METACYCLIC INVARIANTS OF KNOTS AND LINKS

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To each representation ρ on a transitive permutation group P of the group $G = \pi(S - k)$ of an (ordered and oriented) link $k = k_1 \cup k_2 \cup \ldots \cup k_\mu$ in the oriented 3-sphere S there is associated an oriented open 3-manifold $M = M_\rho(k)$, the covering space of $S - k$ that belongs to ρ. The points $o_1, o_2, \ldots$ that lie over the base point o may be indexed in such a way that the elements g of G into which the paths from o_i to o_j project are represented by the permutations g^ρ of the form $(\cdots {}^{i}_{j} \cdots)$, and this property characterizes M. Of course M does not depend on the actual indices assigned to the points $o_1, o_2, \ldots$ but only on the equivalence class of ρ, where two representations ρ of G onto P and ρ' of G onto P' are *equivalent* when there is an inner automorphism θ of some symmetric group in which both P and P' are contained which is such that $\rho' = \theta\rho$.

The covering $M \to S - k$ has a unique completion $\bar{M} \to S$, called the *associated branched covering* [**7**]. The part of $\bar{M}$ that lies over the simple closed curve k_α is the union of a finite or infinite sequence of mutually disjoint sets $k_{\alpha 1}, k_{\alpha 2}, \ldots$ called *branch curves*, each of which is either a simple closed curve or an open arc. If $k_{\alpha\beta}$ has a meridian, it may be obtained by lifting a suitable power of a meridian of k_α.

An orientation-preserving autohomeomorphism φ of S that maps one (ordered and oriented) link k onto another one, k', induces a biunique correspondence $\rho \leftrightarrow \rho'$ between the representations ρ of G and the representations ρ' of G' which is such that, for each corresponding pair, φ lifts to an orientation-preserving homeomorphism $\tilde{\varphi}$ of $\bar{M}_\rho(k)$ onto $\bar{M}_{\rho'}(k')$; this homeomorphism $\tilde{\varphi}$, which is unique up to covering translations, maps branch curves into branch curves. Thus the collection of spaces $M_\rho(k)$ and $\bar{M}_\rho(k)$ that belong to the representations ρ of a distinguishable family, for example the representations ρ of G onto a given permutation group P, is an invariant of the *type* κ of the link k. Such invariants may be called *P-invariants*; thus we have *cyclic invariants* (when P is cyclic), *metacyclic invariants* (when P is metacyclic), *abelian invariants* (when P is abelian), *metabelian invariants* (when P is metabelian), etc. Since the Alexander matrix $A(t_1, \ldots, t_\mu)$ is an invariant of the universal abelian covering $M(\rho) \to S - k$ that belongs to the representation of G onto the free abelian group of rank μ, all the various knot-theoretical invariants that can be derived from the Alexander matrix (e.g. the elementary ideals $E_d(t_1, \ldots, t_\mu)$ and polynomials $\Delta_d(t_1, \ldots, t_\mu)$,

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$d = 1, 2, \ldots$) are *abelian invariants*. (In particular, when $\mu = 1$, the free abelian group of rank μ is cyclic; hence $A(t)$, $E_d(t)$, and $\Delta_d(t)$, etc., are also *cyclic invariants*.)

When κ is tame and P is finite, $\bar{M}$ is an oriented closed 3-manifold, and the linking invariants of $\bar{M}$ are then defined, and they are therefore P-invariants of κ. (When $\bar{M}$ is not a closed manifold, which happens whenever P is not finite, it does not seem to contain any information not already contained in M.)

The reason that so much of the literature is about the cyclic invariants of knots (and the abelian invariants of links) is their relative tractibility. It is simply that the non-abelian invariants are not as well-behaved, and hence not very much has been done with them. However, I have found that it is possible to deal with the metabelian invariants to a certain extent, and I will try to indicate here some of the results that have been obtained about them. In order to present these results in their sharpest form, I shall restrict myself to *metacyclic invariants*, where by "metacyclic" I mean what is often called "K-metacyclic" [**4**, p. 11]. This restriction involves no loss of ideas.

