# 9

# A brief summary of $d = 3$ NAGTs

## 9.1 Introduction

NAGTs in three dimensions have valuable applications in their own right because they are the high-temperature limit of $d = 4$ NAGTs with infrared slavery (see Chapter 11 for more details). They also lead to important insights into $d = 4$ NAGTs at zero $T$, and in many ways, $d = 3$ QCD is more interesting to study to gain this insight than the far more often-invoked two-dimensional theories. It is not a free-field theory (as is a $d = 2$ pure-gauge NAGT), and it has many features strongly analogous to those of $d = 4$ NAGTs that are best understood by applying the pinch technique. In particular, although a $d = 3$ NAGT cannot be asymptotically free (because it is superrenormalizable, not possessing the usual renormalization group), it is still very much infrared unstable, with even worse singularities than those in $d = 4$. Although this $d = 3$ infrared slavery had been strongly suspected before the pinch technique on the basis of conventional Feynman graph calculations, it took the pinch techniqe to settle the issue and demonstrate the existence of infrared slavery in $d = 3$ NAGTs.

Because a $d = 3$ NAGT is the critical nonperturbative part of the high-temperature behavior of its $d = 4$ counterpart, infrared slavery prevents the use of perturbation theory (beyond $\mathcal{O}(g_3^4)$) in understanding all the phenomena of high temperature, including generation of a so-called magnetic mass, which vanishes identically to all orders of perturbation theory. Just as we have already seen at zero temperature, the magnetic mass, found from the PT Schwinger–Dyson equations, cures the otherwise intractable infrared singularities of high-temperature $d = 4$ gauge theories. We study here only the $d = 3$ NAGT part of finite-temperature $d = 4$ NAGTs,

In this chapter, we continue to use the notation introduced in Chapter 7. Also in the present chapter, $g_3$ is the $d = 3$ NAGT coupling, and $g$ continues to be the $d = 4$ coupling.

saving the PT results for other components of finite-temperature field theories for Chapter 11.

In some respects, $d = 3$ NAGTs are somewhat easier technically than their $d = 4$ counterparts. For example, effective field theories of center vortices are fairly simple scalar field theories in $d = 3$ [1] and so are easier than in $d = 4$, where they are string theories. Unfortunately, we cannot cover these effective theories in a book of this length.

We list here a few of the many reasons for being interested in $d = 3$ QCD, most of which are really only understood with the help of the pinch technique, the gauge technique, or both:

1. It is a superrenormalizable theory, very well behaved in the ultraviolet, with corrections to the bare coupling vanishing as inverse powers of large momenta. But the pinch technique reveals infrared slavery, just as in $d = 4$, meaning that the PT propagator has unphysical singularities at finite momentum. Furthermore, a $d = 3$ gauge theory with zero bare mass (no Higgs effect or Chern–Simons (CS) term) is always strongly coupled at low momenta $q$, where the dimensionless expansion parameter is $Ng_3^2/q$ for $SU(N)$. As one might by now expect, infrared slavery is resolved by generation of a gluon mass, which in turn gives rise to a $\langle \mathcal{G}_{ij}^2 \rangle$ condensate and to many of the solitons familiar in $d = 4$: center vortices, nexuses, and sphalerons.
2. In $d = 3$, we will actually prove the existence of this $\mathcal{G}_{ij}^2$ condensate and entropy dominance of the effective action, simply on the hypothesis that the full theory possesses only one mass scale (that of $g_3^2$ itself). We will also show that an approximation based on the pinch technique fully realizes the expected functional form of the exact effective action and the taming of all infrared-slavery singularities. The pinch technique shows that there is a direct connection between the "wrong" sign of the one-loop self-energy, responsible for infrared slavery, and the existence of a minimum in the effective action at a finite condensate VEV.
3. The vacuum wave functional of the functional Schrödinger equation (FSE) for $d = 4$ QCD is expressed in terms of a gauge-invariant effective action whose arguments are background fields given by the coordinate gauge potentials of this wave functional. Certain aspects of the form of the Schrödinger functional are governed by the pinch technique. Gauge invariance gives rise to an infinite tower of QED-like Ward identities, and the gauge technique is effective in exploiting the Ward identities. The two lowest terms in a gauge-technique-inspired expansion of the effective action around small momentum

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lead approximately to $d = 3$ QCD as an effective field theory for calculating matrix elements. (This is by no means obvious because Schrödinger equation functionals depend intrinsically on square roots of operators, which are forms not encountered in conventional effective actions.) This effective field theory shows confinement (because it has a condensate of center vortices), and $d = 3$ estimates of the gluon mass actually lead to an estimate of the $d = 4$ coupling $\alpha_s(m^2) \simeq 0.4 - 0.5$, which is not too far (given the approximations) from what we found in Chapter 6 and in phenomenological evaluations.

4. Although $d = 3$ gauge theory does not have the usual $d = 4$ topological charge, it does admit topologically interesting parity-violating CS terms in the action. The coupling for this term in the action is integrally quantized and called level $k$. In an elegant work, Witten [2] showed that Wilson-loop expectation values in a field theory whose action was just the CS term (a so-called topological field theory) generated some deep results about knots in three dimensions. This Witten theory corresponds to very large values of $k$. When the conventional Yang–Mills action is included along with the CS term, it turns out that gauge bosons get mass $\sim k g_3^2$ in perturbation theory. Perhaps surprisingly, at large $k$, this mass does not lead to well-behaved classical solitons. The pinch technique strongly suggests that at small $k$ ($k \simeq (1 - 2)N$), infrared slavery problems still persist, and there is a phase transition from the large-$k$ Witten phase to a phase that also has a dynamically generated gauge-boson mass. Modified forms of the usual $k = 0$ solitons exist in this phase, which is confining.
5. The PT dynamical mass gives rise to the sphaleron, a soliton of interest purely as a $d = 3$ object. The sphaleron becomes even more interesting when it is coupled to a CS term. Because it is natural for the CS number of a sphaleron to be a half-integral, a condensate of an odd number of sphalerons challenges usual compactnesss assumptions, which suggests challenging the conventional wisdom demanding integral levels $k$ for the CS term, as well. We show that, although noncompact theories could in principle exist, they have infinitely higher energy than the corresponding compact versions. In the process, we show that half-integrality is also related to $d = 3$ knots and to nexuses in $d = 2$. So even though there is no topological charge per se in $d = 3$ gauge theory, there are many interesting and curious topological effects.

We start next with PT perturbation theory at one loop, and then, after finding the exact form of the effective action, we show how the one-loop result realizes the exact functional form of the effective action. This illustrates how infrared slavery is directly related to condensate formation.

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## 9.2 Perturbative infrared instability

We easily see the problems of infrared slavery in $d = 3$ by calculating the one-loop perturbative PT proper self-energy. This goes exactly as in the $d = 4$ case of Section 1.3.3, except for the values of the integrals. The result [3, 4] for the scalar part of the one-loop PT inverse propagator (as defined in Eq. (1.30)) is as follows:

$$\widehat{d}^{-1}(q) = q^2[1 - I_3(q)] = q^2 - \pi b_3 g_3^2 q, \tag{9.1}$$

where

$$I_3(q) = \frac{15Ng_3^2}{4}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^2(q-k)^2}; \qquad b_3 = \frac{15N}{32\pi}. \tag{9.2}$$

Infrared slavery is simply the fact that $I_3$ occurs with a negative sign in the self-energy (or equivalently, that $b_3$ is positive), which has the implication that there is a pole in the propagator for positive $q$. In our metric, where $q$ is the magnitude of an ordinary three-momentum, this indicates a spacelike and thus *tachyonic* pole – a pole corresponding to an imaginary mass.

What could be the cure for this unphysical behavior? At first glance, it could be easy: because the coupling $g_3^2$ has dimensions of mass, the omitted $g_3^4$ term might well provide a sufficiently positive term to overcome the negative one-loop term. This is indeed what happens nonperturbatively, but not to any order of perturbation theory, where the coefficient of $g_3^4$ is identically zero to all orders. (If it were not zero, we could add a bare mass term to the action, which would no longer be perturbatively renormalizable.)

This is only the beginning of the bad perturbative behavior. At $\mathcal{O}(g_3^{2N})$, each perturbative integral, by simple dimensional reasoning, has the infrared behavior $g_3^4(g_3^2/q)^{N-2}$, with poles of infinitely high order in the inverse propagator. But with nonperturbative generation of a (nontachyonic) mass $m$, the infrared behavior of every propagator in a loop is $\sim 1/m^2$, and an easy power counting shows that $q$ in the perturbative ordering expression is replaced by the dynamical mass $m \sim g_3^2$, so all terms are of $\mathcal{O}(m^2)$ for order $N \geq 2$.

A one-loop PT calculation only clearly shows us (i.e., gauge invariantly) the disease, not the cure – which is a dynamical gluon mass. In $d = 4$, this mass is directly related to the gluon condensate, and we now argue that this is so also in $d = 3$.

## 9.3 The exact form of the zero-momentum effective action

Define a condensate operator $\theta$ by

$$\theta(x) = -\frac{1}{2g_3^2}\,\mathrm{Tr}\,(\mathcal{G}_{ij})^2. \tag{9.3}$$

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The key result is Eq. (9.10), giving the precise form of the effective action as a function of the zero-momentum matrix elements of $\theta$. This equation says that this operator must have a (positive) VEV, and so there is a condensate of some sort. It further says that the condensate generates so much entropy that the entropy (a negative contribution to the effective action) overcomes the positive action from whatever is in the condensate – just what we expect for center vortices and nexuses. The condensate is important for the self-consistency of gluon mass generation because it gives [5] the coefficient of $q^{-2}$ in the falloff of the gluon mass at large $q$:

$$m^2(q) \to \frac{58 N g_3^2 \langle\theta\rangle}{15(N^2-1)q^2}. \tag{9.4}$$

Before Lavelle found this result, people were not at all sure of what was going on with the use of the OPE in gauge-boson propagators. The simple reason was that the conventional Feynman propagator was gauge dependent, meaning that not only condensates of gauge-invariant operators, such as $\theta$, appeared in the OPE but also other condensates, such as ghost condensates of the form $\bar{c}c$ and mixed gluon-ghost condensates such as $\partial_i \bar{c} \mathcal{A}_i c$, as explicit computations showed. But in the PT propagator, these gauge-dependent condensates drop out, leaving only Lavelle's simple result.

One can always resort to assuming the existence of a nonvanishing VEV $\langle\theta\rangle$ with no further argument. But in $d = 3$, we can actually prove [6] that there must be such a (positive) VEV by determining the exact dependence of the effective potential on the zero-momentum part of the operator $\theta$. The answer is reminiscent of a similar one-loop result in $d = 4$ QCD [7], showing evidence for a condensate. The only assumption we need to make is that there is only one dimensional parameter in $d = 3$ QCD (without matter fields), and that is the coupling $g_3^2$, which has mass dimension unity. We then show that the effective action $\Gamma(\theta)$ has a minimum for a nonzero value of its argument. We can also show [8] that the exact functional form is actually found in the one-loop PT propagator in the presence of the fields constituting the condensate.

