe this functional, let U_{nr} denote the unipotent radical of the parabolic subgroup of GL_{nr} whose Levi part is $\mathrm{GL}_r\times\mathrm{GL}_r\times\cdots\times\mathrm{GL}_r$. In term of matrices the group U_{nr} consists of all matrices of the form

$$\begin{pmatrix} I & X_{1,2} & X_{1,3} & \ldots & X_{1,n} \\ & I & X_{2,3} & \ldots & X_{2,n} \\ & & I & \ddots & \vdots \\ & & & \ddots & X_{n-1,n} \\ & & & & I \end{pmatrix}. \tag{6}$$

Here I is the $r\times r$ identity matrix, and $X_{i,j}\in\mathrm{Mat}_{r\times r}$.

Let ψ denote an unramified character of F. Define a character $\psi_{U_{nr}}$ of U_{nr} as follows. For $u\in U_{nr}$ as above, define $\psi_{U_{nr}}(u)=\psi(\mathrm{tr}(X_{1,2}+X_{2,3}+\cdots+X_{n-1,n}))$. The stabilizer of $\psi_{U_{nr}}$ inside

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$\mathrm{GL}_r \times \mathrm{GL}_r \times \cdots \times \mathrm{GL}_r$ is the group GL_r^Δ embedded diagonally. The embedding of GL_r^Δ inside GL_{nr} is given by $g \mapsto \mathrm{diag}(g, g, \ldots, g)$.

Given a representation $\sigma_{nr}^{(n)}$ of $\mathrm{GL}_{nr}^{(n)}$, we consider the space of all functionals on $\sigma_{nr}^{(n)}$ which satisfies $l(\sigma_{nr}^{(n)}(u)v) = \psi_{U_{nr}}(u)l(v)$ for all $u \in U_{nr}$ and all vectors v in the space of $\sigma_{nr}^{(n)}$. Given such a functional, we may consider the space of functions $W_v^{(n)}(h) = l(\sigma_{nr}^{(n)}(h)v)$.

Henceforth we shall assume that $\sigma_{nr}^{(n)} = \Theta_{nr}^{(n)}$ and denote the corresponding space of functions by $W_{nr}^{(n)}(h)$. Then, the following proposition is proved in [Cai16, Theorem 1.2].

Proposition 1. *The space of functionals l defined as above on the representation $\Theta_{nr}^{(n)}$ is one-dimensional.*

It is not hard to construct the space of functions $W_{nr}^{(n)}(h)$ explicitly on the space of $\Theta_{nr}^{(n)}$. Indeed, let $f \in \mathrm{Ind}_{B_{nr}^{(n)}}^{\mathrm{GL}_{nr}^{(n)}} \delta_{B_{nr}}^{(n-1)/2n}$. Let U_{nr}^0 denote the subgroup of U_{nr} which consists of all matrices u as in (6) such that $X_{i,j} \in \mathrm{Mat}_{r\times r}^0$ for all i and j. Here $\mathrm{Mat}_{r\times r}^0$ is the subgroup of $\mathrm{Mat}_{r\times r}$ consisting of all matrices X such that $X[l_1, l_2] = 0$ for all $l_1 < l_2$, where $X[l_1, l_2]$ denotes the (l_1, l_2)th entry of X. Then

$$W_{nr}^{(n)}(h) = \int_{U_{nr}^0} f(w_J w_0 u h)\psi_{U_{nr}}(u)\, du \tag{7}$$

defines the space of functions which satisfies the required transformation properties, provided it is not identically zero. Here w_J is the Weyl element $w_J = \mathrm{diag}(J_n, J_n, \ldots, J_n) \in \mathrm{GL}_{nr}$ where J_n is the longest Weyl element of GL_n. The Weyl element w_0 is defined as the element whose $(a + bn, (a-1)r + b + 1)$th entry is one for all $1 \leqslant a \leqslant n$ and $0 \leqslant b \leqslant r-1$, and zero elsewhere. Matrix multiplication implies that $w_J w_0$ is the shortest Weyl element of GL_{nr} with the property that for all $u \in U_{nr}^0$, we have that wuw^{-1} is a lower unipotent matrix.

It is not hard to prove that the function $W_{nr}^{(n)}(h)$ satisfies also the property $W_{nr}^{(n)}(k^\Delta h) = W_{nr}^{(n)}(h)$ for all $k^\Delta \in K_r \subset \mathrm{GL}_r^\Delta$. The group GL_r^Δ was defined right after (6). Indeed, it follows from [CFGK16, p. 4] right after Definition 1 that $W_{nr}^{(n)}(k^\Delta h) = W_{nr}^{(n)}(h)$ for all $k^\Delta \in K_r \cap \mathrm{SL}_r^\Delta$. If k^Δ is any diagonal matrix in K_r, then the property $W_{nr}^{(n)}(k^\Delta h) = W_{nr}^{(n)}(h)$ follows from integral (7). We mention that to derive this identity we use our choice of the maximal abelian subgroup.

By considering the function $W_{nr}^{(n)}(h)$ corresponding to the K_{nr} fixed vector f in the space of $\Theta_{nr}^{(n)}$, one can easily show that $W_{nr}^{(n)}(e) \neq 0$. This will follow from the computation which we will perform in the next proposition.

Before doing that, it will be convenient to perform a simple computation which we will refer to several times. We will do it in the following subsection.