1. The metacyclic representations of G. The K-metacyclic group Γ_p is generated by the two cycles $\omega = (1\,2\ldots p)$, where p is an odd prime, and $\xi = (1\; q^{p-2}\; q^{p-3} \ldots q)$, where q is a primitive root of p. Note that the group Γ_p does not depend on q although, of course, the generator ξ does. A presentation of Γ_p is

$$(\omega, \xi \colon \omega^p = 1,\ \xi^{p-1} = 1,\ \xi\omega = \omega^q\xi);$$

a normal form for the elements of Γ_p is $\omega^a\xi^b$, $0 \leqq a < p$, $0 \leqq b < p - 1$.

It will always be assumed that κ is tame. Let

$$(x_1, \ldots, x_n : r_1 = 1, \ldots, r_{n-1} = 1)$$

be a Wirtinger presentation of G, and denote by $e(j)$ the component of k of which x_j is a meridian, $1 \leqq e(j) \leqq \mu$. If $e(j_1) = e(j_2)$, then $x_{j_1}{}^\rho$ and $x_{j_2}{}^\rho$ must be conjugates in Γ_p. Hence a representation ρ of G into Γ_p is determined by an assignment

$$x_j \xrightarrow{\rho} \omega^{a_j}\xi^{b_{e(j)}}, \qquad j = 1, \ldots, n,$$

provided that $r_i{}^\rho = 1$ for $i = 1, \ldots, n - 1$. This condition can be written in the form

$$(*) \qquad \sum_{j=1}^{n} r_{ij}(q^{b_1}, \ldots, q^{b_\mu})a_j \equiv 0 \pmod{p}, \qquad i = 1, \ldots, n-1,$$

where $||r_{ij}(t_1, \ldots, t_\mu)||$ is the Alexander matrix $A(t_1, \ldots, t_\mu)$. For each μ-tuple $b = (b_1, \ldots, b_\mu)$ let the rank of the matrix $A(q_1{}^{b_1}, \ldots, q_\mu{}^{b_\mu})$ be $n - 1 - d_b$; the number of solutions of the homogeneous system of congruences $(*)$ is then p^{1+d_b}. However, p of these solutions (those for which the ratio $(q - 1)a_j : q^{b_{e(j)}} - 1$ is independent of j) yield representations onto proper subgroups of Γ_p. (These are cyclic representations.) Thus there are $\sum_b p(p^{d_b} - 1)$ representations of G on Γ_p. But each equivalence class of representations on Γ_p contains $p(p-1)$ different representations. This may be summarized by means of the following theorem.

THEOREM 1. *G can be represented on Γ_p if and only if the odd prime p divides $\Delta(q^{b_1}, \ldots, q^{b_\mu})$ for some b. The number of inequivalent representations is equal to*

$$\sum_b \frac{p^{d(b)} - 1}{p - 1},$$

where $d(b)$ is the largest integer d such that p divides $E_d(q^{b_1}, \ldots, q^{b_\mu})$.

In practice, the congruence $(*)$ is most conveniently solved by a simple trial and error procedure carried out on a projection of k. For example, for $p = 3$, one tries to mark on each overpass one of the integers 1, 2, 3 (stand-ins for the transpositions (2 3), (1 3), and (1 2), respectively) in such a way that

(1) all three integers are used, and

(2) at each crossing the three associated integers are either all different or all the same.

An example is shown in Figure 1. A pleasing effect can be obtained by replacing the labels 1, 2 and 3 by, say, the colours red, yellow, and blue.

The 84 prime knots of at most nine crossings are given in the standard knot table [**14**, pp. 70–72], and 167 presumably prime knots of ten crossings and 608 presumably prime, alternating knots of eleven crossings have been listed by Tait and Little [**17**; **10**], although recent rechecking indicates that these latter lists are incomplete. The polynomials of these 608 knots were machine-calculated by Anger [**1**] and M. Syverson [unpublished work done at Princeton (cf. [**8**, p. 127])], and a list of the 294 representations of the 213 of these that have representations on Γ_3 was compiled by Perko [**13**]. The two quadruplets and the one quintuplet were studied by D. A. Gay and T. B. Stoel, Princeton Senior Theses, 1961, 1962, but only partial resolution was achieved.

