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Hecke Operators and Derivatives of *L*-Functions

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Abstract. In this paper, we introduce a new way of studying derivatives of *L*-functions of cusp forms by associating to them cocycles analogous, in function, to the period polynomial. The main result gives a description of the effect of Hecke operators on these cocycles.

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1. Introduction

Let f be a cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. The period polynomial of f is defined by

$$\rho_f(X) = \int_{\infty}^{0} f(z)(z-X)^{k-2}\mathrm{d}z.$$

In [Z], D. Zagier gives a formula describing the action of Hecke operators on period polynomials which generalizes an earlier result of Manin (cf. [M]). The proofs given there are based on a theorem (cf. [CZ]) that algebraically describes certain Hecke operators acting on rational period functions as they have been defined by Knopp in [K].

At the same time, it is possible to generalize the notion of the period polynomial to the case that the cusp form has level higher than one (cf. [A] or [Sk].) Specifically, for a cusp form of (even) weight k for $\Gamma_0(N)$ we can define a map ρ_f from $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ to the space $P_{k-2}(\mathbb{C})$ of polynomials in $\mathbb{C}[X]$ with degree $\leqslant k-2$ such that:

$$\rho_f(\gamma) = \int_{0}^{\infty} (f|_k\gamma)(z)(z-X)^{k-2}\mathrm{d}z.$$

Here, $|_k$ is the usual 'stroke' operator such that

$$(f|_k\gamma)(z) := f(\gamma z)(cz+d)^{-k}, \quad \text{for } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

In [A], Antoniadis has proved a formula giving the values of $\rho_{T_pf}(p \nmid N)$ in terms of values of ρ_f. (Here, T_pf denotes the image of f under the usual Hecke operator T_p.)

In this paper, we first give (Theorem 1) a simpler expression for the action of the Hecke operators on ρ_f, by applying Choie and Zagier's theorem. In fact, the resulting formula is similar to the one proved in [Z] for level 1.

Coming back to the case of level 1, one of the possible interpretations of the period polynomial of a cusp form f of (even) weight k on $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is via the values of the corresponding L-function $L_f(s)$ at $s = 1, \ldots, k-1$. In this case, the period polynomial can be written in the form:

$$\sum_{m=0}^{k-2}\binom{k-2}{n} n!(i/2\pi)^{n+1} L_f(n+1)X^n.$$

Another viewpoint exploits the fact that the period polynomial, in this case, satisfies an 1-cocycle relation with respect to the action $|_{2-k}$ of Γ on $P_{k-2}(\mathbb{C})$. Combining these two interpretations of the period polynomial, we can then obtain a relation between values of $L_f(s)$ and certain cocycles.

On the other hand, D. Goldfeld and I (cf. [G], [D]) have shown that it is possible to construct maps which are related in an similar manner to values of *derivatives* of $L_f(s)$ and which satisfy higher cocycle relations. Motivated by this analogy and the interest of the properties of the usual period polynomial, here we study closer such a map $\sigma_f : \Gamma \times \Gamma \to P_{k-2}(\mathbb{C})$ (somewhat different from the one in [D],[G]) which is associated to $L_f'(s)$ and satisfies a two-cocycle condition.

It turns out, that by identifying elements of $P_{k-2}(\mathbb{C})$ differing by an element of the $\mathbb{Q}$-vector space M_f of $P_{k-2}(K)$-linear combinations of the periods $\int_0^\infty f(z)z^j dz$ ($j = 0, \ldots, k-2$), (where K is the space of $\mathbb{Q}$-linear combinations of πi, $\log n$ ($n = 2, 3, \ldots$)) σ_f induces a map b_f defined on $\Gamma_\infty\backslash\Gamma \cong \mathbb{P}^1(\mathbb{Q})$. Here, Γ_∞ is the subspace of Γ generated by

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The map b_f sends $\Gamma_\infty\gamma$ to the class mod M_f with representative:

$$\int_\infty^0 f(z)(z-X)^{k-2}(u(\gamma z) - u(z))\mathrm{d}z$$

where $u(z) := \log(\eta(z))$.

In Proposition 2 (inspired by a conversation with D. Zagier) we then prove that the projection $\bar{b}_f$ of b_f to the quotient of $(P_{k-2}(\mathbb{C})/M_f)^{\mathbb{P}^1(\mathbb{Q})}$ (the maps $\mathbb{P}^1(\mathbb{Q}) \to P_{k-2}(\mathbb{C})/M_f$) over the group $P_{k-2}(\mathbb{C})/M_f$ satisfies the Eichler–Shimura relations. That is,

$$\bar{b}_f \| (1+T) = \bar{b}_f \| (1+U+U^2) = 0,$$

where $\|$ expresses a naturally defined action and $U = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

The main result of the paper (Theorem 2) is the characterization of the effect of the Hecke operators on $\bar{b}_f$. Again the final expression is completely analogous to the one

Zagier has proved for periods. This suggests a broader applicability of the principle of [CZ], where Knopp's Hecke operators (which is essentially what we use in the statements of both our theorems) are characterized in a completely algebraic manner, without reference to the objects on which they act.

