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APPROXIMATELY RATIONALLY OR ELLIPTICALLY CONNECTED VARIETIES

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Dedicated to Slava Shokurov, on his 60th birthday

Abstract We discuss a possible approach to the study of the vanishing of the Kobayashi pseudo-metric of a projective variety X, using chains of rational or elliptic curves contained in an arbitrarily small neighbourhood of X in projective space for the Euclidean topology.

Keywords: rational curves; rational connectedness; Kobayashi hyperbolicity

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1. Introduction

In [**17**], Lang made some conjectures concerning entire curves in complex projective varieties X. He conjectured, for example, that the Zariski closure of the locus in X swept out by entire curves is equal to the locus swept out by images of abelian varieties under non-constant rational maps $\phi\colon A \dashrightarrow X$. When X is a very general quintic 3-fold in $\mathbb{P}^4$, this has been shown to be incompatible with Clemens's conjecture [**6**] by the following arguments.

(i) X contains countably many families of rational curves, and they are Zariski dense in X.

(ii) On the other hand, X is not swept out by images of non-constant generically finite rational maps $\phi\colon A \dashrightarrow X$ with A abelian, $\dim A \geqslant 2$ (see [**25**]).

(iii) Finally, if X was swept out by elliptic curves, this would contradict Clemens's conjecture on the discreteness of rational curves in X. (See [**8**, Lecture 22] or [**25**].)

The goal of this paper is to discuss and illustrate by examples several possible notions of *approximate rational connectedness* or *approximate elliptic connectedness* concerning complex projective manifolds. The general hope would be that approximately elliptically connected varieties are exactly varieties with trivial Kobayashi pseudo-distance (see [**15**]). The main idea is that, instead of looking at rational or elliptic curves (or abelian varieties)

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sitting in X, we should study rational or elliptic curves contained in arbitrarily small neighbourhoods of X in projective space (for the Euclidean topology).

We assume that X is embedded in some projective space $\mathbb{P}^N$. We start with the following naive definition.

Definition 1.1. X is said to be approximately rationally connected in $\mathbb{P}^N$ *in the naive sense* if, for any neighbourhood (for the Euclidean topology) $U \subset \mathbb{P}^N$ of X, any two points of X are contained in a rational curve $C \subset U$.

Remark 1.2. An equivalent definition is that any two points of X can be joined in an arbitrarily small neighbourhood of X by a chain of rational curves, since such chains can be made irreducible by a small deformation in U, because U has positive tangent bundle.

The reason why this definition cannot be interesting, from the point of view of the study of the Kobayashi pseudo-distance of X, is the following fact.

Lemma 1.3.

(i) *Let Y be a connected projective variety. Then, $X := Y \times \mathbb{P}^1$ is approximately rationally connected in the naive sense in any projective embedding $X \subset \mathbb{P}^N$.*

(ii) *More generally, any connected variety X such that the union of rational curves contained in X is dense in X for the Euclidean topology is approximately rationally connected in the naive sense in any projective embedding.*

(iii) *Assume that $X \subset \mathbb{P}^N$ has the property that, for any neighbourhood U of X, and for any point $x \in X$, there exists a rational curve $C \subset U$ passing through x. Then, X is approximately rationally connected in the naive sense.*

Proof. (i) Indeed, let U be a neighbourhood of X in $\mathbb{P}^N$. For any automorphism g of $\mathbb{P}^N$ sufficiently close to the identity, and any curve $C_y = y \times \mathbb{P}^1 \subset Y \times \mathbb{P}^1 = X$, the curve $g(C_y)$ is then contained in U. It immediately follows that, for any $x \in X$, the set of points z in X such that there exists a rational curve $C \subset U$ passing through x and z contains an open neighbourhood of the curve $C_{pr_1(x)} \setminus \{x\}$ in $X \setminus \{x\}$. Applying this argument to any $z \neq x$ in this neighbourhood, we find that the set of points $x' \in X$ such that there exists a chain of two rational curves $C_1, C_2 \subset U$ passing through x and x' contains an open neighbourhood of x in X. As X is connected and compact, this easily implies that any two points of X can be connected by a chain of rational curves contained in U.

(ii) Let U be a neighbourhood of X in $\mathbb{P}^N$. For some $\epsilon > 0$, U contains $U_\epsilon(X) = \{y \in \mathbb{P}^N, d(y, X) < \epsilon\}$. For any point $x \in X$, there exists by assumption a rational curve $C \subset X$ such that $d(x, C) < \epsilon$. Applying an automorphism g of $\mathbb{P}^N$ such that $d(g, \mathrm{Id}) < \epsilon$, we can thus find a curve $g(C)$ contained in U and passing through x. We then conclude using (iii).

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(iii) For any point $x \in X$, there exists a rational curve $C_x \subset U$ passing through x. Applying to C_x automorphisms of $\mathbb{P}^N$ close to the identity and fixing a point $y \in C_x$, $y \neq x$, we conclude as in (i) that there exists a neighbourhood V_x of x in X such that any point $y \in V_x$ is connected to x by a chain of two rational curves $C_x \cup g(C_x)$ contained in U. By the compactness and connectedness of X, we conclude that any two points of X can be joined by a chain of rational curves contained in U. □

Remark 1.4. The statement in Lemma 1.3 (i) shows that the Kobayashi pseudo-distance $d_{X,K}$ of a subvariety $X \subset \mathbb{P}^N$ may be different from the limit over the open sets $U \subset \mathbb{P}^N$ of the restrictions $d_{U,K}|_X$. Indeed, in the above notation, if one chooses Y to be Kobayashi hyperbolic, then the Kobayashi pseudo-distance of $X = Y \times \mathbb{P}^1$ is non-zero, while the restrictions $d_{U,K}|_X$ are all 0.

The main defect of Definition 1.1 is the fact that it is not stable under surjective morphisms, that is, if $\phi\colon X \to Y$ is surjective and X is approximately rationally connected in the naive sense, Y need not satisfy the same property. Indeed, this follows from Lemma 1.3 (i) and Lemma 1.13. We could try to correct the definition by asking that not only X but also all varieties Y, such that there exists a surjective morphism from X to Y, are approximately rationally connected in the naive sense (say in any projective embedding). However, the following example shows that this is not strong enough.