2.1 A local computation

Let F denote a local field. Given a root α associated with the group GL_b, we will denote by $x_\alpha(l)$ the one-dimensional unipotent subgroup of GL_b associated with this root. Assume that α and β are two roots such that $\alpha + \beta$ is also a root. Assume also that $x_\alpha(z)x_\beta(l) = x_\beta(l)x_\alpha(z)x_{\alpha+\beta}(lz)$. Let $h(a)$ denote a one-dimensional torus of GL_b which satisfies the property $h(a)^{-1}x_\alpha(z)h(a) = x_\alpha(a^{-1}z)$ for all $a \in F^*$.

Let f denote a function defined on $\mathrm{GL}_b(F)$ which satisfies the property

$$f(x_\beta(l_1)x_{\alpha+\beta}(l_2)gk) = \psi(-l_2)f(g) \tag{8}$$

for all $k \in K_b$, where K_b is the standard maximal compact subgroup of GL_b.

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Our goal in this subsection is to compute the integral

$$I=\int_{F^2} f(x_\alpha(z)x_\beta(l)h(a))\psi(\epsilon l)\,dz\,dl.$$

Here $\epsilon=0,-1$.

Lemma 1. *We have $I=f(h(a)x_\alpha(-a^{-1}\epsilon))=f(x_\alpha(-\epsilon)h(a))$.*

Proof. Since f is right K_b invariant,

$$I=\int_{F^2}\int_{|m|\leqslant 1} f(x_\alpha(z)x_\beta(l)h(a)x_\alpha(m))\psi(\epsilon l)\,dm\,dz\,dl.$$

Conjugating $x_\alpha(m)$ to the left, and using the above assumptions on the commutation relations, we obtain the integral

$$I=\int_{F^2}\int_{|m|\leqslant 1} f(x_\alpha(z+am)x_{\alpha+\beta}(-lam)x_\beta(l)h(a))\psi(\epsilon l)\,dm\,dz\,dl.$$

Changing variables in z and using property (8), we obtain $\int\psi(lam)\,dm$ as inner integration. Here m is integrated over $|m|\leqslant 1$. Hence we may restrict the integration domain over the l variable in integral I to the domain $|la|\leqslant 1$.

The next step is to conjugate $x_\beta(l)$ to the left. Using the commutation relations and property (8), we obtain

$$I=\int_F f(x_\alpha(z)h(a))\int_{|la|\leqslant 1}\psi(zl+\epsilon l)\,dl\,dz.$$

Changing variables in l, we obtain

$$I=|a|^{-1}\int_{|(z+\epsilon)a^{-1}|\leqslant 1} f(x_\alpha(z)h(a))\,dz=|a|^{-1}\int_{|(z+\epsilon)a^{-1}|\leqslant 1} f(h(a)x_\alpha(a^{-1}z))\,dz.$$

Writing $a^{-1}z=a^{-1}z+\epsilon a^{-1}-\epsilon a^{-1}$, we obtain

$$I=|a|^{-1}\int_{|(z+\epsilon)a^{-1}|\leqslant 1} f(h(a)x_\alpha(a^{-1}z+\epsilon a^{-1}-\epsilon a^{-1}))\,dz=|a|^{-1}f(h(a)x_\alpha(-a^{-1}\epsilon))\int_{|(z+\epsilon)a^{-1}|\leqslant 1} dz$$

where the last equality is obtained from property (8), and the fact that $x_\alpha(a^{-1}z+\epsilon a^{-1})\in K_b$. From this the lemma follows. $\square$

With the above notation we prove the following corollary.

Corollary 1. *Let $a=0$, or if $a\in F^*$, assume that $|a|\leqslant 1$. Then*

$$\int_F f(x_\alpha(z))\psi(az)\,dz=f(e).$$

Proof. Since f is right invariant under K_b, the above integral is equal to

$$\int_F\int_{|m|\leqslant 1} f(x_\alpha(z)x_\beta(m))\psi(az)\,dm\,dz.$$

Conjugating $x_\beta(m)$ to the left, we obtain, from the left invariant property of f, the integral $\int\psi(mz)\,dm$ as inner integration. Here m is integrated over $|m|\leqslant 1$. Hence, we may restrict the integration over z to the domain $|z|\leqslant 1$. The result follows. $\square$

2.2 On the generating function

In this subsection we will prove that the generating function can be expressed in terms of the function $W_{nr}^{(n)}$.

Embed $g \in \mathrm{GL}_r$ in GL_{nr} as $g \mapsto g_0 = \mathrm{diag}(g, I_r, \dots, I_r)$. We have the following proposition.

Proposition 2. *Let $W_{nr}^{(n)}(h)$ denote the function corresponding to the K_{nr} fixed vector. Then, for $s' = s/n - (n-2)r/2 - 1/2n$, we have*

$$\widetilde{\Delta}_s(g) = \overline{W_{nr}^{(n)}(g_0)} |\det g|^{s'}. \tag{9}$$

Proof. It follows from the discussion right after equation (7) that the function $W_{nr}^{(n)}(g_0)$ is K_r bi-invariant. To prove the proposition it is enough to show that

$$\int_{\mathrm{GL}_r} \omega_{\pi^{(n)}}(g) \overline{W_{nr}^{(n)}(g_0)} |\det g|^{s'} \, dg = L(s, \pi^{(n)}). \tag{10}$$