A connection with geometry (to be discussed in a later work) comes with the observation that $\bar{b}_f$ can be identified with a compactly supported 1-cocycle. Then, the interpretation of the first compactly supported cohomology group in terms of the cohomology of non-compactified modular curve (cf. [H], Kap.2) appears to set $\bar{b}_f$ in an appropriate geometric context.

Because of the analogy the above setup has with the one in Eichler–Shimura–Manin theory, it is natural to ask questions about the linear dependence of values of b_f and its coefficients over a field smaller than $\mathbb{C}$, analogous to the relations holding in the case of the period polynomial (cf. [M]). The natural character of the constructions made here, which parallel the considerations when dealing with L-functions and their derivatives, (e.g. the role of the space generated by periods when we look at the first derivative), further suggests that this kind of questions could be dealt with our methods.

We should finally note that another advantage of our approach is that all the constructions extend to the case of higher derivatives thus enabling us to possibly carry over to that case any insight we derive when we study the first derivative.

2. Hecke Action on Cochains

In [H], Section 2.2, Haberland defines Hecke operators on the space $C^i(\Gamma, V)$ of i-cochains with values in a Γ-module V which commute with the usual coboundary operator d (expressed in terms of the 'bar' resolution).We will describe in some detail how these operators are defined in our setting.

Let $M_2'(\mathbb{Z})$ be the semigroup of 2×2 matrices with non-zero determinant and entries in $\mathbb{Z}$. It is easy to see that, for each $g \in \Gamma$ and $M \in M_2'(\mathbb{Z})$ there is a unique $M_g \in \Gamma M g^{-1}$ of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with a, d positive integers and $0 \leqslant b < d$. By the uniqueness of M_g we deduce that $(M_g)_h = M_{hg}$ $(g, h \in \Gamma)$. Moreover, if $M \in M_2'(\mathbb{Z})$ is upper-triangular with positive diagonal elements, we have

$$\log(j(M_g g M^{-1}, Mz)) = \log(j(g, z)) + \log(d_{M_g}) - \log(d_M) \quad \text{for all } z \in \mathfrak{h}, \tag{1}$$

where $j\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}, z\right) = cz + d$, and d_M, d_{M_g} denote the lower-right entries of M and M_g, respectively. Here and in what follows we have taken log to mean the principal branch of the logarithm.

Now, for a positive integer p, let $\mathfrak{M}_p$ denote the set of matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2'(\mathbb{Z})$ with $ad = p$, $d > 0$ and $0 \leqslant b < d$. Then, following the notation in [Z], we set $T_p^\infty = \sum_{M \in \mathfrak{M}_p} M$. In this notation, Haberland's operators on $C^1(\Gamma, P_{k-2}(\mathbb{C}))$,

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$C^2(\Gamma, P_{k-2}(\mathbb{C}))$ and $C^3(\Gamma, P_{k-2}(\mathbb{C}))$ are given by the formulae

$$\begin{aligned}
(T_p^1 v)(g) &= \sum_{M\in\mathfrak{M}_p} v(M_g g M^{-1})|_{2-k}M,\\
(T_p^2 u)(g,h) &= \sum_{M\in\mathfrak{M}_p} u(M_g g M^{-1}, M_{hg} h M_g^{-1})|_{2-k}M, \qquad (2)\\
(T_p^3 w)(g,h,s) &= \sum_{M\in\mathfrak{M}_p} w(M_g g M^{-1}, M_{hg} h M_g^{-1}, M_{shg} s M_{hg}^{-1})|_{2-k}M,
\end{aligned}$$

for all

$$v \in C^1(\Gamma, P_{k-2}(\mathbb{C})), \qquad u \in C^2(\Gamma, P_{k-2}(\mathbb{C})),$$

$$w \in C^3(\Gamma, P_{k-2}(\mathbb{C})) \quad \text{and} \quad g, h, s \in \Gamma.$$

Here

$$(P|_{2-k}M)(X) = P(MX)j(M, X)^{k-2}, \quad \text{for } P \in P_{k-2}(\mathbb{C})$$

and $M \in M_2'(\mathbb{Z})$. Since $T_p^2 d = dT_p^1$ and $T_p^3 d = dT_p^2$ (cf. [H], Kap. 2), $Z^1(\Gamma, P_{k-2}(\mathbb{C}))$ and $Z^2(\Gamma, P_{k-2}(\mathbb{C}))$ are closed under T_p^1 and T_p^2 respectively. (Since it will always be clear which set these operators act on, from now on we omit the superscript in their notation.)