Define the generating functional for zero-momentum matrix elements of the action density $\theta$:

$$Z(J) \equiv \mathrm{e}^{-W(J)} = \int [\mathrm{d}\mathcal{A}_i] \exp\left[(1-J)\int \mathrm{d}^3x \, \frac{1}{2g_3^2} \mathrm{Tr}\, \mathcal{G}_{ij}^2\right], \tag{9.5}$$

where $W(J)$ is the space-time integral of the vacuum action density in the presence of a space-time constant source $J$ coupled to $\theta$:

$$W(J) = \int \mathrm{d}^3x \, \epsilon_{\mathrm{vac}}(J). \tag{9.6}$$

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In the usual way, multiple derivatives of $W(J)$, evaluated at $J = 0$, give connected matrix elements at zero momentum of the operator $\theta$. In particular, the VEV of $\theta$ at $J = 0$ is

$$\langle\theta\rangle = -\left.\frac{\mathrm{d}\theta}{\mathrm{d}J}\right|_{J=0}. \tag{9.7}$$

Given the assumption that $g_3^2$ is the only mass parameter, it follows that $\epsilon_{\mathrm{vac}} \sim g_3^6$. It is now completely trivial to find $W(J)$ because it differs from $W(J = 0)$ simply by the substitution $g_3^2 \to g_3^2/(1 - J)$. So

$$\epsilon_{\mathrm{vac}} = -\frac{\langle\theta\rangle}{3}(1 - J)^{-3}, \tag{9.8}$$

where the normalization follows from Eq. (9.7).

The next step is to make a Legendre transform to the effective action $\Gamma(\theta)$:

$$\Gamma(\theta) = W(J) + J\int \mathrm{d}^3x\,\theta; \qquad \frac{\mathrm{d}\Gamma}{\mathrm{d}\theta} = J\int \mathrm{d}^3x. \tag{9.9}$$

The effective action has the property that when the current $J$ is turned off, it has an extremum as a function of $\theta$, and its value at the extremum is the vacuum action $W(0)$.

The differential equation for $\Gamma$, plus Eq. (9.8), is elementary to solve:

$$\Gamma(\theta) = \int \mathrm{d}^3x\left[\theta - \frac{4}{3}\theta^{3/4}\langle\theta\rangle^{1/4}\right]. \tag{9.10}$$

This indeed has a minimum at $\theta = \langle\theta\rangle$, and this minimum value of $-\int \mathrm{d}^3x\langle\theta\rangle/3$ is negative.[1] This negative action tells us that the theory is entropy dominated, and so there are interesting nonperturbative effects.

### 9.3.1 The effective action and the pinch technique

Of course, Eq. (9.10) has nothing to say about what the effects are or how large $\langle\theta\rangle$ is in units of $g_3^6$. There are no exact results about the latter, although one can make certain approximations [8] in estimating the effective action. One of two basic approximations is to use the one-dressed-loop effective action, with the structure of the loop supplied by our preceding PT results[2]; the other, commonly used by many authors, is to replace the true condensate fields in $\theta(x)$ with a background field $\mathcal{B}$ that is constant in space-time. This approximation of constancy makes it possible to do the calculations but introduces an unphysical feature, noted long ago [9]:

[1] Could $\langle\theta\rangle$ be zero? Only if $d = 3$ gauge theory is free, which we know it is not.
[2] Or equivalently, the one-loop effective action in the background-field Feynman gauge.

a constant chromomagnetic field is *unstable* to decay into a tangle of space-time-dependent fields. This, not unexpectedly, gives an imaginary part to the effective action. We will simply ignore such features here, knowing that in the real world, the condensate is made of such a tangle of fields and that the effective action is real.

The one-loop effective action, including the classical term, is

$$\Gamma(\theta) = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3k}{(2\pi)^3} \operatorname{Tr} \frac{-1}{2g_3^2} \mathcal{G}_{ij}(k)\mathcal{G}_{ij}(-k)[1 - I_3(k)]. \tag{9.11}$$

We can go beyond the strict one-loop form by allowing the PT function $I_3$ to depend on the condensate. The only feasible way to do this is to assume that the condensate is made of constant fields, so we approximate $\theta$, needed only at zero momentum, by a constant-field condensate such that $\langle\theta\rangle \simeq \bar{B}^2/g_3^2$ for some constant magnetic field of magnitude $\bar{B} \equiv |\mathcal{B}|$.

It turns out [9] that in a constant chromomagnetic field, all gluonic fluctuation modes except one become massive, with $m^2 \sim \bar{B}$. The one exception is a tachyonic mode that carries the instability for decay of the constant field. Without going through the complicated calculations of Nielsen and Olesen [9], we can appreciate such a mass relation from Lavelle's relation in Eq. (9.4), noting[3] that for finite momenta $q^2 \sim m^2$, the mass itself obeys $m^2 \sim [g_3^2\langle\theta\rangle]^{1/2} \sim \bar{B}$. Let us replace (in the spirit of one-loop gap equations, discussed in Section 9.4.2) the free propagators in the integral $I_3$ used for the perturbative one-loop propagator by adding a mass term $k^2 \to k^2 + \bar{B}$, and so on. We omit detailed numerical constants that are not of interest. Then the effective action of Eq. (9.11) is

$$\Gamma(\theta) = \int \mathrm{d}^3x\, \theta[1 - I_3(k = 0)] = \int \mathrm{d}^3x\, \frac{1}{2g_3^2}\left(\bar{B}^2 - b_3 g_3^2 \bar{B}^{3/2}\right) + \cdots, \tag{9.12}$$

where, in calculating $I_3$, the propagators have been modified as discussed earlier. Because $\bar{B} \sim g_3\theta^{1/2}$, this result is a real effective action that has the correct functional form of the exact effective action in Eq. (9.10). No imaginary part shows up because we omitted the tachyonic fluctuation mode.

The minus sign in this approximate effective action is exactly the minus sign coming from infrared slavery, so as promised, infrared slavery and condensate formation are really the same thing. The reader might find it interesting to carry out the same calculation for the $d = 4$ effective action using the $d = 4$ PT self-energy. The result is the famous one-loop effective action $\sim\mathcal{G}^2 \ln \mathcal{G}^2$, which also shows condensate formation, and for the same reason.

[3] In this and what follows, we give explicit formulas only for $SU(2)$. For $SU(N)$, the appropriate scalings follow from Eq. (9.4); for example, $\Gamma$ scales like $N^3(N^2 - 1)$.

## 9.4 The dynamical gauge-boson mass

The next question is estimation of the dynamical mass needed to cure infrared slavery. There are both theoretical estimates [3, 4, 10, 11, 12, 13, 14, 15, 16, 17] and lattice simulations [18, 19, 20, 21]. Some of the theoretical estimates are based on the pinch technique [3, 4, 10, 11, 15, 16] and some on conventional Feynman-graph technology [12, 13]. This is a technically difficult problem, and all authors[4] use some form of one-dressed-loop equations.

### 9.4.1 Early pinch technique work

The early PT papers [3, 4, 10] used the spectral form of the gauge technique to write the three-gluon vertex in terms of the PT propagator. This results in the one-loop integral equation:

$$\widehat{d}^{-1}(q) = q^2 \left[ 1 - \frac{2b_3 g_3^2}{\pi q} \int_0^\infty k \, \mathrm{d}k \, \widehat{d}(k) \ln \left| \frac{2k+q}{2k-q} \right| \right] + \widehat{d}^{-1}(0), \tag{9.13}$$

for the scalar part $\widehat{d}$ of the PT propagator. One can check that if the bare propagator $\widehat{d}(k) = 1/k^2$ and the bare value $\widehat{d}^{-1}(0) = 0$ are used on the right-hand side of this equation, the one-loop propagator of Eq. (9.16), which has a tachyonic pole (to be cured by a positive value of $\widehat{d}^{-1}(0)$), is recovered. Note that this equation, although necessarily approximate, does demand consistency in that the same propagator $\widehat{d}$ appears both on the right-hand side and the left-hand side of this equation (in contrast to the one-loop gap equations discussed next). As it stands, this equation cannot be solved for a gluon mass because the last term $\widehat{d}^{-1}(0)$ is just a placeholder for some dynamical expression. This expression has not yet been worked out because the presence of logarithmically divergent terms that are canceled at two-loop order requires working out the two-loop self-energy, and this remains to be done. However, it is possible to give a lower bound to the gluon mass, or more accurately, $\widehat{d}^{-1}(0)$, because Eq. (9.13) has no solution at all if this quantity vanishes. (If one tries to solve Eq. (9.13) by successive substitution beginning with a massless propagator, infrared singularities from the negative sign of the integral build up uncontrollably.) Numerical investigations give an approximate value of the lower limit, $\widehat{d}^{-1}(0)_{\min}$, as

$$\widehat{d}^{-1}(0)_{\min} = [1.96 b_3 g_3^2]^2, \tag{9.14}$$

[4] Except Karabali et al. [17], whose methods are original and unique. The estimate of Ref. [11] is really an estimate of the ratio of the string tension to the squared mass; it is based on special methods that we will not cover here.

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which is equivalent[5] to $m/(Ng_3^2) \geq 0.29$. Next we will compare this lower limit to estimates based on one-loop gap equations and find some problems.

### 9.4.2 One-loop gap equations and lattice simulations

Since the earlier work, a number of authors [12, 13, 15, 16] have addressed the theoretical issues with one-loop gap equations, in which the internal propagators of the one-loop self-energy are approximated by a simple massive form:

$$\frac{1}{q^2} \to \frac{Z_{\text{in}}}{q^2 + m^2}. \tag{9.15}$$

The one-loop self-energy is calculated with this input, and one demands that the output mass be equal to the input mass. Here one should include the (finite in $d = 3$) input renormalization constant $Z_{\text{in}}$ and check that it, too, is reproduced in the output, but this has not been done in the gap-equation papers, which all use $Z_{\text{in}} = 1$. It might seem reasonable to insist that not only the mass but also the residue of the output propagator agree with the input values, but this is rarely looked at. Part of the reason is that some authors [12, 13] use standard Feynman propagators in the gap equation, and with these, only the pole position, but not its residue, is gauge invariant. The specific concern of Buchmuller and Philipsen [12] and Eberlein [13] is mass generation in finite-$T$ electroweak theory (with the $U(1)$ part dropped), and so these works include Higgs fields. But it is straightforward to suppress these Higgs fields by taking their mass to infinity [16], resulting in a form of dynamical gluon mass generation without Higgs fields.

The results are inconclusive. This happens for two reasons: the first is the use of the conventional Feynman propagator [12, 13] rather than the pinch technique. The position of the pole in the conventional propagator is gauge independent, but otherwise, the propagator, even the pole residue, is gauge dependent. The second reason is that the hypothesized input propagator – even when the one-loop PT self-energy is used [15, 8] – shows no signs of the infrared slavery that motivates the study of dynamical mass in the first place. Recall that the one-loop PT propagator $\widehat{\Delta}_{ij}(q)$ for $d = 3$ QCD is

$$\widehat{\Delta}_{ij}(q) = \left[\delta_{ij} - \frac{q_i q_j}{q^2}\right] \frac{1}{q^2 - \pi b_3 g_3^2 q} + \text{terms} \sim q_i q_j, \tag{9.16}$$

[5] This defines the mass in terms of the behavior of the propagator at zero momentum rather than the pole mass. This leads to minor inaccuracies, but it is still a gauge-invariant mass estimate.

Table 9.1. *Parameter values for three one-loop gap equations.*

| Reference | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|
| [12] | 27/16 | 3/8 | 9/4 |
| [15] | 15/4 | 1/2 | 3/2 |
| [16] | 15/4 | 1/2 | 0 |

with $b_3 = 15N/32\pi$. Roughly speaking, with mass generation, the PT propagator would have the denominator of Eq. (9.16) replaced by something like the following:

$$q^2 - \pi b_3 g_3^2 q \to q^2 - \pi b_3 g_3^2 q + m^2. \tag{9.17}$$

If $m > \pi b g_3^2/2$, there are no tachyonic (real) poles of the propagator.