Notice that this will also imply that the function $W_{nr}^{(n)}(h)$ as defined in (7) is not identically zero on the space of the representation $\Theta_{nr}^{(n)}$. Using the identity $\omega_{\pi^{(n)}}(g) = \int_{K_r} f_{\pi^{(n)}}(kg) \, dk$, we may, after a change of variables, replace in (10) the function $\omega_{\pi^{(n)}}$ by $f_{\pi^{(n)}}$. Here $f_{\pi^{(n)}}$ is the unramified vector in the space of $\pi^{(n)}$. Performing the Iwasawa decomposition, the integral in equation (10) is equal to

$$\int_{T_r} f_{\pi^{(n)}}(t) \int_{V_r} \overline{W_{nr}^{(n)}(v_0 t_0)} |\det t|^{s'} \delta_{B_r}(t)^{-1} \, dv_0 \, dt. \tag{11}$$

Here V_r is the maximal unipotent subgroup of GL_r consisting of upper unipotent matrices. Also $t = \mathrm{diag}(a_1, a_2, \dots, a_r)$. Plug integral (7) into integral (11). Thus, we obtain the integral

$$\int_{V_r} \int_{U_{nr}^0} \overline{f}(w_J w_0 u v_0 t_0) \psi_{U_{nr}}(u) \, du \, dv_0 \tag{12}$$

as an inner integration to integral (11). Let U_{nr}^1 denote the subgroup of U_{nr}^0 consisting of all matrices such that $X_{1,2}[i,j] = 0$ for all $i \neq j$. We claim that integral (12) is equal to

$$\int_{U_{nr}^1} \overline{f}(w_J w_0 u t_0) \psi_{U_{nr}}(u) \, du. \tag{13}$$

We do this by using Lemma 1 several times. It is convenient to use the following notation. For all integers $1 \leqslant a, b \leqslant nr$ and all $m \in F$, let $x_{a,b}(m) = I_{nr} + m e_{a,b}$. Here $e_{a,b}$ is the matrix of size nr which has a one in the (a,b)th entry, and zero elsewhere.

In integral (12) consider the integrations over the variables $X_{1,2}[r, r-1]$ and $v_0[r-1, r]$, where the latter variable indicates the $(r-1, r)$th entry of v_0. In the notation of § 2.1, let $x_\alpha(z) = x_{r,2r-1}(z)$ where $z = X_{1,2}[r, r-1]$, and let $x_\beta(l) = x_{r-1,r}(l)$ where $l = v_0[r-1, r]$. With this notation we have $x_{\alpha+\beta}(m) = x_{r-1,2r-1}(m)$, and from the definition of the character $\psi_{U_{nr}}$ we have $\psi_{U_{nr}}(x_{\alpha+\beta}(m)) \neq 1$. Hence, all the conditions of Lemma 1 are satisfied with $\epsilon = 0$ and $h(a) = h(a_r) = \mathrm{diag}(I_{r-1}, a_r, I_{nr-r})$. From this we deduce that in integral (12) we may restrict the domain of integration to the group V_r with the condition that $v_0[r-1, r] = 0$, and to the group U_{nr}^0 with the condition $X_{1,2}[r, r-1] = 0$.

In general, we apply this process in the following order. Fix $r + 1 \leqslant j \leqslant 2r - 1$. Then for all $j - r + 1 \leqslant i \leqslant r$, set $x_\alpha(z) = x_{i,j}(z)$ with $z = X_{1,2}[i, j-r]$, and $x_\beta(l) = x_{j-r,i}(l)$ with

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$l = v_0[j-r,i]$. With this notation we have $x_{\alpha+\beta}(m) = x_{j-r,j}(m)$. Since $\psi_{U_{nr}}$ is not trivial on $x_{\alpha+\beta}(m)$, we can apply Lemma 1 with $\epsilon = 0$. The end result of this repeated process is that integral (12) is equal to integral (13).

Conjugating by w_0, write $w_0U_{nr}^1w_0^{-1} = U_{nr}^2U_{nr}^3$, where the groups U_{nr}^2 and U_{nr}^3 are defined as follows. First, identify the group U_{nr}^2 with r copies of the group V_n. Here V_n is defined to be the group of all upper unipotent matrices of GL_n. The embedding of U_{nr}^2 inside GL_{nr} is given by $(v_{n,1}, v_{n,2}, \ldots, v_{n,r}) \mapsto \mathrm{diag}(v_{n,1}, v_{n,2}, \ldots, v_{n,r})$. Here $v_{n,i} \in V_n$. To define the group U_{nr}^3, consider the unipotent group generated by all matrices of the form

$$\begin{pmatrix} I &&&& \\ Y_{2,1} & I &&& \\ Y_{3,1} & Y_{3,2} & I && \\ \vdots & \vdots & \ddots & I & \\ Y_{r,1} & Y_{r,2} & \ldots & Y_{r,r-1} & I \end{pmatrix}. \tag{14}$$

Here $Y_{i,j}$ is in $\mathrm{Mat}_{n\times n}$. Then the group U_{nr}^3 is generated by all matrices as in (14) which satisfies the conditions $Y_{i,j}[l_1, l_2] = Y_{i,j}[1,2] = 0$ for all $l_1 \geqslant l_2$.