Example 1.5. Consider the case where $X = (C \times S)/\iota$, where C is a curve of genus greater than or equal to 2 with hyperelliptic involution i, S is a $K3$-surface that is the universal cover $S \to T$ of an Enriques surface, the involution ι acting on $C \times E$ acts as the hyperelliptic involution on C and as the involution over T on S. This involution ι has no fixed points. By Lemma 1.3 (ii), X is approximately rationally connected in any projective embedding, because rational curves are topologically dense in the fibres of $X \to \mathbb{P}^1$. Consider any surjective morphism $X \to Y$, where Y is normal. We claim that Y is approximately rationally connected in the naive sense in any projective embedding. Indeed, if $\dim Y = 1$, one has that $h^{1,0}(Y) = 0$, so $Y = \mathbb{P}^1$. If $\dim Y = 2$, as Y is dominated by $C \times S$, either it is dominated by the $K3$-surface S, or for each $c \in C$ the morphism from $c \times S$ to Y has for image a curve D, and then Y is rationally dominated by a product $C \times \mathbb{P}^1$, since, for any dominating rational map from a $K3$-surface to a smooth curve D, one has that $D \cong \mathbb{P}^1$. In both cases, it is approximately rationally connected in the naive sense in any projective embedding, using Lemma 1.3. The case where $\dim Y = 3$ is worked similarly.

There are several ways to correct the naive definition and we propose two of them.

Definition 1.6. X is *strongly* approximately rationally connected if, for any embedding j of X in a product P of two projective spaces, $j(X)$ is approximately rationally connected inside P in the naive sense (see Definition 1.1).

We now give another variant of Definition 1.1, which might be easier to relate to the vanishing of the Kobayashi pseudo-metric. For any smooth $X \subset \mathbb{P}^N$, the projectivized tangent bundle $\mathbb{P}(T_X)$ is naturally contained in the projectivized tangent bundle $\mathbb{P}(T_{\mathbb{P}^N})$. Any curve $C \subset \mathbb{P}^N$ has a tangent lift $\tilde{C}$ to $\mathbb{P}(T_{\mathbb{P}^N})$. Let $\tilde{U}$ be a neighbourhood of $\mathbb{P}(T_X)$

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in $\mathbb{P}(T_{\mathbb{P}^N})$. We say that a curve $C \subset \mathbb{P}^N$ is $\tilde{U}$-close to X if $\tilde{C} \subset \tilde{U}$. Hence, not only is C close to X, but its tangent space at any point is close to T_X. The following definition takes into account the cohomology classes of the curves considered. Here we use the fact that if U is a tubular neighbourhood of X in $\mathbb{P}^N$, U and X have the same homology. Now let $\tilde{U}$ be a neighbourhood of $\mathbb{P}(T_X)$ in $\mathbb{P}(T_{\mathbb{P}^N})$. Note that if, for any U, $\tilde{U}$ and for any point x of X, there passes a curve $C_x \subset U$ passing through x, which is $\tilde{U}$-close to X, C_x can be chosen to vary locally continuously with x, hence to have a cohomology class $[C] \in H_2(U,\mathbb{Z}) = H_2(X,\mathbb{Z})$ locally independent of x. By considering chains, and by smoothing them, we conclude, using arguments similar to the proof of Lemma 1.3, that, if X is connected, we can assume that the class of the covering curves C_x is in fact independent of x.

Definition 1.7. A connected variety $X \subset \mathbb{P}^N$ is *cohomologically* approximately rationally connected if for any U, $\tilde{U}$, as above, through any point x of X there passes a rational curve C_x contained in U and $\tilde{U}$-close to X, of class $[C]$ independent of x. Furthermore, the convex cone generated by the $(n-1,n-1)$-components of the classes $[C_i] \in H_2(U,\mathbb{Z}) = H_2(X,\mathbb{Z}) = H^{2n-2}(X,\mathbb{Z})$ of such covering curves $C_{i,x}$ contains a strongly positive class.

We say here that a class of type $(n-1,n-1)$ on an n-dimensional variety X is strongly positive if it has a positive intersection with pseudo-effective $(1,1)$-classes (represented by weakly positive currents of type $(1,1)$). When the class belongs to the space $N_1(X)$ generated by curve classes, this is equivalent to being in the interior of the convex cone generated by classes of moving curves (see [**1**]).

Remark 1.8. Since the class of a curve $C \subset U$ is the class of the current of integration over C, this is the class in U of a closed current of type $(N-1,N-1)$, $N = \dim U$. Using a differentiable retraction $\pi\colon U \to X$, we also have the current of integration over $\pi(C)$, whose class is the cohomology class $[C]$ above. This last class is not in general of type $(n-1,n-1)$ (see examples in § 2). However, when $\tilde{U}$ is small, it is close to being of type $(n-1,n-1)$, as C is $\tilde{U}$-close to X.

The cohomological condition in Definition 1.7 addresses the weakness of Definition 1.1; indeed, the rational curves $g(C_y)$ used in the proof of Lemma 1.3 (i) are in the same class as the fibres of pr_1, and this class is not strongly positive.

Remark 1.9. If $H^2(X,\mathbb{Q}) = \mathbb{Q}$, Definitions 1.7 and 1.1 are quite close. Indeed, in this case, the cohomological condition in Definition 1.7 is empty, and we, thus, just ask that, for any neighbourhoods U of X, $\tilde{U}$ of $\mathbb{P}(T_X)$, and any general point $x \in X$, there exists a rational curve in U passing through x and $\tilde{U}$-close to X. The last condition is satisfied by the examples of Lemma 1.3 (i) (but they do not satisfy $H^2(X,\mathbb{Q}) = \mathbb{Q}$).

We believe that these notions should be related to the triviality of the Kobayashi pseudo-distance (see [**15**]) of X, although it is quite hard to establish precise relations. This is due to the notorious difficulty in localizing Ahlfors currents or Brody curves (see [**11, 12, 19**] for important progress on this subject). The motivation for introducing these geometric definitions is the lack of progress on the understanding of complex

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varieties with vanishing Kobayashi pseudo-distance (in contrast with the recent progress made on the Green–Griffiths conjecture, for example, for high-degree hypersurfaces in projective space; see [**10**]).

However, the following easy lemma shows that approximate rational connectedness in either of the above senses is too restrictive topologically.

Lemma 1.10. *Abelian varieties are not approximately rationally connected (in the naive sense). More precisely, if $A \subset \mathbb{P}^N$ is an abelian variety, and U is a tubular neighbourhood of A, U does not contain any rational curve.*

Proof. A and U have the same homotopy type. Hence, as $\pi_2(A) = 0$, we also have that $\pi_2(U) = 0$. Thus, a rational curve contained in U should be homologous to 0 in U, hence in $\mathbb{P}^N$, which is absurd. □

This lemma (and the fact that abelian varieties have trivial Kobayashi pseudo-distance) is the motivation for the following variant of Definitions 1.6, 1.7 and 1.1.

Definition 1.11.

(i) $X \subset \mathbb{P}^N$ is said to be approximately elliptically connected in the naive sense if, for any neighbourhood U of X in $\mathbb{P}^N$, any two points $x, y \in X$ can be joined by a chain of elliptic curves in U.

(ii) $X \subset \mathbb{P}^N$ is said to be strongly approximately elliptically connected if Definition 1.6 holds with rational curves replaced by chains of elliptic curves.