The one-loop gap equation input of Eq. (9.15) differs from the preceding form by not having a negative term, which, of course, comes from infrared slavery. Without this infrared-slavery effect in the input propagator, the self-consistent one-loop masses are lower than they would be with this effect included. This has the effect of giving an output pole residue $Z_{\rm out}$ that is rather different from the input value $Z_{\rm in}$, so that true self-consistency is not achieved. (This comparison of residues only makes sense for the PT gap equations because for non-PT propagators, the residues are not gauge invariant.)

All of the one-loop gap equation results have the same functional form[16]:

$$d^{-1}(q^2) = q^2 + \frac{Ng_3^2}{4\pi}\left[\left(-\alpha q + \frac{\gamma m^2}{q}\right)\arctan\frac{q}{2m} - \beta m\right], \tag{9.18}$$

with the values shown in Table 9.1 for the parameters. Observe that in extrapolating Eq. (9.18) to Minkowski momenta ($q \to iq$), there are only normal threshholds at $-q^2 = 4m^2$. This is to be expected with the gauge-invariant pinch technique, but it only happens in the Feynman gauge otherwise, where the ghosts and Goldstone bosons all have the mass $m$. In other gauges, this is not so, and the self-energy of Buchmuller and Philipsen [12] would have other terms. However, these unphysical threshhold terms do not contribute to the pole mass.

Note that Alexanian and Nair [15] and Ref. [16] have the same values for $\alpha$ and $\beta$; this is because both use the pinch technique. The differing value of $\gamma$ comes from differing treatments of the gauge-invariant mass terms used by these two sets of workers. In contrast, [12] uses the Feynman gauge and has rather different parameters.

Table 9.2. *Estimates of the $SU(N)$ magnetic mass by various techniques*

| Reference | $m/(Ng_3^2)$ | Technique | $Z_{\text{out}}/Z_{\text{in}}$ |
|---|---|---|---|
| [12] | 0.14* | 1-loop gap | N/A |
| [13] | 0.17* | 2-loop gap | N/A |
| [15] | 0.19* | Pinch/gap | 150 |
| [16] | 0.13* | Pinch/gap | <0 |
| [17] | 0.16 | See text | N/A |
| [18] | 0.18* | Lattice | N/A |
| [19] | 0.24* | Lattice | N/A |
| [20] | 0.26* | Lattice | N/A |
| [21] | 0.19 | Lattice | N/A |

Evaluating this and its derivative on the mass shell (at $q = \mathrm{i}m$) yields

$$m = \frac{Ng_3^2}{4\pi}\left(\frac{\alpha+\gamma}{2}\ln 3 - \beta\right) \tag{9.19}$$

$$Z_{\text{out}} = \left(\frac{\alpha+\gamma}{2}\ln 3 - \beta\right)\left[\alpha\left(\frac{1}{4}\ln 3 - \frac{1}{3}\right) - \beta + \gamma\left(\frac{3}{4}\ln 3 - \frac{1}{3}\right)\right]^{-1}. \tag{9.20}$$

The pole masses, following from setting $d^{-1}(q^2 = -m^2) = 0$, seem reasonable when compared to lattice values, as we will see shortly. But the PT residues are not close to self-consistency, as one may easily check. It would be better to use an input propagator of the form of Eq. (9.17), or something like it, but as far as we know, this has not been done with one-loop gap equations; instead, there is the original PT calculation, which demands a self-consistent propagator at all momenta but which has only been carried (so far) to the point of estimating a lower bound for the mass.[6] Ironically, the presumably gauge-dependent parameters of [12] yield a more reasonable value of $Z_{\text{out}}$ than do the PT parameters.

Table 9.2 shows various results for the ratio $m/(Ng_3^2)$, which should be roughly independent of $N$ for gauge group $SU(N)$ (exactly so for one-loop gap equations). Values marked by an asterisk were calculated for $SU(2)$; the rest were calculated for $SU(3)$, and all were assumed to scale linearly in $N$. In Table 9.2, N/A means that no residue factors were given. Reference [17] uses a very interesting approach to $d = 3$ gauge theory that we cannot describe here; it culminates in the formula $m/(Ng_3^2) = 1/(2\pi)$.

[6] Another problem with all the estimates we will discuss is that they do not properly account for the fact that the magnetic mass is really a function of momentum $q$, vanishing like $1/q^2$ (modulo logarithms) at large momentum (see Chapter 2). This is essential for the Schwnger–Dyson equations to yield finite results.

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From Table 9.2, we see that there is some spread in the ratio, with the average lattice value larger than the average gap-equation value. Note that the previously estimated lower bound of 0.29 is larger than any of the masses in the table. The lattice results vary somewhat, in part because the propagators from which the magnetic mass is extracted are in different gauges, and the extracted mass is not exactly the (gauge invariant) pole mass, which is hard to reach on the lattice because it involves extrapolation to negative values of momentum squared.

In any event, there seems to be no question that there is a finite $d = 3$ gluon mass and therefore the solitons (center vortices, nexuses) that we have already discussed.

## 9.5 The functional Schrödinger equation

The FSE is another way, in principle, of expressing the content of a field theory in $d$ dimensions via functional differential equations in $d - 1$ dimensions. There is nothing in the FSE approach that could not be understood directly from the field theory, but sometimes one gains insight by looking at a hard problem in a different way.

For any field theory, the FSE is no more or less than the usual Schrödinger equation, with fields as the coordinates and functional derivatives with respect to these fields as the momenta. The fields, as coordinates, are labeled by (in $d = 3 + 1$) three spatial positions $\vec{x}$. For a gauge theory, the component $\mathcal{A}_0^a$ has no canonical momentum and is set to zero, leaving only the three magnetic potentials $\mathcal{A}_i^a(\vec{x})$ at zero time as coordinates.[7] The canonical momentum is the electric field:

$$\Pi_i^a = \mathcal{E}_i^a(\vec{x}) \to -\mathrm{i}g^2 \frac{\delta}{\delta \mathcal{A}_i^a(\vec{x})}. \tag{9.21}$$

(Note that the commutator term is missing because we set $\mathcal{A}_0^a = 0$.) The Hamiltonian for the NAGT FSE is

$$\mathcal{H} = \int \left\{ -\frac{1}{2} g^2 \left( \frac{\delta}{\delta \mathcal{A}_i^a} \right)^2 + \frac{1}{2g^2} \left[ \frac{1}{2} \left( \mathcal{G}_{ij}^a \right)^2 \right] \right\} \equiv \int \left[ \frac{1}{2} \left( \Pi_i^a \right)^2 \right] + V, \tag{9.22}$$

and the Schrödinger equation is the usual $\mathcal{H}|\psi\rangle = E|\psi\rangle$. We consider the vacuum (ground state) wave functional $\psi\{\mathcal{A}_i^a(\vec{x})\}$ and the time-independent FSE that determines it. This wave functional is the matrix element

$$\psi\{\mathcal{A}_i^a(\vec{x})\} = \langle \mathcal{A}_i^a(\vec{x}) | \psi \rangle, \tag{9.23}$$

[7] See Jackiw [22] for an elegant treatment of the fundamentals of the canonical FSE for gauge theories. We temporarily use group-component notation.

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where the bra vector is an eigenvector of the field operator $\mathcal{A}_i^a$ and $|\psi\rangle$ is the vacuum-state eigenvector, whose energy we normalize to zero for the present. The vacuum wave functional has the form

$$\psi\{\mathcal{A}_i^a(\vec{x})\} = \exp[-S\{\mathcal{A}_i^a(\vec{x})\}], \tag{9.24}$$

where $S$ can be written as a formal power series with infinitely many terms:

$$g^2 S = \frac{1}{2!}\iint \mathcal{A}_i^a \Omega_{ij} \mathcal{A}_j^a + \frac{1}{3!}\iiint \mathcal{A}_i^a \mathcal{A}_j^b \mathcal{A}_k^c \Omega_{ijk}^{abc} + \cdots. \tag{9.25}$$

The $\Omega$ functions relate their associated gauge potentials nonlocally and may have derivatives of high order.

The exponent $S$ is real, bounded below for finite arguments (vacuum wave functionals do not have nodes), and positive for sufficiently large arguments (it is normalizable). Most important, it is a gauge-invariant functional of its arguments. These properties of $S$, plus the usual rules for constructing vacuum matrix elements, allow us to interpret $2S$ as an effective $d = 3$ action. The vacuum matrix elements are of the type[8]

$$\langle\psi|\cdot|\psi\rangle = \int \left[\mathrm{d}\mathcal{A}_i^a\right] \mathrm{e}^{-2S}(\cdot). \tag{9.26}$$

Define the effective $d = 3$ action by

$$I_{d=3} = 2S. \tag{9.27}$$

So FSE matrix elements such as $\langle\psi|W|\psi\rangle$, where $W$ is a spacelike Wilson loop, are expectation values in the $d = 3$ theory with effective action $I_{d=3}$.

The question is: how do we solve for this effective action, and what does it look like? This is not an easy question. In the first place, it is not possible to solve the FSE exactly,[9] so there is little guidance from existing solutions. (Often workers simply postulate what seems to be a reasonable approximate form for $\psi$ – typically Gaussian – to be used for variational estimates, but often, in the process, gauge invariance is lost.) A few low-$N$ terms of the $N$-point coefficients $\Omega_{ijk\ldots}$ can be found order by order in perturbation theory, but that is not very interesting; it is analogous to a bare-loop expansion of the effective action. Much more interesting is the dressed-loop expansion, in which the three-point and higher functions $\Omega_{ijk\ldots}$ are expressed in terms of the dressed two-point function $\Omega_{ij}$; then the FSE (or, equivalently, extremalization of the effective action $S$) yields a nonlinear equation

[8] As usual, we do not explicitly indicate ghost and gauge-fixing terms.

[9] To forestall confusion, there is an exact zero-energy formal solution [23, 24, 25, 26] to the vacuum FSE, which is $\psi \sim \exp(-(8\pi^2/g^2)N_{\mathrm{CS}})$, with $N_{\mathrm{CS}}$ being the CS integral (see Eq. (9.51)). Although this solution is not normalizable because the CS integral does not have a definite sign, it is applicable for certain high-energy, few-to-many scattering processes; see [26].

for the two-point function, quite analogous to the Schwinger–Dyson equation for the PT propagator. This approach is quite successful [27] for the anharmonic oscillator in one dimension (ordinary quantum mechanics) even in the limit where the quadratic term in the potential vanishes and perturbation theory completely fails. Because $I_{d=3}$ is gauge invariant under gauge transformations of its background-field arguments $\mathcal{A}_i(\vec{x})$, it is natural to use the pinch technique and gauge technique to approximate it, along with the dressed-loop expansion. Whether one uses a dressed-loop approximation or a bare-loop expansion, the solution to the FSE always involves the square root of operators. For example, a little experimentation shows that a perturbative expansion involves unfamiliar operators such as $\sqrt{-\nabla^2}$. Because we know that NAGTs show dynamic gluon mass generation, we expect that square-root operators of the form $\sqrt{m^2 - \nabla^2}$ are what turn up in the dressed-loop expansion.