For $v \in V_n$, let $\psi_{V_n}(v)$ denote the Whittaker character of the group V_n. This character is defined as follows. Given $v = (v[i,j]) \in V_n$, then

$$\psi_{V_n}(v) = \psi(v[1,2] + v[2,3] + \cdots + v[n-1,n]). \tag{15}$$

Let $u_2 = \mathrm{diag}(v_{n,1}, v_{n,2}, \ldots, v_{n,r}) \in U_{nr}^2$. Define the character $\psi_{U_{nr}^2}$ of U_{nr}^2 as $\psi_{U_{nr}^2}(u_2) = \psi_{V_n}(v_{n,1})\psi_{V_n}(v_{n,2})\cdots\psi_{V_n}(v_{n,r})$. Then, in the notation of $U_{nr}^2U_{nr}^3$, the character $\psi_{U_{nr}}$ transforms to the character $\psi_{U_{nr}^2}$ on the group U_{nr}^2, and is trivial on the group U_{nr}^3.

Thus, integral (13) is equal to

$$\int_{U_{nr}^3} f_W(u_3w_0t_0w_0^{-1})\,du_3 \tag{16}$$

where

$$f_W(h) = \int_{U_{nr}^2} \overline{f}(w_Ju_2h)\psi_{U_{nr}^2}(u_2)\,du_2.$$

We have $w_0t_0w_0^{-1} = \mathrm{diag}(A_1, A_2, \ldots, A_r)$ where $A_i = \mathrm{diag}(a_i, I_{n-1})$. Conjugating the matrix $w_0t_0w_0^{-1}$ to the left in integral (16), we obtain the factor

$$\alpha(t) = (|a_2||a_3|^2|a_4|^3\cdots|a_r|^{r-1})^{n-2}$$

from the change of variables in U_{nr}^3. Thus, integral (16) is equal to

$$\alpha(t)\int_{U_{nr}^3} f_W(w_0t_0w_0^{-1}u_3)\,du_3. \tag{17}$$

We claim that integral (17) is equal to $\alpha(t)f_W(w_0t_0w_0^{-1})$. This we will show by a repeated application of Corollary 1. Indeed, fix $2 \leqslant i \leqslant r$, where we first start with $i = r$, then $i = r-1$ and so on. Let $1 \leqslant k \leqslant i-1$. Assume that l_1 and l_2 are such that $Y_{i,k}[l_1, l_2]$ is a variable in the domain of integration in integral (17). In the notation of § 2.1, let $x_\alpha(z) = x_{n(i-1)+l_1,n(k-1)+l_2}(z)$ with $z = Y_{i,k}[l_1, l_2]$, and let $x_\beta(m) = x_{n(k-1)+l_2,n(i-1)+l_1+1}(m)$. Then $x_{\alpha+\beta}(l) = x_{n(i-1)+l_1,n(i-1)+l_1+1}(l)$, and from the properties of f_W we have $f_W(x_{\alpha+\beta}(l)g) = \psi(-l)f_W(g)$. Applying Corollary 1 several times in the indicated order, the above claim follows.

Let f be an arbitrary function in the space of $\Theta_{nr}^{(n)}$. Then the integral $f_W(h)$ defines a functional on $\Theta_{nr}^{(n)}$ which by [Cai16, Theorem 3.38] is unique. Applying an argument similar to that in [BG92, pp. 169–170], we deduce that for the unramified vector f in the space of $\Theta_{nr}^{(n)}$ we have the following. First, $f_W(w_0t_0w_0^{-1}) = 0$ unless $|a_i| \leqslant 1$ and $|a_i| = |b_i|^n$ for some $|b_i| \leqslant 1$. On such elements $f_W(w_0t_0w_0^{-1})$ is not zero and is equal to $\delta_{B_{nr}}^{(n-1)/2n}(w_0t_0w_0^{-1})$. Hence, integral (17) is equal to

$$\alpha(t)f_W(w_0t_0w_0^{-1}) = \alpha(t)\delta_{B_{nr}}^{(n-1)/2n}(w_0t_0w_0^{-1}). \tag{18}$$

When $a_i = b_i^n$, we have $f_{\pi^{(n)}}(t) = \prod_{i=1}^{r} \chi_i^n(b_i)\delta_{B_r}^{n/2}(\mathrm{diag}(1,\ldots,1,b_i,1,\ldots,1))$. Combining all this, integral (11) is equal to

$$\prod_{i=1}^{r} \int_{|b_i|\leqslant 1} \chi_i^n(b_i)|b_i|^{ns'+n(n-2)(r-1)/2+(n-1)^2/2}\, db_i. \tag{19}$$

From this the proposition follows. □

3. The Whittaker functional of the generating function

In this section we compute the Whittaker functional of the function $W_{nr}^{(n)}(g_0)$. Here the notation is as in § 2, but we assume that $r < n$. We make this assumption to get a precise proof of [BF99, Conjecture 1.2]. The case when $r \geqslant n$ is similar and will be dealt with in the next section. Embed $g \in \mathrm{GL}_r$ in GL_n as $g \mapsto \mathrm{diag}(g, I_{n-r})$. Let $g_0 = \mathrm{diag}(g, I_{n-r}, I_n, \ldots, I_n) \in \mathrm{GL}_{nr}$, where I_n appears $r-1$ times.

Let V_r denote the standard maximal unipotent subgroup of GL_r, and let $\psi_{V_r}^{-1}$ denote the Whittaker character of V_r. See equation (15) for the definition of ψ_{V_r}. Let $W_{\Theta_n}^{(n)}$ denote the Whittaker function of the theta function defined on $\mathrm{GL}_n^{(n)}$. Our goal is to prove the following theorem.