(iii) $X \subset \mathbb{P}^N$ is said to be cohomologically approximately elliptically connected if Definition 1.7 holds with rational curves replaced by elliptic curves.

Remark 1.12. If X is connected and approximately rationally (respectively, elliptically) connected in the cohomological sense, then it is also approximately rationally (respectively, elliptically) connected in the naive sense, as follows from Lemma 1.3 (iii) (or its obvious extension to the elliptic case).

Properties (i), (ii) and (iii) are satisfied by (very general) Calabi–Yau varieties obtained as the double cover of the projective space $\mathbb{P}^n$ ramified along a degree $2n + 2$ hypersurface, as they are covered (in infinitely many different ways) by families of elliptic curves (see [**22**]).

We give, however, in § 2 (see Theorem 2.5), examples of varieties containing only finitely many rational curves, but which are approximately rationally connected in the cohomological sense. Similarly, abelian varieties are approximately cohomologically elliptically connected (see Theorem 2.1), while the general ones do not contain any elliptic curve. From this, one can deduce that the Fano varieties of lines of very general cubic 4-folds satisfy this property, as they are covered in infinitely many different ways by a two-dimensional family of surfaces birationally equivalent to abelian surfaces (see [**24**]).

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Our hope is that approximately elliptically connected varieties in one of the strengthened senses described above are the same as the 'special varieties' invented by Campana [**4**] (which are also conjectured to be the complex projective varieties with trivial Kobayashi pseudo-distance). Note that Demailly in [**9**] gives a description of the Kobayashi pseudo-metric of X involving algebraic curves in X, together with their intrinsic hyperbolic metric. This says that if X has a trivial Kobayashi pseudo-distance, there exist many algebraic curves in X for which the intrinsic hyperbolic metric is small compared with the metric obtained by restricting a given metric on X. In particular, this compares the genus of these curves with their degree, but this does not say anything about the genus alone.

To start with, we have the following easy lemma.

Lemma 1.13. *If a projective variety X is Kobayashi hyperbolic, it is not approximately elliptically connected (in the naive sense) in any projective embedding.*

Proof. Indeed, if there exists an elliptic curve E_n in any neighbourhood $U_{\epsilon_n}(X)$ of X in $\mathbb{P}^N$, with $\lim_{n\to\infty}\epsilon_n = 0$, we can choose for each n a holomorphic map $f_n\colon \Delta \to E_n \to \mathbb{P}^N$ from the unit disc to E_n, such that $|f_n'(0)| = n$, where the modulus of the derivative is computed with respect to the ambient metric. By Brody's lemma [**2**], there exists an entire curve in $\mathbb{P}^N$ obtained as the limit of a subsequence of the f_n conveniently reparametrized. This entire curve is contained in $\bigcap_n U_{\epsilon_n}(X) = X$ and X is not Kobayashi hyperbolic. $\square$

We also prove, in § 3, the following property.

Proposition 1.14 (see Proposition 3.3). *If X is strongly approximately rationally (respectively, elliptically) connected and $\phi\colon X \to Y$ is a surjective morphism, then Y is approximately rationally (respectively, elliptically) connected in the naive sense. In particular, Y is not Kobayashi hyperbolic.*

We do not know whether this result holds for cohomological approximate elliptic or rational connectedness, but we know by Lemmas 1.3 and 1.13 that it does not hold for naive approximate elliptic or rational connectedness.

Next, as the definitions are stable under étale covers (see Lemma 3.1), one crucial point needed in order to make the class of approximately elliptically connected varieties close to Campana's special manifolds is the following.

Conjecture 1.15. *A variety of general type is not approximately elliptically connected in the naive sense.*

As we do not even know that elliptic or rational curves are not topologically dense in a variety of general type (a weak version of the Green–Griffiths–Lang conjecture), an answer to this question seems to be out of reach at the moment. For example, we know that general hypersurfaces in $\mathbb{P}^n$ of degree greater than or equal to $2n-2$ do not contain any rational curve for $n \geqslant 4$ (see [**21**], giving an optimal bound that is slightly better than [**7**]), and that the only rational curves contained in general hypersurfaces in $\mathbb{P}^n$ of degree $2n-3$, for $n \geqslant 6$, are lines (see [**18**]), but general hypersurfaces in $\mathbb{P}^n$

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of degree $n+2 \leqslant d \leqslant 2n-4$ are not known to carry finitely many families of rational curves, and not even known to not contain a dense set covered by rational curves.

In the other direction, we do not know if rational curves in Calabi–Yau hypersurfaces are topologically dense, except in dimension 2, that is, for $K3$-surfaces, for a Baire second category subset of the moduli space by [**5**]. One question implicitly raised in the present paper is whether it is easier to study rational (or elliptic) curves contained in small neighbourhoods of such hypersurfaces.

2. Some examples

We give two examples of classes of varieties that do not contain many rational (respectively, elliptic) curves but are approximately rationally (respectively, elliptically) connected in the cohomological sense.

Theorem 2.1. *Abelian varieties are approximately elliptically connected (for any projective embedding) in the cohomological sense.*

Proof. Let $A \subset \mathbb{P}^N$ and let $\tilde{U} \subset \mathbb{P}(T_{\mathbb{P}^N})$ be a Euclidean neighbourhood of $\mathbb{P}(T_A)$. For a small deformation A_ϵ of A in $\mathbb{P}^N$, $\mathbb{P}(T_{A_\epsilon}) \subset \mathbb{P}(T_{\mathbb{P}^N})$ is a small deformation of $\mathbb{P}(T_A)$, hence is contained in $\tilde{U}$ when the deformation is small enough. It follows that, for any curve $C \subset A_\epsilon$, its tangent lift $\tilde{C}$ is contained in $\tilde{U}$.

We now use the well-known fact that abelian varieties that are isogenous to a product $E_1 \times \cdots \times E_n$, where each E_i is an elliptic curve, are dense (for the Euclidean topology) in the moduli space of n-dimensional polarized abelian varieties. On the other hand, inside $E_1 \times \cdots \times E_n$, the elliptic curves obtained as the images of E_i under the natural morphisms $\phi_i\colon E_i \to E_1 \times \cdots \times E_n$, $x \mapsto (e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n)$ for given points $e_j \in E_j$ can be chosen to pass through any point. Of course, the same is true for any abelian variety isogenous to E^n.

Now let $A_\epsilon \subset \mathbb{P}^N$ be a sufficiently small deformation of A isogenous to $E_1 \times \cdots \times E_n$. The elliptic curves $\phi_i(E_i)$ contained in A_ϵ then sweep out A_ϵ, and their tangent lift is contained in $\tilde{U}$. For any point $x \in A$, we can find an automorphism of $\mathbb{P}^N$ that is close to the identity and such that $x \in g(A_\epsilon)$. Thus, the curves $g(\phi(E))$ can be chosen to pass through any point of A and to have their tangent lift contained in $\tilde{U}$.