### 9.5.1 The gauge technique and the FSE

The generator of infinitesimal gauge transformations is $\mathcal{D}_j^{ab} \times (-\mathrm{i}\delta/\delta\mathcal{A}_j^b)$, and this must annihilate $\psi$ or, equivalently, $S$. Invariance of $S$ under infinitesimal gauge transformations is trivial for the two-point function $\Omega_{ij}$; this quantity must be conserved (as in an Abelian gauge theory) so that in Fourier space,

$$\Omega_{ij}(k) = \Omega(k)P_{ij}(k) \qquad P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{9.28}$$

For the free theory, $\Omega_0(k) = k$, but for the dressed theory, we expect something like $\Omega(k) = \sqrt{k^2 + m^2}$.

Gauge invariance is more complicated for higher-point functions. Annihilating $\psi$ with the generator of gauge transformations yields a set of ghost-free Ward identities, just as in the pinch technique. For example, the Ward identity for the three-point function is

$$k_{1i}\Omega_{ijk}^{abc}(k_1, k_2, k_3) = f^{abc}\left[\Omega_{jk}(2) - \Omega_{jk}(3)\right], \tag{9.29}$$

where $\Omega_{jk}(2) \equiv \Omega_{jk}(k_2)$, and so on.

Further information comes from the FSE, where one finds that the equation determining the three-point function has the general form

$$\Omega_{il}(1)\Omega_{ljk}^{abc} + \Omega_{jl}(2)\Omega_{lik}^{bac} + \Omega_{kl}(3)\Omega_{lij}^{cab} = f^{abc}\Gamma_{ijk}. \tag{9.30}$$

The right-hand side $\Gamma_{ijk}$ comes from the cubic term in $\mathcal{H}$ plus another term from the five-point function. The Ward identity for $\Gamma_{ijk}$ is determined by the preceding equation plus the Ward identities for the two- and three-point functions, as already

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given, and multiplying both sides of Eq. (9.30) by $k_{1i}$ yields

$$k_{1i}\Gamma_{ijk} = \Omega^2_{jk}(3) - \Omega^2_{jk}(2). \tag{9.31}$$

For free particles with $\Omega = \Omega_0$, this is satisfied by the usual free three-point vertex

$$\Gamma^0_{ijk} = \mathrm{i}(k_1 - k_2)_k\delta_{ij} + \text{c.p.} \tag{9.32}$$

The reader can verify that Eq. (9.30) has a solution of the form

$$\Omega^{abc}_{ijk}(k_1, k_2, k_3) = f^{abc}[\Omega(1) + \Omega(2) + \Omega(3)]^{-1} \\ \times \left\{\Gamma_{ijk} + \left[\Omega(1)\frac{k_{1i}}{k_1^2}\left(\Omega_{jk}(2) - \Omega_{jk}(3)\right) + \text{c.p.}\right]\right\}, \tag{9.33}$$

which respects the Ward identity of Eq. (9.29) by virtue of the massless pole terms of Eq. (9.33). It should now be clear that these longitudinally coupled massless excitations will occur, as a result of enforcing gauge invariance, for every $n$-point function. We will shortly identify these with couplings of the gauged nonlinear sigma (GNLS) field introduced in our conjecture for the infrared-effective action.

So far, the vertex function $\Gamma_{ijk}$ is undetermined, but we will find an approximation to it, useful in the infrared, with the gauge technique. We can read off from Chapter 5 the needed relation

$$\Gamma_{ijk} = \delta_{ij}(k_1 - k_2)_k - \frac{k_{1i}k_{2j}}{2k_1^2k_2^2}(k_1 - k_2)_l\Pi_{lk}(k_3) \\ - [P_{il}(k_1)\Pi_{lj}(k_2) - P_{jl}(k_2)\Pi_{li}(k_1)]\frac{k_{3k}}{k_3^2} + \text{c.p.}, \tag{9.34}$$

where the first term on the right-hand side is the free vertex $\Gamma^0_{ijk}$, and $\Pi_{ij}(k) \equiv P_{ij}(k)\Pi(k)$ is the transverse PT self-energy, related to $\Omega_{ij}$ by

$$\Omega^2_{ij} = P_{ij}[\Omega_0^2 + \Pi\{\Omega\}], \tag{9.35}$$

where $\Omega_0^2 = k^2$ is the free gluon contribution.

In the simple case studied here, $\Pi = m^2$, and the resulting expression for $\Gamma_{ijk}$ is

$$\Gamma_{ijk} = \delta_{ij}(k_1 - k_2)_k + \frac{m^2}{2}\frac{k_{1i}k_{2j}(k_1 - k_2)_k}{k_1^2k_2^2} + \text{c.p.} \tag{9.36}$$

Combining the pinch technique and the gauge technique by solving the Ward identities ensures exact gauge invariance but is nonetheless an approximation (expected to be valid in the infrared regime). Ultimately, it yields a dressed-loop equation for a single transverse operator $\Omega_{ij}(k) \equiv P_{ij}(k)\Omega(k)$. We will not explore this difficult program further here.

The order-by-order appearance of massless longitudinal poles in the gauge-completion process is directly mirrored in the order-by-order solution of the classical GNLS model. Because the notation is more compact, we now switch to anti-Hermitean matrix notation. The local GNLS model, normalized appropriately, has the action

$$I_{\text{GNLS}} = -\frac{m}{g^2}\int \mathrm{d}^3x\, \mathrm{Tr}\,[U^{-1}\mathcal{D}_i U]^2, \tag{9.37}$$

where $U$ is a unitary matrix transforming as $U \to VU$ under the gauge transformation

$$\mathcal{A}_i \to V\mathcal{A}_i V^{-1} + V\partial_i V^{-1}. \tag{9.38}$$

The classical equations for $U$ express this quantity in terms of the $\mathcal{A}_i$ (see Chapter 7), with the result

$$\begin{aligned} U &= \mathrm{e}^{\omega} \\ \omega &= -\frac{1}{\nabla^2}\partial\cdot\mathcal{A} + \frac{1}{\nabla^2}\left\{\left[\mathcal{A}_i, \partial_i\frac{1}{\nabla^2}\partial\cdot\mathcal{A}\right] + \frac{1}{2}\left[\partial\cdot\mathcal{A}, \frac{1}{\nabla^2}\partial\cdot\mathcal{A}\right] + \cdots\right\}, \end{aligned} \tag{9.39}$$

showing the appearance of massless scalars. More generally, because $U^{-1}\mathcal{D}_iU$ is a gauge transformation of $\mathcal{A}_i$, functional integration over $U$ is equivalent to projecting the gauge-invariant part of the mass term. Note that the linear term in $\mathcal{A}_i$ of the GNLS model field $U^{-1}\mathcal{D}_iU$ is the transverse part of $\mathcal{A}_i$. This linear term is the Abelian mass term that began our investigations. All higher-order terms of $\omega$ in Eq. (9.39) are non-Abelian. One can straightforwardly verify that the three-point function of Eq. (9.36) corresponds precisely to the three-point term found by using the expansion of Eq. (9.39) in the GNLS model action. Because the GNLS action is fully gauge invariant, it gives one solution to the all-orders ghost-free Ward identities, and this solution is what is emerging from direct calculations using the gauge technique.

### *9.5.2 The proposed infrared-effective action*

Our proposed form of the effective action answer [28] is that in the infrared regime, where no momenta are large compared to the gluon mass $m$, this action is reasonably well approximated by a $d = 3$ action that is essentially the $d = 3$ massive effective action already studied in the last chapter. This action consists of a gauged, nonlinear sigma model mass term, giving the gluon a mass $m$ and the usual Yang–Mills term

$$I_{d=3} = -\int \mathrm{d}^3x\, \left\{\frac{m^2}{g_3^2}\mathrm{Tr}\,[U^{-1}\mathcal{D}_iU]^2 + \frac{1}{2g_3^2}\mathrm{Tr}\mathcal{G}_{ij}^2\right\}. \tag{9.40}$$

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One finds this form by saving the first two terms in an expansion of a certain approximation to $S$ in powers of the operator $-\nabla^2/m^2$ or, equivalently, $k^2/m^2$ ($k$ is a momentum). The leading term gives the gauged, nonlinear sigma model, and the next leading term gives the conventional Yang–Mills action. As momenta get larger, correction terms with more and more derivatives enter, and finally, in the region of large momenta, expanding in powers of $k^2/m^2$ is useless. Fortunately, because $d = 3 + 1$ QCD is asymptotically free, perturbation theory determines the leading large-momentum terms in $\psi$, but this is not of interest here.

Because this action must describe the same phenomena as, for example, the Schwinger–Dyson equations do, it must be that the gluon mass described by the effective action is the same in $d = 3, 4$. As for the $d = 3$ coupling $g_3^2$, we have already seen that $d = 3$ gauge dynamics determine the dimensionless ratio $m/g_3^2$, so knowing $m$ gives the $d = 3$ coupling.

To understand how such an action might arise, consider just the two-point term in $S$ of Eq. (9.25), called $S_2$. In perturbation theory, one can easily check that choosing

$$\Omega_{ij} = \Omega_0 P_{ij}, \quad \text{with} \;\; P_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \quad \text{and} \;\; \Omega_0 = \sqrt{-\nabla^2}, \tag{9.41}$$

solves the FSE for the free part of the action and also has the crucial property of gauge invariance (in this case, under Abelian $U(1)^{N^2-1}$ gauge transformations). To describe mass generation, make the simple replacement

$$\Omega_0 \to \Omega \equiv \sqrt{m^2 - \nabla^2} \tag{9.42}$$

so that $S_2$ is

$$S_2 = \frac{1}{2g^2} \int \mathcal{A}_i^a \sqrt{m^2 - \nabla^2} P_{ij} \mathcal{A}_j^a. \tag{9.43}$$

This $S_2$ is an exact solution of an FSE with an Abelian gauge Hamiltonian with gauge-invariant mass generation put in by hand:

$$\begin{aligned} H &= \int \left\{ -\frac{1}{2} g^2 \left( \frac{\delta}{\delta \mathcal{A}_i^a} \right)^2 + \frac{1}{2g^2} \left[ \frac{1}{2} (\mathcal{F}_{ij}^a)^2 + m^2 A_i^a P_{ij} \mathcal{A}_j^a \right] \right\} \\ &\equiv \int \left[ \frac{1}{2} \left( \Pi_i^a \right)^2 \right] + V, \end{aligned} \tag{9.44}$$

where $\mathcal{F}_{ij}^a = \partial_i \mathcal{A}_j^a - \partial_j \mathcal{A}_i^a$ are the Abelian field strengths.

Although this is a familiar Hamiltonian, closely related to that of the Abelian Higgs model, the action $I_{d=3} = 2S_2$ is not familiar, involving as it does a square root of

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an operator. Try the infrared expansion

$$\sqrt{m^2 - \nabla^2} \to \frac{1}{m}\left(m^2 - \frac{1}{2}\nabla^2\right) + \cdots . \tag{9.45}$$

Now we do see familiar operators, and saving these two terms in the infrared expansion of $I_{d=3} = 2S_2$ is almost a repeat of the Hamiltonian of Eq. (9.44), divided by $m$:

$$I_{d=3} \to \frac{m}{g^2}\int \mathcal{A}_i^a P_{ij} \mathcal{A}_j^a + \frac{1}{4mg^2}\int [\mathcal{F}_{ij}^a]^2 + \cdots . \tag{9.46}$$

Unfortunately, this action describes gluons of mass $\sqrt{2}m$ and not $m$ because of the 1/2 in the expansion of the square root. The problem is in trying to make a strict expansion around zero momentum when, in fact, momenta of $\mathcal{O}(m)$ are important. Reference [28] describes a least-squares operator approximation, intended to be more or less accurate over the range of momenta from 0 to $\mathcal{O}(m)$, of the form

$$\sqrt{m^2 - \nabla^2} \to \frac{Z}{m}\left(m^2 - \nabla^2 + \cdots\right), \tag{9.47}$$

where $Z$ is perhaps 1.1 or 1.2. This new approximation does describe gluons of mass $m$, as required.