THEOREM 2. *Assume that $r < n$. With the above notation, for all $g \in \mathrm{GL}_r^{(n)}$, we have*

$$\int_{V_r} W_{nr}^{(n)}(v_0g_0)\psi_{V_r}^{-1}(v)\, dv = W_{\Theta_n}^{(n)}\begin{pmatrix} g & \\ & I_{n-r}\end{pmatrix}|\det g|^{(n-1)(r-1)/2}. \tag{20}$$

Proof. We will consider the case when $r = n-1$. This is the hardest case. When $r < n-1$ the computations are similar but simpler. Since we will use some of the notation introduced in the previous sections, we will continue to write r and n even though we assume that $r = n-1$. By the Iwasawa decomposition, it is enough to prove identity (20) for $g = t = \mathrm{diag}(a_1, a_2, \ldots, a_{n-1})$. Notice, that from the left invariant properties of $W_{nr}^{(n)}$ and $W_{\Theta_n}^{(n)}$, we may assume that $|a_i| \leqslant 1$ for all $1 \leqslant i \leqslant n-1$.

We start by plugging integral (7) into the left-hand side of identity (20). Doing so, we obtain the integral

$$\int_{V_r}\int_{U_{nr}^0} f(w_Jw_0uv_0t_0)\psi_{U_{nr}}(u)\psi_{V_r}^{-1}(v)\, du\, dv. \tag{21}$$

As in the proof of Proposition 2, we claim that integral (21) is equal to the integral

$$\int_{U_{nr}^0} f(w_Jw_0u\delta_0t_0)\psi_{U_{nr}}(u)\, du \tag{22}$$

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where $\delta_0=\prod_{i=2}^{n-1}x_{i,r+i-1}(1)$. Here, the definitions of U_{nr}^0 and $x_{a,b}(m)$ are given before and after integral (13). To prove the above claim, we follow exactly the same steps as in the proof that integral (12) is equal to integral (13). The only difference is that because of the character $\psi_{V_r}^{-1}$ in integral (21), for suitable variables in V_r, we need to use Lemma 1 with $\epsilon=-1$ and not with $\epsilon=0$ as in the proof of Proposition 2. This explains the element δ_0. Next we proceed as in the proof of Proposition 2. Following the exact steps which showed that integral (13) is equal to integral (17), we deduce that integral (22) is equal to

$$I_1=\alpha(t)\int_{U_{nr}^3} f_W(w_0t_0w_0^{-1}u_3\delta_1(t))\,du_3.$$

Here the group U_{nr}^3 and $\alpha(t)$ are defined before integral (17), and we remind the reader that we assume that $r=n-1$. Also, we have $\delta_1(t)=\prod_{i=2}^{n-1}x_{(i-1)n+1,(i-2)n+2}(a_i^{-1})$. This element is obtained by conjugating δ_0 by w_0 and t_0.

At this point, for all $2\leqslant j\leqslant n-1$, we will introduce an integral which we denote by I_j. To do that we first fix some notation. Let $t_j=\mathrm{diag}(A_{j,j},A_{j,j+1},\ldots,A_{j,n-1},I_n,\ldots,I_n)$ denote the torus element of GL_{nr} where $A_{j,j}=\mathrm{diag}(a_1,\ldots,a_j,I_{n-j})$, and for all $j+1\leqslant i\leqslant n-1$ we define $A_{j,i}=\mathrm{diag}(I_j,a_i,I_{n-j-1})$. Notice that when $j=n-1$, we get $t_j=\mathrm{diag}(A_{n-1,n-1},I_n,\ldots,I_n)=t_0$. Next we define $\alpha_j(t)=|a_{j+1}a_{j+2}^2a_{j+3}^3\cdots a_{n-1}^{n-j-1}|^{n-2}$, where we set $\alpha_{n-1}(t)=1$. Finally, we define a set of subgroups $U_{n,j}$, and a set of characters $\psi_{U_{n,j}}$ defined on these groups. The definition is inductive, so we start with $U_{n,2}$. Consider the group U_{nr}^3 with $r=n-1$, as was defined right after equation (14). Let $U_{n,2}$ denote the subgroup of U_{nr}^3 with the extra condition that $Y_{n-1,i}=0$ for all $1\leqslant i\leqslant n-2$. Assuming we defined $U_{n,j-1}$, we define $U_{n,j}$ as the subgroup of $U_{n,j-1}$ consisting of matrices of the form (14) such that $Y_{n-j+1,i}=0$ for all $1\leqslant i\leqslant n-j$, and also satisfies the condition $Y_{i,l}[b,j]=0$ for all $2\leqslant i\leqslant n-j$, $1\leqslant l\leqslant i-1$ and $1\leqslant b\leqslant n$. The character $\psi_{U_{n,j}}$ is defined as follows. For $u\in U_{n,j}$ written as in equation (14), we set $\psi_{U_{n,j}}(u)=\psi(\sum_{i=2}^{n-j}Y_{i,i-1}[j-1,j+1])$.

With this notation, for all $2\leqslant j\leqslant n-1$, we set

$$I_j=\alpha_j(t)\int_{U_{n,j}} f_W(t_ju)\psi_{U_{n,j}}(u)\,du.$$

We will prove that $I_2=I_1$, and that for all $2\leqslant j\leqslant n-1$, we have $I_j=I_{j-1}$. This will complete the proof of the theorem. Indeed, proving the above implies that the left-hand side of equation (20) is equal to I_{n-1}. Since $\alpha_{n-1}(t)=1$, the group $U_{n,n-1}$ is the trivial group, and $t_{n-1}=t_0$, we deduce that $I_{n-1}=f_W(t_0)$. But as in equation (18) we obtain that $f_W(t_0)$ equals the right-hand side of equation (20).