Finally, when there is no non-zero morphism between E_i and E_j for $i \neq j$, the classes of the curves $\phi_i(E_i)$ generate the space of the Hodge classes $\mathrm{Hdg}^{2n-2}(A_\epsilon)$. In particular, a convex combination of these classes contains the class $h_{A_\epsilon}^{n-1}$, where $h_{A_\epsilon} = c_1(\mathcal{O}_{A_\epsilon}(1))$. Using the canonical isomorphism $H^{2n-2}(A_\epsilon, \mathbb{R}) \cong H^{2n-2}(A, \mathbb{R})$, we conclude that a convex combination of these classes, transported to A, contains the class h_A^{n-1}, where $h_A = c_1(\mathcal{O}_A(1))$. Taking the $(n-1, n-1)$-component, we also conclude that a convex combination of the $(n-1, n-1)$-components of these classes, transported to A, contains the class h_A^{n-1}, which finishes the proof. □

Remark 2.2. The example of abelian varieties also illustrates why the notion of approximate elliptic (or rational) connectedness might be easier to study than the property of being swept out by entire curves, or of having arbitrarily small neighbourhoods

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in $\mathbb{P}^N$ swept out by entire curves. Indeed, there exist, of course, a lot of entire curves in abelian varieties. However, the elliptic curves E exhibited above, contained in a small deformation of a given abelian variety A in projective space, are much more reasonable, since their induced metric is uniformly equivalent to their flat metric k_E (normalized so the volume is equal to the degree). This is because the flat metric of A is equivalent to the induced metric on A, which easily implies that there exists a flat metric h_{A_ϵ} on A_ϵ equivalent to the induced metric on A_ϵ with constants depending only on A. The restriction $h_{A_\epsilon}|_E$ of this flat metric to E is a flat metric on E, which is then uniformly equivalent to the induced metric $h|_E$. In other words we have $ch|_E \leqslant h_{A_\epsilon}|_E \leqslant Ch|_E$, for some constants c, C depending only on A. Integrating the corresponding Kähler forms over E, we get that

$$c \deg E \leqslant \int_E \omega_{A_\epsilon}|_E \leqslant C \deg E,$$

which tells us, since $\omega_{A_\epsilon}|_E$ is the flat metric, that the normalized metric k_E, which is equal to

$$\frac{\deg E}{\int_E \omega_{A_\epsilon}|_E} h_{A_\epsilon}|_E,$$

satisfies

$$\frac{c}{C} h|_E \leqslant k_E \leqslant \frac{C}{c} h|_E. \tag{2.1}$$

The above arguments suggest the following interesting questions.

Question 2.3.

(i) Does any elliptic curve close enough in the usual topology to an abelian subvariety A of $\mathbb{P}^N$ satisfy (2.1) for some constants depending only on A?

(ii) Does the above question have an affirmative answer for elliptic curves $\tilde{U}$-close to A for a small neighbourhood $\tilde{U}$ of $\mathbb{P}(T_A)$ in $\mathbb{P}(T_{\mathbb{P}^N})$?

An affirmative answer to these questions would have the following consequence.

Proposition 2.4. *Assume that Question 2.3 (i) has an affirmative answer for a given abelian variety $A \subset \mathbb{P}^N$. A subvariety $X \subset A$ that is of general type is then not approximately elliptically connected in the naive sense. If Question 2.3 (ii) has an affirmative answer for $A \subset \mathbb{P}^N$, then a subvariety $X \subset A$ that is of general type is not approximately elliptically connected in the cohomological sense.*

Proof. Indeed, one knows by [**14**] that X satisfies the Green–Griffiths conjecture, so the union of the entire curves contained in X is not Zariski dense in X. On the other hand, assume that, for any $x \in X$, there exists an elliptic curve $E_n \subset \mathbb{P}^N$ passing through x such that $E_n \subset V_{1/n}(X) = \{y \in \mathbb{P}^N,\ zd(y, X) < 1/n\}$. Then consider the flat uniformization $f_n \colon \mathbb{C} \to E_n$ such that $f_n(0) = x_n$ and $f_n^* k_{E_n}$ is the standard metric (so f_n is defined up to the action of $U(1)$). If Question 2.3 (i) has a positive answer, as E_n is close to A, then the derivatives $|f_n'|$ (computed with respect to the ambient metric on $\mathbb{P}^N$) are bounded above and below in modulus, so we can extract a subsequence that converges uniformly

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on compact sets of $\mathbb{C}$ to a non-constant entire curve passing through x and contained in X. As x is arbitrary, this contradicts Kawamata's result. Similarly, if Question 2.3 (ii) has an affirmative answer, elliptic curves $\tilde{U}$-close to A satisfy the above property for $\tilde{U}$ small. This is then also true for elliptic curves $\tilde{U}$-close to X. Hence, by the above argument, one cannot have an elliptic curve $\tilde{U}$-close to X passing through any point of X for $\tilde{U}$ arbitrarily small. □

The second example we consider is the example of elliptic surfaces with finitely many rational curves. More precisely, we consider a very general hypersurface $S \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2l, 3)$ with $l \geqslant 2$.

Theorem 2.5.

(i) *S contains finitely many rational curves, namely, the singular fibres of the elliptic fibration $f := pr_1|_S\colon S \to \mathbb{P}^1$.*

(ii) *S is approximately rationally connected (relative to the Segre embedding) in the cohomological sense.*

Proof. (i) A smooth surface $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 3)$ is a $K3$-surface that contains only countably many rational curves. If Σ is chosen to be very general,

$$\operatorname{Pic} \Sigma = (\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^2))|_\Sigma;$$

hence, no algebraic curve in Σ has degree 1 over $\mathbb{P}^1$. Take such a Σ and consider a very general morphism $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1_0$ of degree l. The surface $S = \Sigma \times_{\mathbb{P}^1_0} \mathbb{P}^1$ is then of bidegree $(2l, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, and, for each rational curve $C \subset \Sigma$ not contained in a fibre of $\Sigma \to \mathbb{P}^1_0$, the curve $C \times_{\mathbb{P}^1_0} \mathbb{P}^1 \subset S$ is irreducible of positive geometric genus. Hence, S does not contain any rational curves beyond those contained in a fibre of $pr_1\colon S \to \mathbb{P}^1$.

(ii) First of all, we apply the criterion for density of the Noether–Lefschetz locus due to Green (see [**23**, Proposition 5.20]) to show that arbitrarily small deformations $S_\epsilon \subset \mathbb{P}^1 \times \mathbb{P}^2$ of a general surface S, as above, admit sections of $f_\epsilon\colon S_\epsilon \to \mathbb{P}^1$.