The next step is to make a gauge completion of this Abelian form by adding the infinitely many terms in the expansion of $S$ in Eq. (9.25) that are required by gauge invariance. It turns out that for any given $\Omega$, there are infrared-useful approximations to all these terms that exactly preserve gauge invariance using the techniques of [11]. The first term in a large-mass expansion of this gauge completion is, as might be expected, equivalent to a gauged nonlinear sigma model mass term. *Equivalent* means that what one actually finds is the $d = 3$ version of the perturbative expansion of this model, as given in Eq. (9.39). The second is the usual Yang–Mills term involving the full field strengths $\mathcal{G}_{ij}^a$, as described in Eq. (9.40). The same problem arises as in the Abelian case – the free mass is $\sqrt{2}M$ – and is approximately resolved in the same way as indicated for the Abelian theory with a modified infrared expansion. The final result is the obvious modification of the Abelian $S_2$:

$$\begin{aligned} -2S &= -I_{d=3} \\ &\to \frac{2mZ}{g^2}\int d^3x\, \mathrm{Tr}\,[U^{-1}\mathcal{D}_i U]^2 + \frac{Z}{mg^2}\int d^3x\, \mathrm{Tr}\,\mathcal{G}_{ij}^2 + \mathcal{O}(M^{-3}). \end{aligned} \tag{9.48}$$

The next question is: what are the consequences of this action? Three are worth mentioning [28]. First, we know already that it has center vortices and nexuses. The center vortices are closed strings, corresponding to the projection of closed

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surfaces in $d = 4$ onto $d = 3$; similarly, the nexuses are points on these strings. Given an entropy-driven condensate of vortices, these will describe confinement through matrix elements of the FSE.

Second, given values of $Ng_3^2/m$ (see Section 9.4), as found strictly in three dimensions, we can actually estimate the on-shell value of the $d = 4$ coupling $\alpha_s(m^2) \equiv g^2/4\pi$. Just compare the two forms of the conjecture, as stated in Eqs. (9.40) and (9.48), and find the equation:

$$g^2 = \frac{2Zg_3^2}{m}. \tag{9.49}$$

This equation expresses a $d = 4$ quantity, $g^2$, in terms of the $d = 3$ ratio $m/g_3^2$, estimates for which we summarized in Table 9.2. Using $Z \simeq 1.2$ and the estimate [16] $Ng_3^2/m \simeq 7.7$ for $N = 3$ gives $\alpha_s(M^2) \simeq 0.5$, a value holding for no quarks. This is close both to an early estimate [4] that comes out of the first PT attempt to find the gluon mass and to modern estimates given in Chapter 6. The early analytic PT estimate is

$$\alpha_s(M^2) = \frac{g^2}{4\pi} = \frac{12\pi}{[11N - 2N_f]\ln[5M^2/\Lambda^2)]} \simeq 0.4, \tag{9.50}$$

where the numerical value comes from $m = 0.6$ GeV, $\Lambda = 0.3$ GeV, and no quarks ($N_f = 0$). Chapter 6 gives a value $\simeq$0.5. We can also compare this result to phenomenological determinations [29, 30, 31] of $\alpha_s(q^2 \simeq 0) \simeq 0.7 \pm 0.3$ coming from studies of infrared-sensitive scattering data. But in the real world to which these data apply, there are three families of light quarks, so we have to modify the FSE estimate. Assuming that the PT formula of Eq. (9.50) applies, we should multiply the result of Eq. (9.49) by 11/9, ending up with an FSE estimate of $\alpha_s(m^2)$ of about 0.6 – near the lower end of the phenomenological range.

The third consequence of this FSE work comes from using it in one less dimension. The same steps go through, yielding an effective action in two dimensions, $I_{d=2}$, that is once again the sum of a gauged nonlinear sigma model and a Yang–Mills term. This action has center vortex solutions, and if they condense, they give confinement as usual. Note the big difference with the usual confinement mechanism in $d = 2$ QCD, which just has the Yang–Mills term. The Yang–Mills action by itself is a free-field theory with a confining propagator, and so all nontrivial group representations of $SU(N)$ are confined. But this is not correct for $d = 3$, the dimensionality where $I_{d=2}$ is supposed to apply; the adjoint and similar representations are not confined but screened. Therefore, it is essential to have the mass term in $I_{d=2}$.

There is another way of creating gluon mass in $d = 3$, which works even in the classical theory: add a CS term to the action.

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## 9.6 Dynamical gluon mass versus the Chern–Simons mass: Two phases

In Chapter 8, we already encountered the CS integral as a time slice of the topological charge density. We repeat the definition of the CS number $N_{\mathrm{CS}}$:

$$N_{\mathrm{CS}} = -\frac{1}{8\pi^2}\int \mathrm{d}^3x\, \epsilon_{ijk}\mathrm{Tr}\left(\mathcal{A}_i\partial_j\mathcal{A}_k + \frac{2}{3}\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\right). \tag{9.51}$$

Form the Yang–Mills–Chern–Simons (YMCS) action by adding the CS action $2\pi i k N_{\mathrm{CS}}$ to the Yang–Mills action. The $d = 3$ functional integral with a CS term (as always, omitting the gauge-fixing and ghost terms) with a new coupling $k$, chosen to be real, is

$$Z = \int[\mathrm{d}\mathcal{A}_i]\exp\left[2\pi i k N_{\mathrm{CS}} + \int \mathrm{d}^3x \frac{-1}{2g_3^2}\mathrm{Tr}\,\mathcal{G}_{ij}^2\right]. \tag{9.52}$$

We can and will always choose the level $k$ to be positive by changing the sign of the gauge potential, that is, by making a parity transformation. In fact, because either sign contributes equally in $Z$, the partition function is an even function of $k$. The factor of i in the Euclidean action comes about because in transforming from Minkowski space (where all actions have an i factor in the path integral) to Euclidean space, no extra factor of i arises, as it usually does. So for real gauge potentials, the resulting pure imaginary action contributes a phase factor to the path integral. But once there is an imaginary part to the Euclidean action, there is no longer any requirement that the dominant contributions to the path integral come from real gauge potentials, and the CS action is generally complex. The partition function is real because there are equal contributions from a complex CS action and its complex conjugate.

The CS term is not gauge invariant under a so-called large-gauge transformation, the kind that carries topological charge. According to Eq. (8.32), the CS term changes by the integer $\mathcal{N}$ of that gauge transformation. We do not need to require that the action itself be gauge invariant; just as with Dirac monopoles, only the functional integrals created from it, such as $Z$, must be gauge invariant. That requires [32, 33] the coupling $k$ to be an integer.[10] This integer is called the *level* of the CS action.

In perturbation theory, the main effect of adding the CS term is that the gluon acquires a mass $m_{\mathrm{CS}}$. This follows from the easily established Euler–Lagrange variation of the CS term:

$$\frac{\delta N_{\mathrm{CS}}}{\delta\mathcal{A}_i(x)} = \frac{1}{16\pi^2}\epsilon_{ijk}\mathcal{G}_{jk}(x) \equiv \frac{1}{8\pi^2}\mathcal{B}_i, \tag{9.53}$$

[10] One might wonder what happens if $k$ is other than integral. If the only large gauge transformations allowed in the path integrals change the CS number by an integer, then the partition function $Z$ vanishes for $k$ nonintegral. We discuss in Section 9.7 [34] that there is no absolute prohibition on considering nonintegral CS numbers, but a theory accommodating nonintegral CS numbers is energetically disfavored.

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from which the classical equations of motion for YMCS theory are

$$[\mathcal{D}_i, \mathcal{G}_{ij}] - \frac{\mathrm{i}kg_3^2}{8\pi}\epsilon_{jkl}\mathcal{G}_{kl} = 0. \tag{9.54}$$

This is a peculiar equation because it is complex. However, it has a perfectly fine perturbative expansion. The linearized version of Eq. (9.54) is

$$\epsilon_{ijk}\partial_j\mathcal{B}_k = \mathrm{i}m_{\mathrm{CS}}\mathcal{B}_i, \tag{9.55}$$

where the CS mass is

$$m_{\mathrm{CS}} = \frac{kg_3^2}{4\pi}, \tag{9.56}$$

and $\mathcal{B}_i \equiv (1/2)\epsilon_{ijk}\mathcal{G}_{jk}$ is the magnetic field. Taking the curl of Eq. (9.55) gives

$$(\nabla^2 - m_{\mathrm{CS}}^2)\mathcal{B}_i = 0. \tag{9.57}$$

Precisely because there is an i in front of the CS term in the equations of motion, this linear propagation equation is nontachyonic and corresponds to a gluon of physical mass $m_{\mathrm{CS}}$, although it is associated with the peculiarities of complexness and parity violation.

We cannot immediately conclude that this mass, present in perturbation theory, removes the infrared instability of ordinary Yang–Mills theory. It turns out, as we will see using the pinch technique, that if the mass is large enough – that is, if $k$ is large enough – infrared slavery is indeed gone. The perturbative expansion parameter $Ng_3^2/m_{\mathrm{CS}}$ behaves like $N/k$, and so large-$k$ perturbation theory should be well defined, as it is in QED. If perturbation theory is to work, there should be no classical solitons – a result proven long ago [35]. This large-$k$ theory, which is in effect the theory without the Yang–Mills term, is a particularly beautiful and mathematically powerful theory that is exactly soluble and beautifully organizes some of the mathematics of $d = 3$ knots [2].

However, if $k$ is small enough, we will show that the CS mass is not large enough to remove infrared-slavery tachyons and that a dynamical gluon mass is required as well. Quantum solitons return [36] along with the expected nonperturbative effects, including confinement. There is a phase transition in YMCS theory at a value $k = k_c$, with $k_c \simeq (1-2)N$. For $k > k_c$, perturbation theory and Witten's results hold, whereas for smaller $k$, infrared slavery must be solved the way we have presented in this book. Of course, this nonperturbative phase is different from that of QCD because of the parity-violating CS term, and the solitons differ in detail. But there are still center vortices, nexuses, and sphalerons.

It is natural, from a physics point of view, to start with the Yang–Mills action as fundamental – something to which we add the CS term. But Witten [2] showed that the theory defined by dropping the Yang–Mills action and keeping only the CS term is not only sensible but is an example of a particularly interesting class of field theories called *topological field theories*. The theory with only the CS term – called CS theory – has no propagating gluonic modes and in fact can be solved exactly. Its observables are completely characterized by the topologically invariant CS numbers and the VEVs of other topological invariants such as Wilson loops. Ultimately it turns out that the phase space of CS theory is finite. Witten showed how the VEVs of multiple, knotted Wilson loops can be calculated to yield the Jones polynomials that characterize the linkings and knottings of the loops. For large $k$, Witten looks at a semiclassical expansion around the classical extrema of the action that, according to Eq. (9.53), comes from configurations of vanishing field strength or pure-gauge potentials. Canonical quantization of the theory requires a choice of gauge; the choice $\mathcal{A}_3 = 0$ reduces the action to a quadratic form, and that is, in part, why the theory is exactly soluble. We will not pursue this fascinating topic any further, except to say that to define rigorously pure CS theory requires a regulator, and the obvious one at large $k$ is the Yang–Mills term. The next sections will focus on the infrared-unstable phase.