Finally, we set

$$(v||M)(g) := v(M_g g M^{-1})|_{2-k}M, \quad \text{for } g \in \Gamma,\ v \in C^1(\Gamma, P_{k-2}(\mathbb{C})),\ M \in M_2'(\mathbb{Z}).$$

In this way, we obtain a map $v||M$ from Γ to $P_{k-2}(\mathbb{C})$. Although $\|$ is not quite an action, we have the following lemma:

LEMMA 1. *For each $v \in C^1(\Gamma, P_{k-2}(\mathbb{C}))$, $g \in \Gamma$ and $M \in M_2'(\mathbb{Z})$, we have*

$$v\|g\|M = v\|gM \quad \text{and} \quad v\||M\|g = v\|Mg.$$

Proof. Since for each $h \in \Gamma$, we have $\Gamma(gM)h^{-1} = \Gamma M h^{-1}$, both $v\|g\|M$ and $v\|gM$ equal $v(M_h h(gM)^{-1})|_{2-k}gM$ when evaluated at $h \in \Gamma$, where by M_h we denote the element of $\Gamma M h^{-1}$ satisfying the conditions set in the definition of $\|$. The second equality is verified similarly.

Obviously, by linearity, we can define $v\|\tilde{M}$ for each element $\tilde{M}$ of the group of finite linear combinations of elements of $M_2'(\mathbb{Z})$ with integer coefficients.

We observe that we can express Haberland's Hecke operators on $C^1(\Gamma, P_{k-2}(\mathbb{C}))$ using the above notation:

$$T_p v = \sum_{M\in\mathfrak{M}_p} v\|M = v\|T_p^\infty. \qquad (3)$$

Also, we note that, for

$$c \in P_{k-2}(\mathbb{C}) = C^0(\Gamma, P_{k-2}(\mathbb{C})) \subset C^1(\Gamma, P_{k-2}(\mathbb{C})),$$

$c\|\gamma$ can be identified with $c|_{2-k}\gamma$ for all $\gamma \in \Gamma$.

3. Periods of Cusp Forms on $\Gamma_0(N)$

Let $f \in S_k(\Gamma_0(N)) = \{\text{cusp forms of weight } k \text{ for } \Gamma_0(N)\}$ and $\gamma \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$. As noted in Section 1, we can define the period polynomial ρ_f of f (cf. [A], [Sk]) setting

$$\rho_f(\gamma) := \int_0^\infty (f|_k\gamma)(z)(X-z)^{k-2}\mathrm{d}z \quad \text{for all} \quad \gamma \in \Gamma.$$

In view of the Eichler–Shimura isomorphism, one might consider as a more natural generalization of the period polynomial the map sending $\gamma \in \Gamma_0(N)$ to the polynomial $\tilde{\rho}_f(\gamma) = \left(\int_\infty^{\gamma\infty} f(w)(w-X)^{k-2}\mathrm{d}w\right)\big|_{2-k}\gamma$. However (using 'Manin's Trick'), we can write $\tilde{\rho}_f(\gamma)$ as a sum of $\rho_f(\gamma_i)|_{2-k}\gamma_i^{-1}\gamma$ for some $\gamma_i \in \Gamma$ which come from the continued fraction expansion of $\gamma\infty$.

If we change variables ($w = \gamma z$) we see that

$$\rho_f(\gamma) = \left(\int_{\gamma 0}^{\gamma\infty} f(w)(w-X)^{k-2}\mathrm{d}w\right)\bigg|_{2-k}\gamma.$$

Let now $\mu_f \in C^1(\Gamma, P_{k-2}(\mathbb{C}))$ be such that

$$\mu_f(\gamma) = \left(\int_\infty^{\gamma\infty} f(w)(w-X)^{k-2}\mathrm{d}w\right)\bigg|_{2-k}\gamma, \quad \text{for} \quad \gamma \in \Gamma.$$

(From now on, the stroke operator will be applied only to elements of $P_{k-2}(\mathbb{C})$, so we will denote it by $|$ rather than $|_{2-k}$.)