We recall the statement of the criterion in the form that we use here: consider the universal family $\pi\colon \mathcal{S} \to B$, $\mathcal{S} \subset B \times \mathbb{P}^1 \times \mathbb{P}^2$, of such smooth surfaces. Let $0 \in B$ and let S_0 be the fibre $\pi^{-1}(0)$. We then have the following.

Proposition 2.6. *Assume that there exists a $\lambda \in H^1(S_0, \Omega_{S_0})$ such that the map $\bar{\nabla}_\lambda\colon T_{B,0} \to H^2(S_0, \mathcal{O}_{S_0})$ is surjective. For any open set $U \subset B$ (for the Euclidean topology), the set of classes $\alpha \in H^2(S_0, \mathbb{Z})$ that become algebraic on some fibre $\mathcal{S}_t$ for some $t \in U$ then contains the set of integral points in a non-empty open cone of $H^2(S_0, \mathbb{R})$.*

In this statement, the map $\bar{\nabla}_\lambda$ is the composition of the Kodaira–Spencer map $T_{B,0} \to H^1(S_0, T_{S_0})$ and the cup-product/contraction map $\lambda\lrcorner\colon H^1(S_0, T_{S_0}) \to H^2(S_0, \mathcal{O}_{S_0})$.

Using the Griffiths description of the infinitesimal variations of Hodge structure (IVHS) of hypersurfaces (see [**23**, 6.2.1]), we find that this map identifies to the multiplication

$$\mu_{P_\lambda}\colon R_{2l,3}(S_0) \to R_{6l-2,6}(S_0)$$

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by a certain polynomial $P_\lambda \in H^0(S_0, \mathcal{O}_{S_0}(4l-2,3))$, where $R(S_0)$ is the Jacobian ring of the defining equation of S_0. One checks explicitly that, for generic S_0 and generic P_λ, the map μ_{P_λ} is surjective.

Using Proposition 2.6, we conclude that, for generic S_0 and for any small simply connected neighbourhood U of 0 in B, there exists a non-empty open cone C in $H^2(S_0,\mathbb{R})$ such that any integral class $\alpha \in H^2(S_0,\mathbb{Z}) \cap C$ becomes (by parallel transport) algebraic on some fibre S_t, for some $t \in U$.

Now let $F \in H^2(S_0,\mathbb{Z})$ be the class of a fibre of f_0. Its parallel transport to S_t is then the class F_t of a fibre of f_t, and the elliptic fibration $f_t\colon S_t \to \mathbb{P}^1$ admits a section if and only if there exists an algebraic class $\alpha \in H^2(S_t,\mathbb{Z}) \cong H^2(S_0,\mathbb{Z})$ such that $\langle \alpha, F\rangle = 1$. Note, furthermore, that the class $h := c_1(pr_2^*\mathcal{O}_{\mathbb{P}^2}(1))$, which is of degree 3 on the class F, remains algebraic on any deformation S_t of S_0 in $\mathbb{P}^1 \times \mathbb{P}^2$. Hence, f_t has a section if and only if there exists an algebraic class $\alpha \in H^2(S_t,\mathbb{Z}) \cong H^2(S_0,\mathbb{Z})$ such that $\langle \alpha, F\rangle = 1 \mod 3$. To conclude that the set of surfaces S_t having a section is dense, we then use the following lemma (which is used implicitly in [**20**, Remark 1]).

Lemma 2.7. *For any non-empty open cone $C \subset H^2(S_0,\mathbb{R})$, there exist elements in $C \cap \{\alpha \in H^2(S_0,\mathbb{Z}),\ \langle \alpha, F\rangle = 1 \mod 3\}$.*

The second and final step of the proof is the following lemma, due to Chen and Lewis [**5**].

Lemma 2.8. *Let $f\colon S \to \mathbb{P}^1$ be an elliptic fibration, and let L be a line bundle on S_t, of degree $d \neq 0$ on fibres. Assume that the fibres of f are irreducible and reduced and that the monodromy group $\operatorname{Im} \pi_1(\mathbb{P}^1_{\mathrm{reg}}, t_0) \to \operatorname{Aut} H^1(S_{t_0},\mathbb{Z})$ is the full symplectic group $\mathrm{SL}(2,\mathbb{Z})$. For any section $\sigma\colon \mathbb{P}^1 \to S$ of f such that the class $d[\sigma] - c_1(L)$ is not of torsion, the curves $C_n := \sigma_n(\mathbb{P}^1)$ then have the property that $\bigcup_n C_n$ is dense for the Euclidean topology in S.*

Here $\sigma_n := \mu_n \circ \sigma$, where $\mu_n\colon S \dashrightarrow S$ is the self-rational map that associates the point y of the fibre S_u to $x \in S$ with $f(x) = u \in \mathbb{P}^1$ such that $(dn+1)x - nL|_{S_u} = y$ in $\operatorname{Pic} S_u$.

The proof of Theorem 2.5 is now concluded as follows. Let S_0 be generic. Let U be a neighbourhood of S_0 in $\mathbb{P}^5$ and let $\tilde{U}$ be a neighbourhood of $\mathbb{P}(T_{S_0})$ in $\mathbb{P}(T_{\mathbb{P}^5})$. As S_0 is generic, the fibration $f_0\colon S_0 \to \mathbb{P}^1$ is a Lefschetz fibration with irreducible fibres and with monodromy equal to the full symplectic group. These properties remain true for a small deformation of S_0.

By Lemma 2.7, there exists a surface S_t that is a small deformation of S_0 in $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ such that $f_t\colon S_t \to \mathbb{P}^1$ has a section C. In particular, S_t is contained in U, and $\mathbb{P}(T_{S_t})$ is contained in $\tilde{U}$. Thus, any curve contained in S_t is $\tilde{U}$-close to S_0. By Lemma 2.8, the union $\bigcup_n C_n$ is dense in S_t. Applying automorphisms of $\mathbb{P}^5$ close to the identity, if needed, we conclude that, for any point $x \in S_0$, there exists an arbitrarily small deformation S_t of S_0 in $\mathbb{P}^5$ containing a section C_n passing through x. These curves are contained in U and are $\tilde{U}$-close to S_0.