### 9.6.1 The nonperturbative phase uncovered by the pinch technique

First, we make a heuristic argument. Because $Z$ of Eq. (9.52) is even in $k$, we can write it as

$$Z = \int [\mathrm{d}\mathcal{A}_i] \cos(2\pi \mathrm{i} k N_{\mathrm{CS}}) \exp\left[-\int \mathrm{d}^3x \frac{1}{2g_3^2} \mathrm{Tr}\mathcal{G}_{ij}^2\right] \equiv \mathrm{e}^{-\Gamma(\theta,k)}. \qquad (9.58)$$

In the formal limit of small $k$, this equation says that

$$\Gamma(\theta, k) \simeq \Gamma(\theta) + 2\pi^2 k^2 \langle N_{\mathrm{CS}}^2 \rangle; \qquad (9.59)$$

the CS term increases the effective action and at some point can be expected to overcome the entropic effects that tend to make $\Gamma$ negative. So there might be a phase transition at some value of $k = k_c$ of $\mathcal{O}(N)$ separating a nonperturbative phase with all the usual phenomena (gluon mass, a condensate, solitons) from a perturbative phase.

The first step is to calculate the one-loop PT propagator $\widehat{\Delta}(p)_{ij}$ for YMCS theory. The corresponding bare propagator has a new parity-violating term:

$$\widehat{\Delta}_0^{-1}(p)_{ij} = (p^2\delta_{ij} - p_i p_j) + m_{\mathrm{CS}}\epsilon_{ija} p_a + \frac{1}{\xi} p_i p_j; \qquad (9.60)$$

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here $m_{\text{CS}} = kg_3^2/4\pi$ is the classical CS mass and $\xi$ is a gauge-fixing parameter. The PT self-energy, which enters the full propagator through

$$\widehat{\Delta}^{-1}(p)_{ij} = \Delta_0^{-1}(p)_{ij} - \widehat{\Pi}(p)_{ij}, \tag{9.61}$$

also has two conserved terms, a parity-conserving term and a parity-violating term:

$$\widehat{\Pi}(p)_{ij} = (p^2\delta_{ij} - p_i p_j)\widehat{A}(p) + m_{\text{CS}}\epsilon_{ija} p_a \widehat{B}(p). \tag{9.62}$$

These equations yield the propagator

$$\begin{aligned}\widehat{\Delta}(p)_{ij} = &\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)\frac{1}{(1-\widehat{A})(p^2 + m_{\text{R}}^2)}\\ &- m_{\text{R}}\epsilon_{ija} p_a \frac{1}{p^2(1-\widehat{A})(p^2+m_{\text{R}}^2)} + \xi \frac{p_i p_j}{p^4}\end{aligned} \tag{9.63}$$

in terms of a running mass

$$m_{\text{R}}(p) = m_{\text{CS}}\left(\frac{1-\widehat{B}}{1-\widehat{A}}\right). \tag{9.64}$$

A lengthy calculation [36] gives equally lengthy results for $\widehat{A}(m_{\text{CS}}/p)$, $\widehat{B}(m_{\text{CS}}/p)$, and we will not quote them in full here. Both positive and negative powers of $m_{\text{CS}}/p$ appear, but owing to cancellations, the propagator is finite both in the $m_{\text{CS}} = 0$ limit and in the $p = 0$ limit. One simple-looking result is the one-loop PT propagator in the limit of no CS mass. One might expect this to reduce to the usual QCD expression of Eq. (9.16), but in the limit $m_{\text{CS}} = 0$, there is a term $\sim 1/m_{\text{CS}}$ that leads to a cancellation:

$$\widehat{\Delta}^{-1}(p)_{ij} = (p^2\delta_{ij} - p_i p_j)(1 - \pi b g_3^2 p) + \epsilon_{ija} p_a g_3^2 \left(\frac{k+N}{4\pi}\right). \tag{9.65}$$

Note the replacement of $k$ by $k + N$, which happens also in Witten's pure CS topological theory. The $N$ here arises from the mass cancellation. There is also the infrared-slavery term already uncovered in Eq. (9.16), with $b_3 = 15N/(32\pi)$.

So the question now is whether some finite value of $m_{\text{CS}}$ can overcome the infrared slavery problem. By looking at the full expressions for $\widehat{A}$, $\widehat{B}$, one can check that the relevant self-energy $\widehat{A}$ is positive and monotone decreasing in momentum $p$, vanishing like $1/p$ at large momentum. So the infrared-slavery problem is solved if $\widehat{A}(p = 0)$ is less than 1. The result for this quantity is as follows:

$$1 - \widehat{A}(p = 0) = 1 - \frac{29N}{12k}. \tag{9.66}$$

It then follows that, at the one-loop level, YMCS theory is consistent and free of tachyons only if $k$ is larger than a critical value $k_c$, where

$$k_c = \frac{29N}{12}. \tag{9.67}$$

What happens for higher loops has not yet been studied, but a fair guess is that, for example, the denominator $k$ in Eq. (9.66) would be replaced by $k + N$, in which case $k_c$ would be $17N/12$. If so, these two values of $k_c$ suggest that we know the critical value of $k_c$ to within a factor of 2 and that higher loops do not change the fact that there is a critical value.

So the infrared slavery problem persists if $k < k_c$, in which case, we solve it just as before: there has to be a dynamical gluon mass $m$ generated, and this mass generation is self-consistently supported by condensates of solitons of the massive effective action. If so, the resulting infrared-effective action has both a CS term and a GNLS term (see Eq. (9.71)). In the following, we argue that as $k \to k_c$ from below, the dynamical gluon mass along with the condensate supporting it must vanish, and the solitons composing the condensate no longer exist. There are [36] qualitative arguments suggesting that the exact form, given in Eq. (9.10), of the zero-momentum effective action as a function of the operator $\theta$ of Eq. (9.3) gets modified in a certain way by a CS term. This modified effective action $\Gamma(\theta, k)$ has all the right qualitative properties, including correct scaling in $N$ and $g$ for all quantities appearing in it, a dynamical gluon mass consistent with the operator product expansion of Eq. (9.4), a phase transition at a critical value of $k$ at which the condensate vanishes, a quadratic increase in $\Gamma$ as a function of $k$ for small $k$, and the correct zero-$k$ limit. We will not detail the arguments, all based on the one-loop equations given so far, but will simply state the result here:

$$\Gamma(\theta, k) = \int \theta \left\{ 1 - k_c \left[ k^2 + \left( \frac{4\pi}{g_3^2} \right)^2 \left( a_3 g_3^2 \theta \right)^{1/2} \right]^{-1/2} \right\}; \tag{9.68}$$

here $a_3$ is the Lavelle constant of Eq. (9.4), and the critical value $k_c$ is proportional to $N$. The pure numbers in this expression are not to be taken seriously.

Now let us check the properties of this modified effective action. First, in the limit $k = 0$, it is of the necessary form given in Eq. (9.10), with $\langle\theta\rangle \simeq (Ng_3^2)^3(N^2 - 1)$. Second, for small $k$, the leading correction term is positive and quadratic, as expected from Eq. (9.59). Third, at $k = k_c$, the minimum of $\Gamma$ moves to $\theta = 0$ so there is no condensate, whereas for positive $\theta$, the effective action $\Gamma$ is also positive, indicating that entropy effects are no longer dominant.

The order parameter for this phase transition is the dynamical gluon mass $m(k)$, which now depends on $k$. When $k \lessapprox k_c$, some simple algebra shows that the

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minimum of $\Gamma$ in Eq. (9.68) obeys the following:

$$\theta^{1/2} \sim (k_c - k), \tag{9.69}$$

and so

$$m(k) \sim \theta^{1/4} \sim (k_c - k)^{1/2}, \tag{9.70}$$

characteristic of a second-order phase transition. If there really is a phase transition at $k = k_c$, then one would expect solitons to appear for smaller $k$.

### 9.6.2 YMCS solitons

D'Hoker and Vinet [35] long ago looked for classical solitons of YMCS theory. Their result is that there are no finite-action classical solitons of the vortex or sphaleron type in classical YMCS theory. There are solitons, but they have a curious instability that creates a singularity, preventing them from having finite action.

The remaining questions are as follows: are there any finite-action solitons when there is both a dynamical mass $m$ and a CS mass in the effective action? How do these solitons behave when $m \to 0$? For $k < k_c$, the full effective action with dynamical mass term is

$$\int \mathrm{d}^3x \left[2\pi \mathrm{i} k N_{\mathrm{CS}} - \frac{1}{2g_3^2}\int \mathrm{Tr}\,\mathcal{G}_{ij}^2 - \frac{m^2}{g_3^2}\int \mathrm{Tr}[U^{-1}\mathcal{D}_i U]^2\right]. \tag{9.71}$$

First, let us look for center vortices using the classical equations from Eq. (9.71). Center vortices are Abelian, and this action leads to the Abelian solution [36]:

$$\begin{aligned}\mathcal{A}_i^a(x) = \frac{2\pi Q^a}{\mu}\oint \mathrm{d}z_k \big\{&\epsilon_{ijk}\partial_j\left[\mu_-(\Delta_+(x-z) - \Delta_0(x-z)) + (+\leftrightarrow -)\right] \\ &+ \mathrm{i}\delta_{ik}\mu_+\mu_-\left[\Delta_+(x-z) - \Delta_-(x-z)\right]\big\},\end{aligned} \tag{9.72}$$

where the masses $\mu$, $\mu_\pm$ are

$$\mu_\pm = \frac{1}{2}[\pm m_{\mathrm{CS}} + (m_{\mathrm{CS}}^2 + 4m^2)^{1/2}] \qquad \mu = \mu_+ + \mu_-, \tag{9.73}$$

and $\Delta_\pm$ is the free Feynman propagator for mass $\mu_\pm$. This is a peculiar soliton because it has a (parity violating) imaginary part. But the total action, including the CS term, is real, and the action per unit length is finite (given that the running mass $m(q)$ decreases as in Eq. (9.4)). This soliton is twisted and has a nonzero $\langle N_{\mathrm{CS}}\rangle$. The twist (or $N_{\mathrm{CS}}$) changes sign when $k$ changes sign, and so the action is even in

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$k$. In the limit $m_{\text{CS}} \to 0$, this vortex reduces to the usual center vortex, and in the limit $m \to 0$, the soliton vanishes.[11]

Next, look for spherical solitons in $SU(2)$, which should resemble modified sphalerons. The usual spherical decomposition of the gauge potential is

$$2\mathrm{i}\mathcal{A}_i = \epsilon_{iak}\tau_a\widehat{x}_k\left[\frac{\phi_1(r)-1}{r}\right] - (\tau_i - \widehat{x}_i\widehat{x}\cdot\vec{\tau})\frac{\phi_2(r)}{r} + \widehat{x}_i\widehat{x}\cdot\vec{\tau}H_1(r) \quad (9.74)$$

$$U = \exp\left[\mathrm{i}\beta(r)\frac{\vec{\tau}\cdot\widehat{x}}{2}\right]. \quad (9.75)$$

Inserting these into the effective YMCS plus mass action of Eq. (9.71) gives [36, 34] what looks like four equations of motion, one for each of the four functions in Eq. (9.74):

$$\begin{aligned} 0 &= (\phi_1' - H_1\phi_2)' + \frac{1}{r^2}\phi_1(1-\phi_1^2-\phi_2^2) \\ &\quad + (\mathrm{i}m_{\text{CS}} - H_1)(\phi_2' + H_1\phi_1) - m^2(\phi_1 - \cos\beta) \end{aligned} \quad (9.76)$$

$$\begin{aligned} 0 &= (\phi_2' + H_1\phi_1)' + \frac{1}{r^2}\phi_2(1-\phi_i^2-\phi_2^2) \\ &\quad - (\mathrm{i}m_{\text{CS}} - H_1)(\phi_1' - H_1\phi_2) - m^2(\phi_2 + \sin\beta) \end{aligned} \quad (9.77)$$

$$\begin{aligned} 0 &= \phi_1\phi_2' - \phi_2\phi_1' + H_1(\phi_1^2+\phi_2^2) \\ &\quad + \left(\mathrm{i}m_{\text{CS}}(1-\phi_1^2-\phi_2^2) + \frac{1}{2}m^2r^2(H_1-\beta'\right) \end{aligned} \quad (9.78)$$

$$0 = \frac{1}{r^2}[r^2(\beta'-H_1)]' - \frac{2}{r^2}(\phi_1\sin\beta + \phi_2\cos\beta). \quad (9.79)$$

Here primes indicate radial derivatives. In fact, there are only three independent equations; Eq. (9.78), which comes from varying the matrix $U$, is (as we already know) not independent and can be derived from the other three.