If we denote by T, S the matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ respectively, we have

$$\begin{aligned}
&(\mu_f\|(T-1))(\gamma) \\
&\quad = \mu_f(\gamma T^{-1})|T - \mu_f(\gamma) \\
&\quad = \left(\int_\infty^{\gamma 0} f(w)(w-X)^{k-2}\mathrm{d}w\right)|\gamma T^{-1}|T - \left(\int_\infty^{\gamma\infty} f(w)(w-X)^{k-2}\mathrm{d}w\right)|\gamma \\
&\quad = -\rho_f(\gamma),
\end{aligned}$$

and

$$\begin{aligned}
(\mu_f\|(S-1))(\gamma) &= \mu_f(\gamma S^{-1})|S - \mu_f(\gamma) \\
&= \left(\int_\infty^{\gamma\infty} f(w)(w-X)^{k-2}\mathrm{d}w\right)|\gamma S^{-1}|S - \mu_f(\gamma) = 0.
\end{aligned}$$

On the other hand, for $n \in \mathbb{N}$ with $(n, N) = 1$, equalities (3), $M(\infty) = M_\gamma(\infty) = \infty$

imply that $(\mu_f||T_n^\infty)(\gamma)$ equals

$$\begin{aligned}
&\sum_{M\in\mathfrak{M}_n}\mu_f(M_\gamma\gamma M^{-1})|M\\
&\quad=\sum_{M\in\mathfrak{M}_n}\left(\int_{M_\gamma\infty}^{M_\gamma\gamma\infty}f(z)(z-X)^{k-2}\mathrm{d}z\right)|M_\gamma\gamma\\
&\quad=\sum_{M\in\mathfrak{M}_n}\int_{\infty}^{\gamma\infty}f(M_\gamma z)(M_\gamma z-M_\gamma\gamma X)^{k-2}j(M_\gamma\gamma,X)^{k-2}\mathrm{d}(M_\gamma z)\\
&\quad=\sum_{M\in\mathfrak{M}_n}\frac{n^{k-1}}{(d_{M_\gamma})^k}\int_{\infty}^{\gamma\infty}f(M_\gamma z)(z-\gamma X)^{k-2}j(\gamma,X)^{k-2}\mathrm{d}z,
\end{aligned}$$

where d_{M_γ} denotes the lower-right entry of M_γ. Since M_γ ranges over $\mathfrak{M}_n$ as M ranges over $\mathfrak{M}_n$,

$$\sum_{M\in\mathfrak{M}_n}\frac{n^{k-1}}{(d_{M_\gamma})^k}f(M_\gamma z)=(T_nf)(z)$$

and, hence, $\mu_f\|T_n^\infty=\mu_{T_nf}$.

Now, in [CZ] it is proven that there exist $X_n, Y_n\in\mathbb{Z}[M_n]$ (where M_n denotes the set of matrices in $M_2'(\mathbb{Z})$ with determinant n) such that

$$T_n^\infty(T-1)=(T-1)X_n+(S-1)Y_n. \tag{4}$$

Moreover, an example of such a X_n is given explicitly. Specifically, if, for a positive n, we denote by Man_n the set of $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in M_n$ such that $a>|c|$, $d>|b|$, $bc\leqslant 0$, $c=0\Rightarrow -d/2<b\leqslant d/2$, and $b=0\Rightarrow -a/2<c\leqslant a/2$, then we can set $X_n=\sum_{M\in Man_n}M$.

From the equalities we just proved, together with Lemma 1,we then deduce that, for all $\gamma\in\Gamma$,

$$\begin{aligned}
\rho_{T_nf}(\gamma)&=-(\mu_{T_nf}\|(T-1))(\gamma)=-(\mu_f\|T_n^\infty(T-1))(\gamma)\\
&=-(\mu_f\|(T-1)X_n)(\gamma)-(\mu_f\|(S-1)Y_n)(\gamma)=(\rho_f\|X_n)(\gamma)\\
&=\sum_{M\in Man_n}\rho_f(M_\gamma\gamma M^{-1})|M.
\end{aligned}\tag{5}$$

Obviously, for $N=1$, $\rho_f(\gamma)=$ Constant and (5) gives Zagier's formula (cf. [Z]).

Now, we set $\rho_f^{\pm}(\gamma)=\frac{1}{2}(\rho_f(\gamma)+\rho_f(\varepsilon\gamma\varepsilon)|\varepsilon)$, where $\varepsilon:=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$. Since, for odd n, Man_n is invariant under

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\to\varepsilon\begin{pmatrix}a&b\\c&d\end{pmatrix}\varepsilon=\begin{pmatrix}a&-b\\-c&d\end{pmatrix},$$

(5) implies that

$$\rho_{T_nf}(\varepsilon\gamma\varepsilon)=\sum_{M\in Man_n}\rho_f((\varepsilon M\varepsilon)_{\varepsilon\gamma\varepsilon}(\varepsilon\gamma\varepsilon)(\varepsilon^{-1}M^{-1}\varepsilon^{-1}))|(\varepsilon M\varepsilon).$$