We prove that $I_2=I_1$. Since $|a_i|\leqslant 1$, we obtain the Iwasawa decomposition $\delta_1(t)=\prod_{i=2}^{n-1}x_{(i-2)n+2,(i-1)n+1}(a_i)\prod_{i=2}^{n-1}h_i(a_i)k$. Here $k\in K_{nr}$, and we have $\prod_{i=2}^{n-1}h_i(a_i)=\mathrm{diag}(B_{2,1},B_{2,2},\ldots,B_{2,n-1})$. Here $B_{2,1}=\mathrm{diag}(1,a_2,I_{n-2})$, for $2\leqslant i\leqslant n-2$ we have $B_{2,i}=\mathrm{diag}(a_i^{-1},a_{i+1},I_{n-2})$, and $B_{2,n-1}=\mathrm{diag}(a_{n-1}^{-1},I_{n-1})$. Conjugating in I_1 the matrix $\delta_1(t)k^{-1}$ to the left, and using the left invariant properties of f_W, we obtain by matrix multiplication

$$I_1=\alpha_2(t)\int_{U_{nr}^3} f_W(t_2u_3)\psi_{U_{nr}^3}(u_3)\,du_3. \tag{23}$$

Here we use the fact that $w_0t_0w_0^{-1}\prod_{i=2}^{n-1}h_i(a_i)=t_2$. The factor of $|a_2a_3\cdots a_{n-1}|^{-(n-2)}$ is obtained from a change of variables when we conjugate the torus $\prod_{i=2}^{n-1}h_i(a_i)$ across U_{nr}^3. The product of this factor by $\alpha_1(t)$ is equal to $\alpha_2(t)$. The character $\psi_{U_{nr}^3}$ is defined as follows. For $u_3\in U_{nr}^3$, define $\psi_{U_{nr}^3}(u_3)=\psi(\sum_{i=2}^{n-1}Y_{i,i-1}[1,3])$. To complete the proof that $I_2=I_1$, we need to show that we can restrict the support of integration from U_{nr}^3 to $U_{n,2}$. In other words, we need to show that for all $1\leqslant i\leqslant n-2$, the integration over all variables in $Y_{n-1,i}$ is in K_{nr}. This is done as in the proof of Proposition 2 while showing that integral (17) reduces to the left-hand side of identity (18). Indeed, from the definition of the torus t_2, given a variable $Y_{n-1,i}[l_1,l_2]$, we can find a one-dimensional unipotent subgroup $x_\beta(m)$ so that we can apply Corollary 1. Thus $I_2=I_1$.

The next step is to prove that $I_j=I_{j-1}$. The first step is to prove that we can integrate over a smaller unipotent group. Let $U_{n,j-1,1}$ denote the subgroup of $U_{n,j-1}$ consisting of all matrices which also satisfies $Y_{i,l}[b,j]=0$ for all $3\leqslant i\leqslant n-j+1$, $1\leqslant l\leqslant i-2$ and $1\leqslant b\leqslant j-1$. To show that we can reduce the domain of integration from $U_{n,j-1}$ to $U_{n,j-1,1}$ we apply Corollary 1. In the notation of this corollary, let $x_\alpha(z)=x_{n(i-1)+b,n(l-1)+j}(z)$ with $z=Y_{i,l}[b,j]$, and let $x_\beta(m)=x_{nl+j-2,n(i-1)+b}(m)$. Notice that in this case the root $\alpha+\beta$ corresponds to the one-dimensional unipotent subgroup $x_{nl+j-2,n(l-1)+j}(c)$, which is a subgroup of $U_{n,j-1,1}$. Moreover, the character $\psi_{U_{n,j-1}}$ is not trivial on this subgroup. Hence, the conditions of the Corollary 1 are satisfied. We mention that the order for which we apply this corollary is important. We first vary $3\leqslant i\leqslant n-j+1$ and fix $l=1$. Then we repeat the same process with $l=2$ and so on.

The second step is to show that I_{j-1} is equal to

$$\alpha_{j-1}(t)\int_{U_{n,j-1,2}}f_W(t_{j-1}u\delta_{j-1}(t))\psi_{U_{n,j-1}}(u)\,du. \tag{24}$$

Here $U_{n,j-1,2}$ is the subgroup of $U_{n,j-1,1}$ which satisfies the condition that $Y_{i,i-1}[j-1,j]=0$ for all $2\leqslant i\leqslant n-j+1$. The matrix $\delta_{j-1}(t)=\prod_{i=2}^{n-j+1}x_{(i-1)n+j-1,n(i-2)+j}(a_{j+i-2}^{-1})$. To derive integral (24) we apply Corollary 1 with $x_\alpha(z)=x_{n(i-1)+j-1,n(i-2)+j}(z)$ with $z=Y_{i,i-1}[j-1,j]$ and $x_\beta(l)=x_{(i-1)n+j-2,(i-1)n+j-1}(l)$. The next step is to perform an Iwasawa decomposition for $\delta_{j-1}(t)$ in integral (24). This is done as with $\delta_1(t)$ and we obtain

$$\delta_{j-1}(t)=\prod_{i=2}^{n-j+1}x_{n(i-2)+j,(i-1)n+j-1}(a_{j+i-2})\prod_{i=2}^{n-j+1}h_i'(a_{j+i-2})k$$