Why are there no classical solitons (at $m = 0$) but there are (quantum) solitons for finite $m$? To a large extent, the answer to this question appears in Eq. (9.77) for the amplitude $H_1$ or, equivalently, $A$ of Eq. (9.82). This equation is algebraic, not differential, and has the solution

$$A = \frac{1}{\phi_1^2 + \frac{m^2r^2}{2} - \frac{m_{\text{CS}}^2B^2}{m^2}}\left[\frac{1}{m}(B\phi_1' - B'\phi_1) + 1 - \phi_1^2 + \frac{m_{\text{CS}}^2}{m^2}B^2\right]. \quad (9.80)$$

[11] Because the Abelian equations are linear, there are other linear combinations of the preceding $\pm$ solutions without this property, but they do not have finite action per unit length.

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D'Hoker and Vinet [35] have an analogous equation, but in the gauge $B = 0$ and with no dynamical mass, so their equation is recovered by setting $B$, $B/m$ and $m$ to zero. Their denominator, then, is just $\phi_1^2$. They show that there is at least one zero of this denominator and that the existence of one zero leads to an infinite number of zeros and a "soliton" having an accumulation point of zeroes at $r = 0$. In our case, if $m$ is large enough, it is possible that the $m^2r^2/2$ term in the denominator prevents the denominator from vanishing, and this does happen, at least numerically [36]. The numerics show that for small enough $m$, there is at least one zero, and the D'Hoker–Vinet disease arises: there are no sphaleron-like solitons.

So several different lines of investigation, all of them qualitative, lead to the same conclusion: for large $k$, YMCS theory is in the Witten phase and can be solved exactly, but for $k < k_c$, with $k_c \simeq (1-2)N$, there is a second-order phase transition to a nonperturbative phase with a dynamical gluon mass in addition to the CS mass.

## 9.7 Compactness and the Chern–Simons number of YMCS solitons

The developments so far provide a setting for investigating whether the assumption of compactness, which quantizes various topological indices, is physically necessary [34]. We know already that topological charge may consist of localized lumps of nonintegral charge whose sum over all Euclidean space-time is integral, but this does not challenge the notion of compactness, which is only needed for infinite spaces and their boundaries at infinity. Compactness requires that quantum numbers defined on such boundaries be integral, but here we assume otherwise and look for the consequences, using a model of a dilute condensate of YMCS sphalerons. The result is that the noncompact model has a vacuum energy density higher than that of the compact theory by a finite amount and hence a vacuum energy higher by an infinite amount after integrating over all three-space [34]. So compactness is energetically preferred.

The model begins with sphalerons from Eqs. (9.76), (9.76), (9.77), and (9.78), with boundary conditions

$$\begin{aligned} r = 0:&\ \phi_1(0) = 1, \qquad \phi_2(0) = H_1(0) = \beta(0) = 0; \\ r = \infty:&\ \phi_1(\infty) = \cos\beta(\infty), \qquad \phi_2(\infty) = -\sin\beta(\infty), \qquad H_1 \to \beta'. \end{aligned} \tag{9.81}$$

These equations are again complex, but in one case, they can be reduced to real equations for real functions, and this is the case of interest. Set $\beta = \pi$ and $\alpha = 0$. Then one can choose $H_1$, $\phi_2$ to be pure imaginary, with all other functions real:

$$H_1 = \mathrm{i}m_{\text{CS}}A(r); \qquad \phi_2 = \mathrm{i}\frac{m_{\text{CS}}}{m}B(r). \tag{9.82}$$

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The equations become real, and so $\phi_1$, $A$, $B$ are all real. Generally, solitons with both real and imaginary field components have conjugate solitons found by complex conjugation (which changes the sign of the CS number), but the soliton here is self-conjugate, and its ordinary CS number vanishes. For this choice of boundary conditions, the CS action $2\pi ikN_{\text{CS}}$ is real (the integral in Eq. (9.51) is imaginary). The general form of $N_{\text{CS}}$ for a spherical soliton is

$$N_{\text{CS}} = \frac{1}{8\pi^2}\int \frac{\mathrm{d}^3x}{r^2}\,[\phi_1\phi_2' - \phi_2\phi_1' - \phi_2' - H_1(1-\phi_1^2-\phi_2^2)]. \tag{9.83}$$

Substitute the forms of Eq. (9.82) to find a purely imaginary $N_{\text{CS}}$ and so a purely real CS action. (There is no reason that the contribution of solitons to $N_{\text{CS}}$ should be integral or even real.) This action is $\mathcal{O}(k^2)$ and positive for small $k$, as we argued earlier on general grounds.

The only interpretation we can make of a pure imaginary $N_{\text{CS}}$ is that what we usually think of as the (topological) CS number vanishes. This soliton is very much like the QCD sphaleron of Chapter 7, which, considered only as a $d=3$ soliton, has no CS number. However, by making a special gauge transformation on this solution, we can endow it with a genuine (and nonintegral) CS number. The spherical equations of motion have a residual $U(1)$ gauge invariance that preserves spherical symmetry:

$$\begin{aligned}
\phi_1(r) &\to \phi_1(r)\cos\alpha(r) + \phi_2(r)\sin\alpha(r)\\
\phi_2(r) &\to \phi_2(r)\cos\alpha(r) - \phi_1(r)\sin\alpha(r)\\
\beta(r) &\to \beta(r) + \alpha(r)\\
H_1(r) &\to H_1(r) + \alpha'(r).
\end{aligned} \tag{9.84}$$

These transformations can be read off from the gauge transformation:

$$\mathcal{A}_i^a \to V\mathcal{A}_i^a V^{-1} + V\partial_i V^{-1}, \tag{9.85}$$

with[12]

$$V(\alpha) = \exp\left[\frac{\mathrm{i}}{2}\vec{\tau}\cdot\hat{r}\alpha(r)\right]. \tag{9.86}$$

Of course, the subgroup generated by all group elements of the form in Eq. (9.86) is Abelian, but it does not commute with the general vector potential. We call gauge transformations of the type in Eq. (9.86) spherical gauge transformations.

For any gauge transformation, as in Eq. (9.85), $N_{\text{CS}}$ changes according to Eq. (8.33). With the assumption of compactness, the gauge transformation $V$ approaches the identity on the sphere at infinity, and the gauge potential $\mathcal{A}_i$ vanishes at least as

[12] To avoid singularities at the origin, choose $\alpha(0) = 0$.

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fast as $1/r$, and so the change in $N_{\text{CS}}$ reduces to the winding number of Eq. (9.87), with the added requirement that $\alpha(r = \infty) = 2\pi L$ for an integer $L$. When $V$ approaches $I$ on the sphere at infinity, the space of gauge transformations is really defined on the three-sphere $S^3$ rather than on $\mathcal{R}_3$ because all the points at infinity are mapped to a single point. This integral winding number is that of the map of the group space $S^3$ onto the spatial $S^3$ that we just identified, or in other words, the homotopy $\Pi^3(S^3) \simeq Z$, and this winding number is $L$. The winding number is topological, which means two things: it is independent of a choice of metric, and it can be expressed as a boundary-value integral. It is not completely elementary to find the function whose divergence is the winding-number integrand; the answer is in the work of Deser et al. [32] for general gauge transformations.[13] For the spherical gauge transformation of Eq. (9.86), a straightforward calculation (easiest with Eq. (9.83)) gives

$$\frac{1}{8\pi^2}\int \epsilon_{ijk}\,\mathrm{Tr}\frac{1}{3}V(\alpha)^{-1}\partial_i V(\alpha)V^{-1}(\alpha)\partial_j V(\alpha)V^{-1}(\alpha)\partial_k V(\alpha)$$
$$= \frac{1}{2\pi}\int_0^\infty \mathrm{d}r\,\alpha'(r)[1-\cos\alpha(r)] = \frac{1}{2\pi}\,[\alpha(r)-\sin\alpha(r)]|_0^\infty\,. \tag{9.87}$$

The answer depends only on the boundary value $\alpha(r = \infty)$ because we choose $\alpha(0) = 0$.

Now we abjure compactness and let $\alpha(r = \infty)$ be arbitrary. There is a (real) CS number that is not integral. A particularly interesting case removes an integrable singularity arising from $\beta(0) \neq 0$; this singularity can be removed by invoking a spherical gauge transformation with $\alpha(0) = -\pi,\ \alpha(\infty) = 0$. The CS number is $1/2$, just as we would expect for a sphaleron.

We assume that for entropic reasons, there is a dilute (noninteracting) condensate of sphalerons in the vacuum so that all solitons are essentially independent. When a CS term is present in the action, the dilute-gas partition function $Z$ is the usual expansion as a sum over sectors of different sphaleron number:

$$Z(k) = \sum_J Z_J \qquad Z_J(k) = \sum_{\text{c.c.}} \frac{1}{J!}e^{-\sum I_c} + \cdots, \tag{9.88}$$

where $Z_J(k)$ is the partition function in the sector with $J$ sphalerons; the subscript c.c. indicates a sum over collective coordinates of the sphalerons; $I_c$ is the action (including CS action) of a sphaleron, and the omitted terms indicate corrections to the dilute-gas approximation. To be explicit, separate the sum over collective

[13] We know without calculation that the winding number for spherical gauge transformations has to be a total divergence because it changes the action without changing the equations of motion.

coordinates into kinematic coordinates, such as spatial position (the $a$th soliton is at position $\vec{r} - \vec{a} \equiv \vec{r}(a)$) and gauge-collective coordinates. The former we represent in the standard dilute-gas way; the latter, we indicate as a functional integral over spherical gauge transformations $U$:

$$Z(k) = \int [\mathrm{d}U] \sum \frac{1}{J!} \left(\frac{V}{V_c}\right)^J \exp\left\{-J\Re\mathrm{e}\, I_c - 2\pi \mathrm{i} k \left[J N_{\mathrm{CS}}(\mathcal{A}) + N_{\mathrm{CS}}(U)\right]\right\}. \quad (9.89)$$

Here $V$ is the volume of all three-space; $V_c$ is a finite collective-coordinate volume; $\Re\mathrm{e}\, I_c$ is the real part of the action; $N_{\mathrm{CS}}(\mathcal{A})$ is the CS number of each individual soliton of gauge potential $\mathcal{A}$; and $N_{\mathrm{CS}}(U)$ is the CS number of the large gauge transformation.