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However, $\varepsilon M_\gamma \varepsilon$ is an upper-triangular element of $M_2'(\mathbb{Z})$ (with positive diagonal entries) such that

$$(\varepsilon M_\gamma \varepsilon)(\varepsilon\gamma\varepsilon)(\varepsilon^{-1}M^{-1}\varepsilon^{-1}) = \varepsilon(M_\gamma \gamma M^{-1})\varepsilon \in \Gamma.$$

Hence, for some $n \in \mathbb{N}$, $(\varepsilon M \varepsilon)_{\varepsilon\gamma\varepsilon} = S^n \varepsilon M_\gamma \varepsilon$ and (since $\rho_f(S^n\delta) = \rho_f(\delta)$ for all $\delta \in \Gamma$),

$$\rho_f((\varepsilon M\varepsilon)_{\varepsilon\gamma\varepsilon}(\varepsilon\gamma\varepsilon)\varepsilon^{-1}M^{-1}\varepsilon^{-1}) = \rho_f(\varepsilon M_\gamma \gamma M^{-1}\varepsilon)$$

so,

$$\begin{aligned}\rho^{\pm}_{T_n f}(\gamma) &= \frac{1}{2}\left(\sum_{M\in Man_n} \rho_f(M_\gamma\gamma M^{-1})|M \pm \sum_{M\in Man_n} \rho_f(\varepsilon M_\gamma \gamma M^{-1}\varepsilon)|\varepsilon M\varepsilon|\varepsilon\right)\\ &= \sum_{M\in Man_n} \frac{1}{2}\Big(\rho_f(M_\gamma\gamma M^{-1}) \pm \rho_f(\varepsilon M_\gamma\gamma M^{-1}\varepsilon)|\varepsilon\Big)|M.\end{aligned}$$

From this, we obtain the following result:

THEOREM 1. *Let f be a cusp form of even weight k for $\Gamma_0(N)$. Then for any odd positive integer n with $(n, N) = 1$, we have:*

$$\rho^{\pm}_{T_n f}(\gamma) = \sum_{M\in Man_n} \rho^{\pm}_f(M_\gamma\gamma M^{-1})|M \quad \textit{for all} \quad \gamma \in \Gamma.$$

4. Derivatives of L-functions

We now restrict ourselves to the case that $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ is a cusp form of (even) weight k for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The periods $\rho_n(f)$ of f are defined by

$$\rho_n(f) = \int_0^{\infty} f(z)z^n dz = n!(i/2\pi)^{n+1}L_f(n+1), \quad (n = 0, \ldots, k-2).$$

The period polynomial can then be written in the form

$$\rho_f(X) = \sum_{k=0}^{k-2}\binom{k-2}{n}\rho_n(f)X^n = \int_0^{\infty} f(z)(zX+1)^{k-2}\mathrm{d}z.$$

(Notice that this is consistent with the definition of the period polynomial given in the Introduction because $\rho_n(f) = (-1)^{n-1}\rho_{k-2-n}(f)$.) We observe that $\rho_f(X)$ equals $-r_f(T)$ where r_f is a map from Γ to $P_{k-2}(\mathbb{C})$ such that

$$r_f(g) := \int_{\infty}^{g\infty} f(z)(z-gX)^{k-2}j(g, X)^{k-2}\mathrm{d}z, \quad \text{for all} \quad g \in \Gamma.$$

The map r_f is an element of the group of 1-cocycles $Z^1(\Gamma, P_{k-2}(\mathbb{C}))$ (with respect to

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the action $|_{2-k}$). In fact, it is a representative of the image of f under the Eichler–Shimura map from $S_k(\Gamma)$ to $H^1(\Gamma, P_{k-2}(\mathbb{C}))$. (cf. [S], Ch. 8).

We now construct a map associated to $L'_f(j)$'s ($j = 1, \ldots, k-1$) which satisfies a 2-cocycle condition. We first set $u(z) := 2\log(\eta(z))$ where $\eta(z)$ denotes Dedekind's η-function

$$\eta(z) = \mathrm{e}^{\pi iz/12} \cdot \prod_{n=1}^{\infty}(1 - \mathrm{e}^{2\pi inz}), \qquad z \in \mathfrak{h}(= \text{upper-half plane}).$$

The function u satisfies the transformation rule:

$$u(\gamma z) = u(z) + \log(j(\gamma, z)) + \kappa_\gamma, \quad \text{for all} \quad z \in \mathfrak{h}, \gamma \in \Gamma \tag{6}$$

where κ_γ is a constant depending only on γ and belonging to $\mathbb{Q}[\pi i]$. In particular, we have $\kappa_T = -\pi i/2$. We can then define a map $\sigma_f : \Gamma \times \Gamma \to P_{k-2}(\mathbb{C})$ such that:

$$\sigma_f(g_1, g_2) = \left(\int_\infty^{g_1\infty} f(z)(z - X)^{k-2}(u(g_2 z) - u(z))\mathrm{d}z\right)|g_1, \quad (g_1, g_2 \in \Gamma).$$