where $k\in K_{nr}$. Here $\prod_{i=2}^{n-j+1}h_i'(a_{j+i-2})=\operatorname{diag}(B_{j-1,1},B_{j-1,2},\dots,B_{j-1,n-j+1},I_n,\dots,I_n)$, where $B_{j-1,1}=\operatorname{diag}(I_{j-1},a_j,I_{n-j})$, $B_{j-1,i}=\operatorname{diag}(I_{j-2},a_{j+i-2}^{-1},a_{j+i-1},I_{n-j})$ for $2\leqslant i\leqslant n-j$, and $B_{j-1,n-j+1}=\operatorname{diag}(I_{j-2},a_{n-1}^{-1},I_{n-j+1})$. Plugging this into integral (24) and conjugating the matrix $\delta_{j-1}(t)k^{-1}$ to the left, we obtain

$$\alpha_j(t)\int_{U_{n,j-1,2}}f_W(t_ju)\psi_{U_{n,j}}(u)\,du. \tag{25}$$

Here, we obtain the factor of $|a_ja_{j+1}\cdots a_{n-1}|^{-(n-2)}$ from the conjugation of the toral part of $\delta_{j-1}(t)k^{-1}$ across $U_{n,j-1,2}$. This combined with $\alpha_{j-1}(t)$ gives the factor $\alpha_j(t)$ in integral (25). Notice also that the conjugation by the unipotent part of $\delta_{j-1}(t)k^{-1}$ changes the additive character to $\psi_{U_{n,j}}$. This is well defined. Indeed, notice that $U_{n,j}$ is a subgroup $U_{n,j-1,2}$ and we can view $\psi_{U_{n,j}}$ as a character of $U_{n,j-1,2}$ by extending it trivially. Finally, we have the identity $t_{j-1}\prod_{i=2}^{n-j+1}h_i'(a_{j+i-2})=t_j$. To show that integral (25) equals I_j, we need to show that we may restrict the domain of integration from $U_{n,j-1,2}$ to $U_{n,j}$. We do so using Corollary 1. Indeed, the group $U_{n,j}$ is the subgroup of $U_{n,j-1,2}$ obtained by setting $Y_{n-j+1,l}=0$ for all $1\leqslant l\leqslant n-j$ and $Y_{i,i-1}[b,j]=0$ for all $2\leqslant i\leqslant n-j$ and $1\leqslant b\leqslant j-2$. To show that we may restrict the integration over $U_{n,j-1,2}$ to the subgroup obtained by setting $Y_{n-j+1,l}=0$, we argue in a similar way to the reduction from the group U_{nr}^3 to $U_{n,2}$ as was done right after integral (23). Then, finally, to obtain the group $U_{n,j}$ we use Corollary 1 with $x_\alpha(z) = x_{n(i-1)+b,n(i-2)+j}(z)$ where $z = Y_{i,i-1}[b,j]$ and $x_{n(i-2)+j,n(i-1)+b+1}(m)$. Here $2 \leqslant i \leqslant n-j$ and $1 \leqslant b \leqslant j-2$. □

4. The case when $r \geqslant n$

As mentioned in the introduction, the authors of [BF99] were well aware that the situation when $r \geqslant n$ is similar. Since they do not specify this case explicitly, we briefly mention the global constructions and show how a similar result to Theorem 2 holds in this case.

Assume first that $r > n$. Let $\pi^{(n)}$ denote a cuspidal representation of the group $\mathrm{GL}_r^{(n)}(\mathbf{A})$. Let $\Theta_n^{(n)}$ denote the theta representation of the group $\mathrm{GL}_n^{(n)}(\mathbf{A})$. Then we consider the global integral (1) introduced in the introduction. The group $V_{r,n}$ is defined as follows. Recall that V_r is the standard maximal unipotent subgroup of GL_r. Then $V_{r,n}$ is the subgroup of V_r consisting of all matrices $v = (v_{i,j}) \in V_r$ such that $v_{i,j} = 0$ for all $2 \leqslant j \leqslant n+1$. The character $\psi_{V_{r,n}}$ is defined by $\psi_{V_{r,n}}(v) = \psi(v_{n+1,n+2} + v_{n+2,n+3} + \cdots + v_{r-1,r})$. It follows from the cuspidality of ϕ that integral (1) converges for all s. A similar unfolding to that in [BF99, § 2] implies that for $\mathrm{Re}(s)$ large, integral (1) is equal to integral (2).

Next we consider the case when $r = n$. In this case the global integral is given by

$$\int_{Z_r(\mathbf{A})\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbf{A})} \phi(g)\overline{\theta(g)}E(g,s)\,dg. \tag{26}$$

Here Z_r is the subgroup of Z, the center of GL_r, which consists of scalar matrices which are r powers. For simplicity we assume that all representations have a trivial central character. Also, $E(g,s)$ is the Eisenstein series defined on the group $\mathrm{GL}_r(\mathbf{A})$ and is associated with the induced representation $\mathrm{Ind}_{P(\mathbf{A})}^{\mathrm{GL}_r(\mathbf{A})}\delta_P^s$. Here P is the maximal parabolic subgroup of GL_r whose Levi part is $\mathrm{GL}_{r-1} \times \mathrm{GL}_1$. Unfolding this integral, by first unfolding the Eisenstein series, we obtain for $\mathrm{Re}(s)$ large that integral (26) is equal to

$$\int_{Z_r(\mathbf{A})V_r(\mathbf{A})\backslash \mathrm{GL}_r(\mathbf{A})} W_\phi(g)\overline{W_\theta(g)}f(g,s)\,dg. \tag{27}$$

Here $f(g,s)$ is a section in the above induced representation.