Even when we consider the apparently innocuous case of sphalerons of CS number $1/2$, choose $k$ integral, and allow only compact gauge transformations, problems arise. The $N_{\mathrm{CS}}$ contribution to a term in the sum in the partition function is a phase factor $\exp(\mathrm{i}\pi k J)$, which is –1 if both $k$ and $J$ are odd. When $k$ is odd, the odd $J$ terms in $\ln Z$ have the opposite sign to those of a normal dilute-gas condensate, which means that the free energy, which for a normal dilute-gas condensate is negative, has turned positive. So the noncompactified theory splits into two sectors, one with even numbers of sphalerons and the other with odd numbers, and the odd-number sector has infinitely higher free energy than the (compactified) even-number sector. (Noncompactification also leads to a number of other unphysical results in the dilute-gas approximation not considered here.)

Now generalize to arbitrary noncompact gauge transformations. Begin with potentials $\mathcal{A}$ (all indices suppressed) of the self-conjugate soliton given earlier, satisfying the equations of motion with boundary conditions of Eq. (9.81) and fixed to a standard spherical gauge (to be specific, we use the self-conjugate soliton with zero CS number). Introduce a different CS number for each soliton (labeled by a set of indices $a$) by making a spherical gauge transformation characterized by $\alpha(a;\infty)$ for the $a$th soliton. If we assume compactness, these gauge transformations, with integral CS numbers, have no effect (as long as the CS index $k$ is integral). But we give up compactness, so these gauge transformations do have an effect, and each $\alpha(a;\infty)$ is a collective coordinate.

Because the total CS number of all $J$ sphalerons comes from a surface contribution, we can immediately write the phase factor in the action by using Eq. (9.87):

$$Z(k) = \sum_J \frac{1}{J!} \left(\frac{V}{V_c}\right)^J \exp[-J\Re\mathrm{e}\, I_c] \exp[\mathrm{i}k(\alpha - \sin\alpha)], \quad (9.90)$$

where

$$\alpha = \sum_{a=1}^{J} \alpha(a;\infty). \tag{9.91}$$

The $\alpha(a;\infty)$ are collective coordinates, and we integrate over them:

$$Z(k) = \sum_J Z_{RJ} \times \left\{ \prod_a \int_0^{2\pi} \frac{\mathrm{d}\alpha(a;\infty)}{2\pi} \right\} \exp[\mathrm{i}k(\alpha - \sin\alpha)], \tag{9.92}$$

where $Z_{RJ}$ indicates the explicitly real terms in the summand of Eq. (9.90). This integral is reduced to a product by using the familiar Bessel identity

$$\mathrm{e}^{\mathrm{i}z\sin\theta} \equiv \sum_{-\infty}^{\infty} J_N(z) e^{\mathrm{i}N\theta}, \tag{9.93}$$

and the integral becomes, for integral $k$, $[J_k(k)]^J$. So the dilute-gas partition function is

$$Z(k) = \sum_J \frac{1}{J!} \left( \frac{V}{V_c} \right)^J \exp\left\{ J[-\Re\mathrm{e}\, I_c + \ln J_k(k)] \right\} = \exp\left[ \frac{V}{V_c} e^{-\Re\mathrm{e}\, I_c} J_k(k) \right]. \tag{9.94}$$

Because $1 \geq J_k(k) > 0$ for all levels $k$, integrating over the collective coordinates has increased the free energy (the negative logarithm of $Z$). This shows that by compactifying the sphalerons, we lower the free energy, yielding something like the usual dilute-gas partition function (which is Eq. (9.94) without the $J_k(k)$ factor). This simply requires that the total CS number $\alpha$ be an integer, not that each contributing soliton have integral CS number.

There are a number of other issues concerning compactness that we will not discuss here; for example, what happens if the CS level $k$ is nonintegral [34]? The upshot is that compactness is more than just a mathematical assumption because compact theories always have infinitely less free energy than noncompact theories.

### *9.7.1 Sphalerons, knots, and compactness*

We conclude this chapter by noting [34] that sphalerons of CS number 1/2 can be topologically mapped onto the linkages of $d = 3$ knots (see Kaufmann [37]) – the same sort of linkages that occur between center vortices and Wilson loops. For compact knots (knots whose links are closed strings), the total link number is integral and is composed of a sum of an even number of terms, each $\pm 1/2$, one for each crossing (defined subsequently). But noncompact knots, involving nonclosed strings, may have half-integral link numbers.

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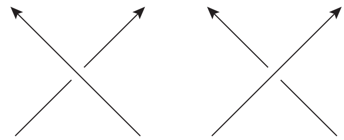


Figure 9.1. Overcrossings or undercrossings of knot components; the sign $\varepsilon(p)$ distinguishes the two possibilities shown.

The connection between the non-Abelian gauge potentials of a sphaleron and link numbers is an Abelian gauge potential formed from the sphaleron gauge potential, whose Abelian CS integral describes the linkages in terms of knots that occur in the Abelian magnetic field lines. For gauge group $SU(2)$, there is a deep relation between the CS integral and these Abelian gauge potentials and field strengths. This turns the non-Abelian CS form, with its characteristic cubic term, into an Abelian CS form that measures the linkages of the closed lines of Abelian field strength. The Abelian CS form, when described in terms of its Dirac string as in the confinement picture, has [34] exactly the form of the integral used in knot theory to describe over- and undercrossings, as shown in Figure 9.1. Aside from the topological characteristics we briefly note here, there is no particular physical meaning to this Abelian gauge potential.

The simplest way to think of knots in three dimensions is to project them onto a $d = 2$ plane, carefully distinguishing the various overcrossings and undercrossings that arise (see Kaufmann [37] for details). Knots are made of links or closed oriented loops (such as those occurring in $d = 3$ center vortices). A single link can be self-knotted, but the description of such knots is ambiguous until either the twist or the writhe of the single link is prescribed. Spread a system of linked closed strings out on a table to see undercrossings and overcrossings, such as idealized in Figure 9.1.

For the crossings of two distinct curves, the link number $Lk$, which is an integer, is defined as

$$Lk = \frac{1}{2}\sum_{p\in C}\epsilon(p), \tag{9.95}$$

where $C$ is the set of crossing points of one curve with the other. It turns out that a single sphaleron corresponds (through the knot structure of its associated Abelian field lines) to a single term in this sum for $N_{Lk}$, and the 1/2 for every term is precisely the CS number for the sphaleron.

The projection of two links formed from (nonpathological) closed loops has to have an even number of terms in the sum for $N_{Lk}$, so the sum yields an integer. But if the links are open, there can be an odd number (e.g., either of the crossings in Figure 9.1). An open link is equivalent to a closed link with one closure at infinity or, in other words, a noncompact link. So compactness of the knotted links implies integrality of $N_{\text{CS}}$.

The underlying topology comes from the perhaps surprising result that there is a topological map $S^3 \to S^2$, with the homotopy $\Pi^3(S^2) \simeq Z$. (As usual, that this map be described by integral indices requires compactness.) This map and homotopy were found by Hopf, and the map is called the Hopf fibration. The simplest way to look at the Hopf fibration (which describes, in essence, a total bundle space $S^3$ as the fibration of the base space $S^2$ by a fiber in $S^1$) is to begin with an $SU(2)$ matrix $U(\vec{r})$ in the fundamental representation and from it construct a unit vector $\widehat{n}$ by

$$U\tau_3 U^{-1} = \vec{\tau} \cdot \widehat{n} \tag{9.96}$$

(the $\vec{\tau}$ are the Pauli matrices). The unit vector lives on $S^2$, and the group space of $SU(2)$ is $S^3$. Something must be redundant in such a map, and it is that $U$ can be right multiplied by $\exp[\mathrm{i}\alpha(\vec{r})\tau_3/2]$ without changing $\widehat{n}$; the unit vector field corresponds to the coset $SU(2)/U(1)$. This redundancy, parametrized by $\alpha(\vec{r})$, will turn out to be a change of gauge for the fictitious Abelian potential (a shift by $\partial_i \alpha$).

The fictitious Abelian gauge potential is

$$A_i = \mathrm{i}\,\mathrm{Tr}\left(\tau_3 U \partial_i U^{-1}\right), \tag{9.97}$$

and its field strength is

$$B_i = \epsilon_{ijk}\partial_j A_k = -\mathrm{i}\epsilon_{ijk}\mathrm{Tr}\left(\tau_3 U\partial_j U^{-1} U\partial_k U^{-1}\right) = \frac{1}{2}\epsilon_{ijk}\epsilon_{abc}n^a\partial_j n^b \partial_k n^c. \tag{9.98}$$

Some manipulations using the antisymmetric property of the $\epsilon$ symbol lead to a very elegant formula, expressing the pure-gauge form of $N_{\text{CS}}$ as an Abelian CS integral:

$$\frac{-1}{12\pi^2}\int \mathrm{d}^3r\, \epsilon_{ijk}\mathrm{Tr}\left(U\partial_i U^{-1} U\partial_j U^{-1} U\partial_k U^{-1}\right) = \frac{1}{16\pi^2}\int \mathrm{d}^3r\, A_i B_i. \tag{9.99}$$

We have already encountered such an Abelian CS integral in interpreting the link number of a center vortex and a Wilson loop in Eq. (7.41). For the Hopf map, the link number is the integer in the homotopy $\Pi^3(S^2) \simeq Z$. However, for the standard sphaleron, this link number is half-integral because the standard sphaleron is noncompact; the field lines of $B_i$ terminate only at spatial infinity. Half-integrality of the link number of Eq. (9.95) also occurs when the links are noncompact. Reference [34] gives full details of this relation between sphalerons and knots.

Each $1/2$, with appropriate sign, marks an overcrossing or undercrossing of knots in its $d = 2$ projection. Although knots are uniquely a property of Euclidean three-space, because $d = 1$ strings do not link in any other space,[14] a great deal of knot theory is essentially two-dimensional, based on projecting the knots' overcrossings and undercrossings onto a $d = 2$ space (see Figure 9.1).

This leads to a useful $d = 2$ interpretation of knots and sphalerons that is quite analogous to the $d = 4$ interpretation of topological charge as counting, through a nonoriented intersection number, the linkages of center vortices and nexuses. An analogous nonoriented $d = 2$ intersection-number integral of closed $d = 1$ strings, with $SU(2)$ nexuses put by hand on the strings, yields some elementary knot properties.

A special ribbon framing is often used. In this, called the *Frenet–Serret* framing, the unit vector field of Eq. (8.28) is the principal normal vector $\widehat{e}_2$, lying in the direction of the curve's curvature vector (or derivative of the tangent vector $\widehat{e}_1$). With this framing in the $\epsilon \to 0$ limit, the twist is

$$Tw = \frac{1}{2\pi} \oint_\Gamma \mathrm{d}s\, \widehat{e}_2 \cdot \frac{d\widehat{e}_3}{ds}, \tag{9.100}$$

where $\widehat{e}_3$ is the principal binormal vector, given by $\widehat{e}_3 = \widehat{e}_1 \times \widehat{e}_2$. This expression for twist is well defined, provided that the curvature of the curve is not zero somewhere (in which case, $\widehat{e}_2$ is not defined).

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