2. The coverings of companions. Let $\bar{L}_1$ be a (tame) solid torus in S that contains k in its interior, and let $L_1 = \bar{L}_1 - k$. Denote by λ the type of a core l of $\bar{L}_1$. The closure L_2 of the complement of $\bar{L}_1$ is a compact submanifold of S that intersects $\bar{L}_1$ in their common boundary L_0. If k is not a core of $\bar{L}_1$ and is not contained in any 3-cell subset of $\bar{L}_1$, then k is called a *companion* [**15**] of λ. Assuming that the base point o lies in L_0, let us consider the fundamental groups of L_1, L_2, $L_0 = L_1 \cap L_2$ and $L = L_1 \cup L_2 = S - k$.

We know [**5**] that $\pi(L)$ is the direct limit of the system

$$\begin{tikzcd} & \pi(L_0) \arrow[dl] \arrow[dr] & \\ \pi(L_1) & & \pi(L_2) \end{tikzcd}$$

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Since k is not contained in any 3-cell subset of $\bar{L}_1$, the injection $\pi(L_0) \to \pi(L_1)$ is an isomorphy. If λ is not trivial, the injection $\pi(L_0) \to \pi(L_2)$ is also an isomorphy, and thus $\pi(L)$ is the free product with amalgamation

$$\pi(L) = \pi(L_1) \underset{\pi(L_0)}{*} \pi(L_2).$$

A transitive representation ρ of $\pi(L)$ on a permutation group P induces, for $i = 0, 1, 2$, a representation ρ_i of $\pi(L_i)$ on a subgroup P_i of P. In general, the part M_j of the covering space M that lies over L_i is the union of several components, each of which is a covering space of L_i. It is easy to see that M_i is connected if and only if P_i is transitive, and that M_i is then the covering of L_i that belongs to ρ_i. Thus *if* P_0 *is transitive*, so that P_1 and P_2 are also transitive, M_0, M_1, *and* M_2 *are connected* and $\pi(M)$ *is the direct limit of the system*

$$\begin{array}{ccc} & \pi(M_0) & \\ \swarrow & & \searrow \\ \pi(M_1) & & \pi(M_2) \end{array}$$

Since the injections $\pi(M_0) \to \pi(M_i)$, $i = 1, 2$, are isomorphies if and only if the injections $\pi(L_0) \to \pi(L_i)$ are isomorphies, we see that

$$\pi(M) = \pi(M_1) \underset{\pi(M_0)}{*} \pi(M_2) \quad \textit{if and only if} \quad \pi(L) = \pi(L_1) \underset{\pi(L_0)}{*} \pi(L_2).$$

3. Metacyclic invariants of doubled knots. To each integer γ there is associated to a tame knot l a companion $k = l_\gamma$ called the *double* [**18**] of l with *twist* γ. If l^* is any other tame knot, and a tubular neighbourhood $\bar{L}_1$ of l is mapped faithfully onto a tubular neighbourhood $\bar{L}_1^*$ of l^*, then l_γ is thereby mapped onto l_γ^*; hence the definition of doubling is reduced to description of the doubles of a trivial knot. Although l_γ and l_γ^* are inequivalent whenever l and l^* are inequivalent [**16**], l_γ and l_γ^* have exactly the same cyclic invariants (since the group of l_γ modulo its second commutator subgroup does not depend on l but only on γ). This raises the question of distinguishing between l_γ and l_γ^* by algebraic invariants.

The Alexander polynomial of l_γ is

$$\Delta(t) = \gamma t^2 + (1 - 2\gamma)t + \gamma$$

and the second elementary ideal is $E_2(t) = (1)$. Hence, by Theorem 1, the number of inequivalent representations on Γ_p is equal to the number of roots of the congruence

$$\gamma t^2 + (1 - 2\gamma)t + \gamma \equiv 0 \pmod{p},$$

i.e. there is just one representation if p divides $1 - 4\gamma$, there are two representations if $1 - 4\gamma$ is a quadratic residue of p, and there are no representations if $1 - 4\gamma$ is a non-residue.