To conclude, it only remains to check that a convex combination of the $(1,1)$-components of the classes of (a variant of) the curves C_n (transported to S_0) contains an ample

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class on S_0. For this we observe that the sum $2[C_n]+[C_{-2n-1}]\in H^2(S_t,\mathbb{Z})=H^2(S_0,\mathbb{Z})$ is a combination of the class h and the class F. The coefficient in h is obviously positive. This class may not be ample, but we observe that the class F is, in S_0, the class of a rational curve (namely, a singular fibre). Instead of the curves C_n, we can thus consider the curves C'_n obtained by smoothing, in $\mathbb{P}^5$, the union of C_n and a covering of large degree of a singular fibre. The resulting curves can be chosen to stay $\tilde{U}$-close to S_0, and the sum $2[C'_n]+[C'_{-2n-1}]\in H^2(S_t,\mathbb{Z})=H^2(S_0,\mathbb{Z})$ is an ample class. □

These two examples might give one the feeling that the natural way to produce elliptic or rational curves in an (arbitrarily) small neighbourhood of a subvariety $X\subset\mathbb{P}^N$ is by studying elliptic or rational curves lying in some small deformation X_ϵ of X in $\mathbb{P}^N$. This is, however, not true at all, as the following example, obtained by mimicking the trick of [**26**], shows. Start with an abelian variety $A\cong\mathbb{C}^n/\Gamma$ admitting an endomorphism ϕ such that the corresponding endomorphism $\phi_{\mathbb{Q}}$ of $\Gamma_{\mathbb{Q}}=H_1(A,\mathbb{Q})$ has only eigenvalues of multiplicity 1. It is then immediate to see that the pair (A,ϕ) is rigid. Furthermore, we can construct this ϕ such that A does not contain any elliptic curve (one considers, for example, simple abelian varieties with complex multiplication).

Starting from such a pair (A,ϕ), we consider the projective variety X obtained by successively blowing up $A\times A\times\mathbb{P}^1$ along $A\times x_0\times t_1$, $x_1\times A\times t_2$, $\operatorname{diag}A\times t_3$, $\operatorname{graph}\phi\times t_4$, for generic choices of x_0, x_1 and distinct points $t_1,\dots,t_4\in\mathbb{P}^1$. Choose a projective embedding of X in $\mathbb{P}^N$. As A does not contain elliptic or rational curves, the only rational or elliptic curves contained in X are contained in the union D of the exceptional divisors of the blow-ups or in proper transforms of the fibres of the map $p_{A\times A}\circ\tau\colon X\to A\times A$, where $\tau\colon X\to A\times A\times\mathbb{P}^1$ is the blow-up map. Furthermore, the deformations of X preserve the exceptional divisors, hence are all of the same type as X, and, as the pair (A,ϕ) is rigid, it follows that elliptic or rational curves contained in a small deformation X_ϵ of X in $\mathbb{P}^N$ are close (for the usual topology) to either a curve contained in D_ϵ, or to a fibre of the map $p_{A\times A}\circ\tau\colon X\to A\times A$. Hence, for a general point x of X, elliptic curves passing through x and contained in a small deformation X_ϵ of X have for homology class a multiple of the class of a fibre of $p_{A\times A}\circ\tau$. As this class is not strongly positive, we cannot use such curves to prove that X is approximately elliptically connected in the cohomological sense.

We have, however, the following result.

Lemma 2.9. *X is approximately elliptically connected (for any projective embedding $X\subset\mathbb{P}^N$) in the cohomological sense.*

Proof. Indeed, recall that $\tau\colon X\to A\times A\times\mathbb{P}^1$ is the blow-up map. Choose a neighbourhood $\tilde{U}$ of $\mathbb{P}(T_X)$ in $\mathbb{P}(T_{\mathbb{P}^N})$ and let $x\in X$. Let $\tau(x)=(y,t)$ with $y\in A\times A$ and $t\in\mathbb{P}^1$. We choose x such that $t\notin\{t_1,\dots,t_4\}$, so, in particular, x does not belong to the exceptional divisor of τ, and there exists a copy $A\times A\times t\subset X$ passing through x. By Theorem 2.1, $A\times A\times t$ is then approximately elliptically connected in $\mathbb{P}^N$ in the infinitesimal and cohomological sense. Thus, there exists an elliptic curve E_x passing through x, whose tangent lift is contained in $\tilde{U}$. Furthermore, the class of these elliptic curves can be chosen to be independent of $x\in X$, and a convex combination of them generates an

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ample class on $A \times A \times t \subset X$. On the other hand, assume now that the line $y \times \mathbb{P}^1$ does not meet the locus of $A \times A \times \mathbb{P}^1$ blown up under τ. This line is then a rational curve C_x contained in X and passing through x. In $\mathbb{P}^N$, we can smooth the curve $E_x \bigcup_x C_x$ and it is easy to see that the smoothed curve can be chosen to pass through x and to stay $\tilde{U}$-close to X. This proves the result since a convex combination of the classes $[C_x] + [E_x]$ is strongly positive. □

3. Stability results and further questions

We start with the following results concerning stability under étale covers. Here, P is any smooth complex projective variety.

Proposition 3.1. *Assume that $X \subset P$ is connected and approximately rationally or elliptically connected in the naive (respectively, cohomological) sense. Let $U \subset P$ be a neighbourhood of X that has the same homotopy type as X (e.g. a tubular neighbourhood). Any étale connected proper cover $X' \to X$ is then approximately rationally connected in the corresponding neighbourhood $f\colon U' \to U$ of U, in the naive (respectively, cohomological) sense.*

Proof. We give the proof for the rational case, the elliptic case being similar, due to the fact that étale covers of elliptic curves are again elliptic curves. We first consider approximate connectedness in the naive sense. Any small neighbourhood $V_\epsilon(X')$ of X' in U' contains a neighbourhood of the form $f^{-1}(U_{\epsilon'}(X))$ for some ϵ'. Now let $x', y' \in X'$ and let x, y be their images in X. There exists a smooth rational curve C contained in $U_{\epsilon'}(X)$ (we assume here that $\dim U \geqslant 3$, since the case where $\dim X = 1$ is completely understood by Lemma 1.13) and containing x and y. The inverse image of C in $f^{-1}(U_{\epsilon'}(X))$ is a finite union of rational curves, and one of them, say $C_{x'}$, passes through x'. As $C_{x'}$ maps onto C, it contains one point y'' of X' over y. In conclusion, the set of points $y'' \in X'$ that are joined to x' by a rational curve in $V_\epsilon(X')$ contains an open subset $W_{x'} \subset X'$ that maps onto X. For any point $z \in W_{x'}$, the open set W_z must be equal to $W_{x'}$, since a point in W_z is joined to x' by a chain of two rational curves passing through z, and this chain can be smoothed. We may assume that the cover $f\colon X' \to X$ is Galois. Let $g \in \mathrm{Gal}(X'/X)$. Then $gW_{x'} = W_{gx'}$, and, by the above, we conclude that X' is the finite disjoint union of open sets of the form $W_{x'}$. As X' is connected, it follows that $X' = W_{x'}$.