The following lemma proves that σ_f is a 2-coboundary and, hence it automatically satisfies a 2-cocycle condition:

LEMMA 2. *Let f be a cusp form of weight k for Γ. For all $g, h \in \Gamma$, we have:*

$$\sigma_f(h, g) = dv_f(h, g) = v_f(gh) - v_f(g)|h - v_f(h),$$

where $v_f : \Gamma \to P_{k-2}(\mathbb{C})$ is defined by

$$v_f(g) = \left(\int_\infty^{g(\infty)} f(z)(z - X)^{k-2}u(z)\mathrm{d}z\right)|g$$

and where d denotes the coboundary operator on 1-cochains (again with respect to the 'bar' resolution.)

Proof. For all $g, h \in \Gamma$, we have:

$$\begin{aligned}\sigma_f(h, g) &= \int_\infty^{h(\infty)} f(z)\big(u(gz) - u(z)\big)(z - hX)^{k-2}j(h, X)^{k-2}\mathrm{d}z \\ &= -v_f(h) + \int_\infty^{h(\infty)} f(z)u(gz)(z - hX)^{k-2}j(h, X)^{k-2}\mathrm{d}z \end{aligned}\tag{7}$$

By a simple computation we can see that for $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and $z_1, z_2 \in \mathfrak{h}$, we have:

$$(z_1 - \gamma z_2)j(\gamma, z_2) = (\gamma^{-1}z_1 - z_2)j(\gamma^{-1}, z_1).$$

If we then set $\gamma = g^{-1}$, $z_1 = z$ and $z_2 = ghX$, this becomes:

$$(z - hX)j(g^{-1}, ghX) = (gz - ghX)j(g, z),$$

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or

$$(z - hX) = (gz - ghX)j(g, z)j(g, hX).$$

Moreover, $f(\gamma z)d(\gamma z) = f(z)j(\gamma, z)^{k-2}dz$ and hence the integral in (7) equals:

$$\int_{\infty}^{h(\infty)} f(gz)u(gz)(gz - ghX)^{k-2}j(g, hX)^{k-2}j(h, X)^{k-2}d(gz).$$

By a change of variables ($z' = gz$), we obtain:

$$\begin{aligned}&\int_{g(\infty)}^{gh(\infty)} f(z')u(z')(z' - ghX)^{k-2}j(g, hX)^{k-2}j(h, X)^{k-2}d(z')\\ &\quad= \left(\int_{\infty}^{gh(\infty)} f(z)u(z)(z - ghX)^{k-2}j(gh, X)^{k-2}\mathrm{d}z\right) -\\ &\qquad - \left(\int_{\infty}^{g(\infty)} f(z)u(z)(z - g(hX))^{k-2}j(g, hX)^{k-2}\mathrm{d}z\right)j(h, X)^{k-2}.\end{aligned}$$

Since this equals $v_f(gh) - v_f(g)|h$, the proof of the lemma is complete.

Some obvious equalities we will be using tacitly in the sequel are: $v_f(gT^{-1}) = v_f(gT)$, $\sigma_f(T^{-1}, g) = \sigma_f(T, g)$, $\sigma_f(g, T^{-1}) = \sigma_f(g, T)$ for all $g \in \Gamma$. The relation of σ_f with the values of $L_f'(s)$ at $s = 1, \ldots, k-1$ is given by the following

PROPOSITION 1. *For all* $f \in S_k(\Gamma)$, *we have:*

$$\sigma_f(T, T) = -\sum_{j=0}^{k-2} \frac{(k-2)!}{(k-2-j)!}(i/2\pi)^{j+1}X^j(L_f'(j+1) + \lambda_j L_f(j+1)),$$

where $\lambda_j := \Gamma'(j+1)/\Gamma(j+1) - \log(2\pi)$.

Proof. Equality (6) implies that $\sigma_f(T, T)$ equals:

$$\begin{aligned}&\int_{\infty}^{0} f(z)(zX+1)^{k-2}(\log z + \kappa_T)\mathrm{d}z\\ &\quad= \sum_{j=0}^{k-2}\binom{k-2}{j}X^j\int_{\infty}^{0} f(z)z^j(\log z + \kappa_T)\mathrm{d}z.\end{aligned}$$

Differentiating both sides of the 'Mellin inverse transformation'

$$\int_0^{\infty} f(iy)y^{s-1}\mathrm{d}y = \Gamma(s)(2\pi)^{-s}L_f(s),$$

with respect to s, we eventually obtain:

$$\int_0^{\infty} f(z)\log(z)z^{s-1}\mathrm{d}z$$
$$= \Gamma(s)i^s(2\pi)^{-s}L_f'(s) + \Gamma(s)i^s(2\pi)^{-s}(\Gamma'(s)/\Gamma(s) + \log(i/2\pi))L_f(s).$$

From these identities (with $s = j+1$), the result follows immediately.