Let us consider a representation ρ of $\pi(L)$ onto Γ_p. Since a longitude of l belongs to the second commutator subgroup of $\pi(L)$, it must lie in the kernel of ρ. Since the winding number of k in $\bar{L}_1$ is equal to 0, a meridian of l must lie in the commutator subgroup of $\pi(L)$ and must therefore be mapped into a power of ω. A properly chosen meridian cell of $\bar{L}_1$ cuts k in two points, and two corresponding meridians of k are mapped by ρ into, say, $\omega^{a_1}\xi^b$ and $\omega^{a_2}\xi^b$. Since $\rho(\pi(L))$ is not allowed to be a proper subgroup of Γ_p, the integer b must be prime to $p - 1$, and the integers a_1 and a_2 must be incongruent modulo p. (If $a_1 \equiv a_2 \pmod p$, then all of $\pi(L_2)$ would lie in the kernel of ρ, and consequently every meridian of k would be mapped by ρ into $\omega^{a_1}\xi^b$.) It follows that $\pi(L_0)$ is mapped onto the cyclic subgroup generated by ω, and therefore that the representation ρ_0 is transitive; consequently, by the preceding paragraph, M_0, M_1, and M_2 are connected. Moreover, if l is not trivial, then

$$\pi(M) = \pi(M_1) \underset{\pi(M_0)}{*} \pi(M_2).$$

Now $\pi(M_0)$ is the free abelian group generated by x^p and y, where the meridian x and the longitude y generate the free abelian group $\pi(L_0)$. The orientable surface X (of genus 1) bounded by x in L_1 is covered by an orientable surface bounded by x^p in M_1, and the orientable surface Y bounded by y in L_2 is covered by an orientable surface in M_2 whose boundary is y. Thus $H(M)$ is the direct product

$$H(M) = \frac{H(M_1)}{\langle y \rangle} \times \frac{H(M_2)}{\langle x^p \rangle}.$$

If λ^* is the trivial type 0, then, as shown above, $M_0{}^*$, $M_1{}^*$, and $M_2{}^*$ are connected, but in this case

$$H(M^*) = \frac{H(M_1{}^*)}{\langle y \rangle} = \frac{H(M_1)}{\langle y \rangle}.$$

On the other hand, $H(M_2)/\langle x^p \rangle$ is clearly the homology group of the p-fold cyclic branched covering of l; hence [**6**, p. 416, (6.1)], $H(M_2)$ is the homology group of the p-fold cyclic unbranched covering of l and $\langle y \rangle$ is a direct summand of it. Thus we have the following result.

The torsion numbers of a pth metacyclic (unbranched) covering of l_γ are the torsion numbers of the corresponding pth metacyclic (unbranched) covering of 0_γ together with the torsion numbers of the pth cyclic (unbranched) covering of l; the betti number of a pth metacyclic (unbranched) covering of l_γ is equal to 1 *less than the sum of the betti number of the corresponding pth metacyclic (unbranched) covering of 0_γ and the betti number of the pth cyclic (unbranched) covering of l.* (This result was first proved by direct calculation by Artin [**2**].)

4. Fundamental group of metacyclic covering. There is a very simple algorithm for the fundamental group $\pi(\bar{M})$ of the branched covering that belongs to a metacyclic representation onto Γ_3. If the group of the link k is $G = |x_1, \ldots, x_n : r_1 = 1, \ldots, r_{n-1} = 1|$ and ω is a representation of G onto Γ_3, then each x_j^ω must be one of the transpositions $(1\,2)$, $(1\,3)$, $(2\,3)$, say $(u_j\, v_j)$. By the Reidemeister-Schreier theorem (cf. [**8**, pp. 144–147]), the fundamental group $\pi(M)$ of the unbranched covering M has the presentation

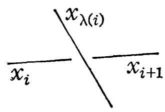
$$(x_{j\beta} : r_{i\alpha}\ (j = 1, \ldots, n, \beta = 1, 2, 3; i = 1, \ldots, n-1, \alpha = 1, 2, 3), \quad x_{j'1} = 1, x_{j''1} = 1),$$

where $x_{j'}$ and $x_{j''}$ are any two of the generators $x_1, \ldots, x_n$ which are such that $x_{j'}^\omega = (1\,2)$, $x_{j''}^\omega = (1\,3)$, say, and $r_{i\alpha}$ are as described in [**8**, p. 146]. A presentation of the group $\pi(\bar{M})$ is obtained by adjoining the branch relations $x_{ju_j}x_{jv_j} = 1$, $x_{jw_j} = 1$, (where $u_j\, v_j\, w_j$ is some permutation of $1\,2\,3$ and $u_j < v_j$). Of course, this set of branch relations is very redundant but let us adjoin them all and use them to eliminate x_{jv_j} and x_{jw_j}. Writing x_j again for x_{ju_j} there results a presentation of the following form:

$$(x_1, \ldots, x_n : s_1 = 1, \ldots, s_{n-1} = 1, x_{j'} = 1, x_{j''} = 1),$$

where s_i can be described as follows:

(1) If the ith relation $r_i = 1$



is $x_{i+1} = x_{\lambda(i)}{}^{\epsilon} x_i x_{\lambda(i)}{}^{-\epsilon}$ and $x_i^\omega, x_{i+1}^\omega, x_{\lambda(i)}^\omega = (uv)$, then the relation $s_i = 1$ is $x_i = x_{\lambda(i)} x_i^{-1} x_{\lambda(i)}$;

(2) if $x_i^\omega = (uv)$, $x_{\lambda(i)}^\omega = (uw)$, $x_{i+1}^\omega = (vw)$, then the relation $s_i = 1$ is $a = bc$, where a is that one of x_i, x_{i+1}, $x_{\lambda(i)}$ that is represented by $(1\,3)$, b that one that is represented by $(1\,2)$, and c that one that is represented by $(2\,3)$. (If the two relations $x_{j'1} = 1$ and $x_{j''1} = 1$ are omitted, the group presented is the free product of $\pi(\bar{M})$ and the free group F_2 of rank 2; cf. [**6**, p. 413].) An example will show how easy this algorithm is to use:

$$\begin{aligned}
\pi(\bar{M}) &= |a, b, A, B, C, \alpha, \beta, \gamma : \alpha = aB, \gamma = aA, \beta = bA, \alpha = bC, \beta = aC, \\
&\qquad \gamma = bB, \beta = \gamma\bar{\alpha}\gamma, B = C\bar{A}C, b = 1, C = 1| \\
&= |a, b, A, B, C : aB = bC, bA = aC, aA = bB, (A = \bar{a}bB), \\
&\qquad B = C\bar{A}C| \qquad b = 1, C = 1 \\
&= |a, b, A : a\bar{b}a = b\bar{a}b, \bar{b}a = \bar{a}b\bar{a}b| \qquad b = 1, A = a \\
&= |a : a^3 = 1|.
\end{aligned}$$

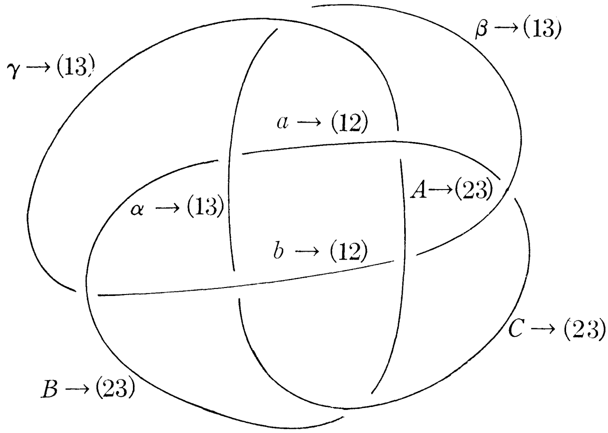


FIGURE 1

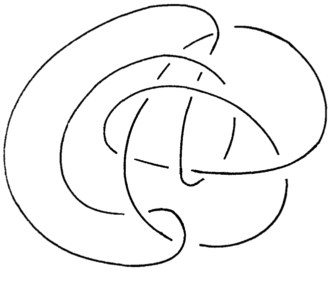
I have calculated $\pi(\bar{M})$ for the 294 branched covering spaces $\bar{M}$ of the 213 tabulated knots that have representations on Γ_3. It turns out that in this range, $\pi(\bar{M})$ is always either cyclic or the free product of two cyclic groups, which indicates that because of redundancies the number of different manifolds occurring grows rather slowly with n (the number of crossings). $\bar{M}$ turns out to be in this range simply connected 90 times. By a method that I plan to discuss in a later paper, I have shown that these are in fact all homeomorphic to S^3.

The method of this section may easily be modified to deal with any representation ρ that maps meridians into transpositions. It is only necessary to remark that in $\pi(\bar{M})$ there may be a third type of relation, namely:

(3) If $x_i^{\omega} = x_{i+1}^{\omega} = (uv)$ and $x_{\lambda(i)}^{\omega} = (wq)$, then the relation $s_i = 1$ is $x_i = x_{i+1}$.