For approximate rational connectedness in the cohomological sense, we have to add the following argument. Recall from Definition 1.7 that we need to have, for any tubular neighbourhood V' of X', any neighbourhood $\tilde{V}'$ of $\mathbb{P}(T_{X'})$ in $\mathbb{P}(T_{U'})$ and any point $x' \in X'$, a finite number of rational curves $C_{i,x'}$ passing through x', contained in V', $\tilde{V}'$-close to X', such that the class of the curve $C_{i,x'}$ does not depend on x' and the $(n-1, n-1)$-part of the class $\sum_i n_i [C_{i,x'}] \in H_2(V', \mathbb{Z}) = H_2(X', \mathbb{Z})$ is strongly positive for some $n_i > 0$. Of course, we may assume that V' and $\tilde{V}'$ are inverse images under f of similar neighbourhoods V, $\tilde{V}$ for $X \subset U$. If we start from such data $C_{i,x}$, for $x \in X$ and for the neighbourhoods V, $\tilde{V}$, we observe now that the class in $H_2(X', \mathbb{Z})$ of the unique lift $C_{i,x'}$ of $C_{i,x}$ passing through x' does not depend on x', because it can be

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chosen to vary continuously with x', and X' is connected. In particular, we find that the sum $\sum_{g\in \mathrm{Gal}(X'/X)} g_*[C_{i,x'}]$ is equal to card $G[C_{i,x'}]$ and it is the pullback under f of the class $[C_{i,x}] \in H_2(X,\mathbb{Z})$. The fact that there exist such $C_{i,x'}$ with $\sum_i n_i[C_{i,x'}]^{n-1,n-1} \in H_2(V',\mathbb{R}) = H_2(X',\mathbb{R})$ strongly positive is, thus, equivalent to the fact that there exist such $C_{i,x}$ with $\sum_i n_i[C_{i,x}] \in H_2(V,\mathbb{R}) = H_2(X,\mathbb{R})$ strongly positive. □

The following consequence of Proposition 3.1 illustrates the power of the cohomological condition in Definition 1.11.

Corollary 3.2. *The varieties X in Example 1.5 are not approximately elliptically connected in the cohomological sense in any projective embedding.*

Proof. Recall that X is a quotient of a product $S \times C$ by a free involution ι, where $g(C) \geqslant 2$. If it was approximately elliptically connected in the cohomological sense in some projective embedding, by Proposition 3.1, the product $S \times C$ would be approximately rationally connected in the cohomological sense in some embedding. In particular, there would exist elliptic curves E_i contained in a tubular neighbourhood U of $S \times C$, with the property that some combination of the $(2,2)$-component of the classes $[E_i] \in H_2(U,\mathbb{Z}) = H_2(S\times C,\mathbb{Z}) = H^4(S\times C,\mathbb{Z})$ be strongly positive. But, for any continuous map ϕ from an elliptic curve E to a genus greater than or equal to 2 curve C, the induced map $\phi_*\colon H_2(E_i) \to H_2(C)$ is trivial. Thus, the classes $pr_{2*}[E_i]$ vanish in $H_2(C,\mathbb{Z})$, and, for any line bundle L of positive degree on C, $\langle pr_2^* c_1(L), [E_i]\rangle = 0$. Thus, $\sum_i n_i \langle pr_2^* c_1(L), [E_i]^{2,2}\rangle = 0$ for any n_i, which provides a contradiction. □

Concerning the stability under morphism, we have the following easy result.

Proposition 3.3. *Let $\phi\colon X \to Y$ be a surjective morphism, where X and Y are smooth projective and* $\dim Y > 0$. *If X is strongly approximately rationally (respectively, elliptically) connected, Y is approximately rationally (respectively, elliptically) connected in the naive sense in any projective embedding.*

Proof. Let $j_Y\colon Y \hookrightarrow \mathbb{P}^N$ be a projective embedding. Choose a projective embedding j_X of X in some projective space $\mathbb{P}^M$, and consider the embedding

$$j'_X = (j_X, j_Y \circ \phi)\colon X \hookrightarrow P = \mathbb{P}^M \times \mathbb{P}^N.$$

By assumption, $j'_X(X)$ is approximately rationally (respectively, elliptically) connected in the naive sense in P. The morphism $pr_2\colon P \to \mathbb{P}^N$ sends rational curves (respectively, a chain of elliptic curves) passing through any two given points of $j'_X(X)$ and contained in a sufficiently small neighbourhood of $j'_X(X)$ to rational curves (respectively, a chain of elliptic curves) passing through any two given points of $j_Y(Y)$ and contained in a given neighbourhood of Y in $\mathbb{P}^N$. □

Corollary 3.4. *A fibration $X \to Y$ over a Kobayashi hyperbolic variety Y is not strongly approximately elliptically connected.*

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Proof. Indeed, if it were strongly approximately elliptically connected, the variety Y would be approximately elliptically connected, hence, in particular, not Kobayashi hyperbolic by Lemma 1.13. This gives a contradiction. □

As we mentioned in § 1, Proposition 3.3 is not true for naive approximate rational or elliptic connectedness. This implies a negative answer to the following question.

Question 3.5. Let $Z \subset \mathbb{P}^N$ be the Segre embedding of $\mathbb{P}^k \times \mathbb{P}^l$ for some integers k, l. Fix a distance d on $\mathbb{P}^N$. Is it true that, for any $\epsilon > 0$, there exists $\eta(\epsilon) > 0$ such that $\lim_{\epsilon\to 0}\eta(\epsilon) = 0$, and that, for any rational (respectively, elliptic) curve C contained in $U_\epsilon(Z)$, there exists a rational (respectively, elliptic) curve $C' \subset Z$ such that $d(C, C') := \mathrm{Sup}_{c\in C,\, c'\in C'}\{d(c, C'), d(c', C)\} \leqslant \eta$?

If we look at the proof of Lemma 1.3 (i), a counter-example is obtained by constructing, in a small neighbourhood of the union of a large number of lines $l_i = x_i \times \mathbb{P}^1$, $i = 1, \ldots, M$, contained in Z, with $d(l_i, l_{i+1}) < \epsilon$, a chain $\bigcup_i l_{i,\epsilon}$ of rational curves in $\mathbb{P}^N$ obtained by deforming the l_i in such a way that $l_{i,\epsilon}$ meets $l_{i+1,\epsilon}$, and then by smoothing the resulting chain to a rational curve C. If the diameter of the set $\{x_i,\ i = 1, \ldots, M\}$ is large, and the points x_i are taken in a Kobayashi hyperbolic subvariety $Y \subset \mathbb{P}^k$, the distance $d(C, C')$ between C and any elliptic or rational curve C' in Z is bounded below by a positive constant.

3.1. Further questions and remarks

The first obvious question is the following.

Question 3.6. Let $X \subset \mathbb{P}^N$ be approximately elliptically connected (in the strong or cohomological sense). Is the Kobayashi pseudo-distance of X trivial?

As our motivation was to understand the class of varieties with trivial Kobayashi pseudo-distance, which includes conjecturally Calabi–Yau manifolds (see [**15**]), it is also natural to ask the following.

Question 3.7. Let X be a Calabi–Yau manifold. Is X approximately rationally or elliptically connected (in any of the senses introduced in this paper)?

Another important question is the following.