Finally, using formula (2), we obtain a relation between T_pf and σ_{T_pf}.

LEMMA. 3. *For all $g_1, g_2 \in \Gamma$, we have*

$$(T_p\sigma_f)(g_1, g_2) = \sigma_{T_pf}(g_1, g_2) + \sum_{M\in\mathfrak{M}_p} C_M\left(\int_{\infty}^{g_1\infty} f(M_{g_1}z)(z-X)^{k-2}\mathrm{d}z\right)|g_1,$$

where

$$C_M := \frac{p^{k-1}}{(d_{M_{g_1}})^k}\left(\kappa_{M_{g_2g_1}g_2M_{g_1}^{-1}} - \kappa_{g_2} + \log\left(\frac{d_{M_{g_2g_1}}}{d_{M_{g_1}}}\right)\right).$$

Proof. From equality (2), we see that $(T_p\sigma_f)(g_1, g_2)$ equals

$$\sum_{M\in\mathfrak{M}_p}\int_{\infty}^{M_{g_1}g_1M^{-1}\infty} f(z)(z - M_{g_1}g_1X)^{k-2}A(M, X)^{k-2}(u(M_{g_2g_1}g_2M_{g_1}^{-1}z) - u(z))\mathrm{d}z,$$

where

$$A(M, X) := j(M_{g_1}g_1M^{-1}, MX)^{k-2}j(M, X)^{k-2}.$$

Because of the 'cocycle condition' satisfied by $j(g, z)$ and the fact that $j(M_{g_1}, z)$ is independent from z, we have

$$A(M, X) = j(M_{g_1}, X)^{k-2}j(g_1, X)^{k-2}.$$

Since M and M_{g_1} fix ∞, if we substitute $M_{g_1}z$ for z, equality (6) implies that the summand corresponding to M equals:

$$\int_{\infty}^{g_1\infty} f(M_{g_1}z)(M_{g_1}z - M_{g_1}g_1X)^{k-2}(\log(j(M_{g_2g_1}g_2M_{g_1}^{-1}, M_{g_1}z))+$$
$$+\kappa_{M_{g_2g_1}g_2M_{g_1}^{-1}})j(M_{g_1}, X)^{k-2}j(g_1, X)^{k-2}d(M_{g_1}z).$$

Since the M_g's are upper-triangular, equality (1) implies that this can be written in the form:

$$\int_{\infty}^{g_1\infty}\frac{p^{k-1}}{(d_{M_{g_1}})^k}f(M_{g_1}z)(z - g_1X)^{k-2}j(g_1, X)^{k-2}(u(g_2z) - u(z) + \tilde{C}_M)\mathrm{d}z,$$

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where

$$\tilde{C}_M := \kappa_{M_{g_2g_1}g_2M_{g_1}^{-1}} - \kappa_{g_2} + \log\left(\frac{d_{M_{g_2g_1}}}{d_{M_{g_1}}}\right).$$

As noted in the proof of Theorem 1,

$$\sum_{M\in\mathfrak{M}_p} \frac{p^{k-1}}{(d_{M_g})^k} f(M_g z) = (T_p f)(z)$$

so we obtain the desired result.

5. A Special Set of Values of σ_f

According to the 2-cocycle condition σ_f satisfies, we have,

$$\sigma_f(\gamma_2\gamma_1,\gamma) = \sigma_f(\gamma_1,\gamma\gamma_2) - \sigma_f(\gamma_1,\gamma_2) + \sigma_f(\gamma_2,\gamma)|\gamma_1 \quad \text{for all} \quad \gamma,\gamma_1,\gamma_2 \in \Gamma.$$

This, together with the equality $\sigma_f(S,\gamma) = 0$, $(\gamma \in \Gamma)$ implies (by induction) that the set $\{\sigma_f(T,g); g \in \Gamma\}$ generates (over $\mathbb{Z}[\Gamma]$) the set of values of σ_f (in the same way that the period polynomial generates the set of values of the 1-cocycle r_f.) In view of this observation, we can concentrate on the subset $\{\sigma_f(T,g); g \in \Gamma\}$ of the values of σ_f.