Particularly simple is the case of the representation onto Γ_2, i.e. the case of the 2-fold cyclic covering, since in this case all the relations $s_i = 1$ are of type (1).

Since the developments of this section are obviously an outgrowth of [**9**], it may be appropriate here to note that S. Kinoshita and K. Perko have each kindly pointed out to me that diagram (b) of [**9**, p. 215] is in error, and that a correct diagram is, for example, the following one.



5. Linking of the branch curves. When a representation ρ of a knot group is cyclic, the branch curve in $\bar{M}$ is a knot; in other cases it is a link.† The idea of attaching the linking numbers of such a link to the type of the knot or link goes back to Reidemeister [**14**, Chapter III, § 15] and Bankwitz and Schumann [**3**], but there has been a lack of systematic study of this invariant. In his senior thesis, Perko [**13**] set forth an algorithm for the combinatorial calculation of these linking numbers, and, in particular, a computer program for their calculation in the case of representation onto Γ_3. This program was run for the 213 knots mentioned above, and the results for the knots of at most nine crossings are shown in the table below. These linking numbers are, of course, rational numbers whose denominators divide the largest torsion number of $\bar{M}$, and they exist only if the cycles carried by the branch curves are torsion cycles. Since reversing the orientation of s reverses the orientation of $\bar{M}$, and hence the sign of the linking number of the branch curves, the knot will certainly be non-amphicheiral whenever this linking number exists and is different from zero; (when there is more than one representation, the condition is that the set of these linking numbers calculated for the various representations should be unsymmetric). Using this criterion, Perko observed that the knots 6_1, 9_1, 9_{23}, 9_{37}, and 9_{46} are not amphicheiral, the non-amphicheirality of the last two of these appears not to have been known previously.

The following table gives the structure of the group $\pi(\bar{M})$ and the linking

κ	o	v	κ	o	v	κ	o	v
3_1	1	± 2	9_1	1	± 6	9_{37}	0	———
			9_2	1	± 2		3	$\pm 10/3$
6_1	1	± 2	9_4	1	± 2		3	$\pm 14/3$
			9_6	1	± 6		3	$\pm 10/3$
7_4	1	± 2	9_{10}	1	± 6	9_{38}	1	± 14
7_7	1	± 6	9_{11}	1	± 6	9_{40}	4	± 4
			9_{15}	1	± 6	9_{46}	0	———
8_5	2	± 4	9_{16}	2	± 4		3	$\pm 2/3$
8_{10}	2	0	9_{17}	1	± 10		3	$\pm 2/3$
8_{11}	1	± 6	9_{23}	1	± 6		3	$\mp 2/3$
8_{15}	2	± 4	9_{24}	2	0	9_{47}	3	$\pm 2/3$
8_{18}	3	± 2	9_{28}	2	± 4		3	$\pm 2/3$
	3	± 2	9_{29}	1	± 10		3	$\pm 2/3$
	3	∓ 2	9_{34}	5	$\pm 14/5$		3	$\mp 2/3$
	3	∓ 2	9_{35}	0	———	9_{48}	0	———
8_{19}	2	± 4		3	$\pm 2/3$		3	$\pm 10/3$
8_{20}	2	0		3	$\pm 2/3$		3	$\pm 10/3$
8_{21}	2	± 4		3	$\pm 2/3$		3	$\pm 10/3$

†In general, the expected number of components lying over a knot with meridian x will be the number of cycles in x^ρ. However, Perko [**13**] has given an example showing that it may in fact be fewer than expected.

number v of the two branch curves for those prime knot types κ of at most nine crossings that can be represented on Γ_3; in this range, $\pi(\bar{M})$ is always a cyclic group of order $o \geqq 0$, and thus the cases where v is undefined occur only when $o = 0$. Note that in the cases where there are several representations, reorientation changes the signs of the corresponding numbers v *simultaneously*. The representations ρ on Γ_3 of a composite knot type κ and the corresponding linking numbers v are determined by the representations ρ and linking numbers v of the individual factors of κ [**13**]. The numbers v are even integers whenever $\bar{M}$ is simply connected [**13**]; when $\bar{M}$ is not simply connected, it has always turned out that the numerator of v is an even integer but this has not been proved.

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