Next we study the local unramified computation corresponding to integrals (2) and (27). Since the Whittaker coefficient of the representation $\pi^{(n)}$ is not factorizable, it is not clear that these integrals are Eulerian. However, as explained in [BF99], if we can prove similar results to Proposition 2 and Theorem 2, the so-called 'new way' would imply that these integrals are indeed factorizable. As for Proposition 2, it is clear that it holds for all values of r and n.

As for Theorem 2, this is not the case for all matrices $g \in \mathrm{GL}_r^{(n)}$. Assuming $r \geqslant n$, denote by T_0 the subgroup of GL_r which consists of all diagonal matrices $t = \mathrm{diag}(a_1, a_2, \ldots, a_r)$ such that $|a_i| \leqslant 1$ for $1 \leqslant i \leqslant n$ and $|a_i| = 1$ for all $n+1 \leqslant i \leqslant r$. Let $T_0^{(n)}$ denote the inverse image of T_0 inside $\mathrm{GL}_r^{(n)}$. Let $\mathrm{GL}_{r,0}^{(n)}$ denote all elements $g \in \mathrm{GL}_r^{(n)}$ which can be written as $g = vtk$ where $v \in V_r$, $t \in T_0^{(n)}$ and $k \in K_r$. Here K_r is the standard maximal compact subgroup of GL_r. With this notation we have the following theorem.

Theorem 3. *Assume that $r \geqslant n$. Then, for all $g \in \mathrm{GL}_{r,0}^{(n)}$, we have*

$$\int_{V_r} W_{nr}^{(n)}(v_0 g_0)\psi_{V_r}^{-1}(v)\,dv = W_{\Theta_n}^{(n)}(g)|\det g|^{(n-1)(r-1)/2+n-r}. \tag{28}$$

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Proof. The proof of this theorem is the same as the proof of Theorem 2, and so we will only indicate the end result. Using the Iwasawa decomposition, we may assume that $g_0 = t_0 = \mathrm{diag}(t, I_r, \ldots, I_r)$ where $t = (a_1, \ldots, a_r)$. This factorization produces a factor of $\delta_{B_r}^{-1}(t)$.

Defining similar integrals I_j as in Theorem 2, we prove that the left-hand side of integral (28) is equal to $f_W(t_0')$ where $t_0' = \mathrm{diag}(A_1, A_2, \ldots, A_{r-n+1}, I_n, \ldots, I_n)$. Here $A_1 = \mathrm{diag}(a_1, \ldots, a_n)$, and for all $2 \leqslant i \leqslant r-n+1$ we have $A_i = \mathrm{diag}(I_{n-1}, a_{n+i-1})$. Applying the factorization of equation (18), we obtain the identity

$$f_W(t_0') = \delta_{P_{r,n}}^{(n-1)/2n}(t_0') \prod_{i=1}^{r-n+1} W_{\Theta_n}^{(n)}(A_i).$$

From the properties of the Whittaker function, we deduce that for all $2 \leqslant i \leqslant r-n+1$ we have $W_{\Theta_n}^{(n)}(A_i) = 0$ unless $|a_{n+i-1}| = 1$. Notice that $\delta_{B_r}^{-1}(A_1, I_{n-r}) = \delta_{B_n}^{-1}(A_1)|\det A_1|^{n-r}$. From this the theorem follows. $\square$

Notice that this theorem is enough to prove that the corresponding local versions of integrals (2) and (27) are Eulerian. Indeed, the local version of integral (2) is given by

$$\int_{V_n\backslash \mathrm{GL}_n} W_\phi \begin{pmatrix} g & \\ & I_{r-n} \end{pmatrix} \overline{W_\theta(g)} |\det g|^{s-(r-n)/2}\, dg. \tag{29}$$

Here ϕ is a vector in the local component of $\pi_\nu^{(n)}$ where ν is a place where all data are unramified. Similarly for θ. Also, W_ϕ is any local Whittaker functional defined on the representation $\pi_\nu^{(n)}$. Similarly, W_θ is the Whittaker functional defined on the space of $\Theta_{n,\nu}^{(n)}$. It is known that for the representation $\Theta_{n,\nu}^{(n)}$ this Whittaker functional is unique (see [KP84]). However, this need not be the case for the representation $\pi_\nu^{(n)}$.

Applying the Iwasawa decomposition to the quotient $V_r\backslash \mathrm{GL}_r$, the domain of integration in integral (29) is reduced to the torus T_r of GL_r. However, because of the Whittaker functional properties, $W_\phi\left(\begin{smallmatrix} t & \\ & I_{r-n}\end{smallmatrix}\right)$ is zero unless $t \in T_0$. Hence, we can apply Theorem 3 to deduce that integral (2) is indeed Eulerian.

A similar argument applies to integral (27). Indeed, using the properties of the Whittaker function, we can choose representatives for the quotient $Z_r\backslash T_r$ to be in the group T_0. Hence, once again we can apply Theorem 3.

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David Ginzburg ginzburg@post.tau.ac.il

School of Mathematical Sciences, Tel Aviv University,
Ramat Aviv, Tel Aviv 6997801, Israel