Question 3.8. Is the property of cohomological approximate rational or elliptic connectedness independent of the choice of projective embedding?

The following question is related to the work of Graber *et al.* [**13**].

Question 3.9. Let $\phi\colon X \to Y$ be a surjective morphism. Assume that the fibres of ϕ are rationally connected (see [**16**]) and that Y is approximately rationally (respectively, elliptically) connected in the strong or cohomological sense. Is X then approximately rationally (respectively, elliptically) connected in the same sense?

We give one result in this direction. Let $P = \mathbb{P}^n$ and let $Q = \mathbb{P}(H^0(P, \mathcal{O}_P(d)))$, where $d \leqslant n$ and $n \geqslant 2$ are such that the general hypersurface of degree d in P is Fano of

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dimension greater than or equal to 1. There exists a universal subvariety $Z \subset Q \times P$, defined by the tautological equation $F_d \in H^0(\mathcal{O}_{Q\times P}(1,d))$. Via the second projection, Z is a fibration in projective spaces over P.

Proposition 3.10. *If $Y \subset Q$ is rationally (respectively, elliptically) connected in the cohomological sense, and $X := Y \times_Q Z \to Y$ is smooth of the expected dimension (hence, the generic fibre of $X \to Y$ is a smooth hypersurface in P), X is approximately rationally (respectively, elliptically) connected in the cohomological sense in Z, hence in $Q \times P$.*

Proof. Let $V \subset Z$ be a tubular neighbourhood of X and let $\tilde{V} \subset \mathbb{P}(T_Z)$ be a neighbourhood of $\mathbb{P}(T_X)$. There exist neighbourhoods $U \subset Q$ of Y and $\tilde{U} \subset \mathbb{P}(T_Q)$ of $\mathbb{P}(T_Y)$ such that V contains $\pi^{-1}(U)$ and $\tilde{V}$ is contained in $\pi_*^{-1}(\tilde{U})$, where $\pi := pr_1|_Z : Z \to Q$. As $Y \subset Q$ is approximately rationally (respectively, elliptically) connected in the cohomological sense, there exists a curve E that is rational (respectively, elliptic), contained in U and passing through any point y of Y. Furthermore, E can be chosen to be $\tilde{U}$-close to Y, of class independent of y, and, finally, a convex combination of the $(n-1, n-1)$-part of these classes contains a strongly positive class in Y, where $n = \dim Y$. Moving E if needed, we may assume that Z_E is smooth with irreducible fibres and $Z_E \to E$ is a smooth Fano complete intersection over the generic point of E. By the Tsen–Lang theorem, the family $Z_E \to E$ has a section. Results of [**16**] even show that such a section $\tilde{E}$ can be chosen to have an arbitrarily positive class in Z_E. These sections produce elliptic curves $\tilde{E} \subset V$ that are then $\tilde{V}$-close to X, and pass through the general point of X. Finally, under our assumptions (and because we may assume that $\dim X \geqslant 3$, since otherwise the result is obvious), the Lefschetz hyperplane section theorem states that $H^2(X,\mathbb{Z}) = H^2(Y,\mathbb{Z}) \oplus H^2(P,\mathbb{Z})$. It is then immediate to conclude that if the curves $\tilde{E}$ have an ample class in Z_E, and a convex combination of the $(n-1, n-1)$-components of the pushforward of their classes in $H_2(U,\mathbb{Z}) = H_2(Y,\mathbb{Z}) = H^{2n-2}(Y,\mathbb{Z})$ contains a strongly positive class, then a convex combination of the $(m-1, m-1)$-components of their classes in $H_2(V,\mathbb{Z}) = H_2(X,\mathbb{Z}) = H^{2m-2}(X,\mathbb{Z})$ contains a strongly positive class. □

Remark 3.11. The analogous result, if one only assumes that the fibres of $X \to Y$ are approximately elliptically or even rationally connected in the strong sense, is not true. Indeed, consider Example 1.5, where $X = (C \times S)/\iota$, and $Y = \mathbb{P}^1$, where C is a curve of genus greater than or equal to 2 with hyperelliptic involution i, S is a $K3$-surface that is the universal cover $S \to T$ of an Enriques surface. The morphism $\phi\colon X \to Y$ is induced by passing to the quotient from the projection $p_2\colon C \times S \to C$ using the isomorphisms $X \cong (C \times S)/\iota$, $C/i \cong \mathbb{P}^1$ and the equivariance of p_2. The fibres of ϕ are isomorphic to S or to T, hence are strongly approximately rationally connected. However, X is not strongly approximately rationally or elliptically connected by Corollary 3.2.

To finish, we pose the following questions.

Question 3.12 (Campana). Assume that X is approximately rationally connected (in the adequate sense). Is $\pi_1(X)$ finite?

The following similar question is very much related to the results of [**3**].

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Question 3.13 (Campana). Assume that X is strongly approximately elliptically connected (in the adequate sense). Is $\pi_1(X)$ virtually abelian?

The two questions (for cohomological approximate connectedness) are related as follows.

Proposition 3.14. *Suppose that Question 3.13 has a positive answer for cohomological approximate connectedness; Question 3.12 then also has a positive answer for cohomological approximate connectedness.*

Proof. Let $X \subset \mathbb{P}^N$ be approximately rationally connected in the cohomological sense. We know, assuming that Question 3.13 has a positive answer, that $\pi_1(X)$ is virtually abelian. Passing to an étale cover of X, and using Lemma 3.1, we may assume that X is approximately rationally connected in the cohomological sense in an adequate variety U, and, furthermore, has torsion-free abelian π_1. We want to prove that $\pi_1(X)$ is trivial. Equivalently, if $a_X : X \to \operatorname{Alb} X$ is the Albanese map, letting $Y := a_X(X) \subset \operatorname{Alb} X$, one wants to prove that Y is a point. Assume the contrary. Then, choosing an ample line bundle on $\operatorname{Alb} X$, its pullback $a_X^* L$ to X is a semi-positive line bundle that is not numerically trivial. Consider now the rational curves C in a tubular neighbourhood U of X in projective space. Their class $[C] \in H_2(U, \mathbb{Z}) \cong H_2(X, \mathbb{Z})$ then factors through $\pi_2(X)$, hence vanishes in $H_2(\operatorname{Alb} X)$ under the map a_{X*}. Hence, we conclude that $\langle [C], \phi^* c_1(a_X^* L)\rangle = 0$.

This contradicts the fact that X is approximately rationally connected in the cohomological sense, because the latter implies, in particular, the existence of rational curves C_i in any small neighbourhood of X in U, with the property that the class $\sum_i n_i [C_i]^{n-1,n-1} \in H^{n-1,n-1}(X)$ is strongly positive, so

$$\sum_i n_i \langle [C_i], \phi^* c_1(a_X^* L)\rangle \neq 0.$$

□

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