Let K be the $\mathbb{Q}$-vector space generated (over $\mathbb{Q}$) by the set $\{i\pi, \log 2, \log 3, \ldots\}$. We consider (for a fixed $f \in S_k(\Gamma)$), the $\mathbb{Q}$-vector space $M_f = \sum_{j=0}^{k-2} P_{k-2}(K)\int_0^\infty f(z)z^j\mathrm{d}z$ and the projection map $\pi_f : P_{k-2}(\mathbb{C}) \to P_{k-2}(\mathbb{C})/M_f$. Obviously the action of $M_2'(\mathbb{Z})$ on $P_{k-2}(\mathbb{C})$ induces an action on $P_{k-2}(\mathbb{C})/M_f$ so that $\pi_f(P|\gamma) = \pi_f(P)|\gamma$ (denoted again by $|$). Since $\kappa_\delta \in \mathbb{Q}[\pi i]$,

$$\begin{aligned}&\pi_f(\sigma_f(\gamma,\delta))\\ &\quad= \pi_f\left(\int_\infty^{\gamma\infty} f(z)(z-\gamma X)^{k-2} j(\gamma,X)^{k-2}\log(j(\delta,z))\mathrm{d}z\right) \quad \text{for all } \gamma,\delta \in \Gamma.\end{aligned}$$

In particular, $\pi_f(\sigma_f(\gamma,S^n)) = 0$. At the same time, the 2-cocycle condition implies

$$\sigma_f(T,S^n\gamma) = \sigma_f(\gamma T,S^n) + \sigma_f(T,\gamma) - \sigma_f(\gamma,S^n) \mid T$$

and, hence, we have $\pi_f(\sigma_f(T,S^n\gamma)) = \pi_f(\sigma_f(T,\gamma))$. In view of this fact, we define a map $b_f\colon \Gamma_\infty\backslash\Gamma \to P_{k-2}(\mathbb{C})/M_f$ such that:

$$b_f(\Gamma_\infty\gamma) = \pi_f(\sigma_f(T,\gamma))|T, \quad \text{for all} \quad \gamma \in \Gamma.$$

Defining $\|$ on $C^1(\Gamma, P_{k-2}(\mathbb{C})/M_f)$ by the formula used in the case of $C^1(\Gamma, P_{k-2}(\mathbb{C}))$, we can prove that it induces an action on the group of maps $\mathbb{P}^1(\mathbb{Q}) \to P_{k-2}(\mathbb{C})/M_f$, denoted by $(P_{k-2}(\mathbb{C})/M_f)^{\mathbb{P}^1(\mathbb{Q})}$. Indeed, identifying $(P_{k-2}(\mathbb{C})/M_f)^{\mathbb{P}^1(\mathbb{Q})}$ with the group of maps $v : \Gamma \to P_{k-2}(\mathbb{C})/M_f$ such that

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$v(S\gamma) = v(\gamma)$ $(\gamma \in \Gamma)$ we have,

$$(v||M)(S\gamma) = v(M_{S\gamma}S\gamma M_\gamma^{-1})|M = v(((M_\gamma)_S SM_\gamma^{-1})(M_\gamma^{-1}\gamma M^{-1}))|M,$$

for

$$v \in \left(\frac{P_{k-2}(\mathbb{C})}{M_f}\right)^{\mathbb{P}^1(\mathbb{Q})}, \quad M \in M_2'(\mathbb{Z}) \text{ and } \gamma \in \Gamma.$$

Since $(M_\gamma)_S SM_\gamma^{-1}$ fixes infinity and belongs to Γ, we deduce that, in fact, it belongs to Γ_∞. But v is defined on $\Gamma_\infty\backslash\Gamma$, so

$$(v\|M)(S\gamma) = v(M_\gamma\gamma M^{-1})|M = (v\|M)(\gamma), \quad \text{i.e.} \quad v\|M \in \left(\frac{P_{k-2}(\mathbb{C})}{M_f}\right)^{\mathbb{P}^1(\mathbb{Q})}.$$

It should be noted that, on $(P_{k-2}(\mathbb{C})/M_f)^{\mathbb{P}^1(\mathbb{Q})}$, $\|$ is an action, because for $M_1, M_2 \in M_2'(\mathbb{Z})$, $\Gamma_\infty(M_1M_2)_\gamma = \Gamma_\infty(M_1)_{(M_2)_\gamma\gamma M_2^{-1}}(M_2)_\gamma$. (Indeed, $(M_1)_{(M_2)_\gamma\gamma M_2^{-1}}$, $(M_2)_\gamma$ are upper-triangular elements with positive diagonal elements and

$$(M_1)_{(M_2)_\gamma\gamma M_2^{-1}}(M_2)_\gamma\gamma(M_1M_2)^{-1} \in \Gamma).$$

So, for $v \in (P_{k-2}(\mathbb{C})/M_f)^{\mathbb{P}^1(\mathbb{Q})}$, we have

$$\begin{aligned}
&(v\|M_1M_2)(\gamma)\\
&\quad = v((M_1M_2)_\gamma\gamma M_2^{-1}M_1^{-1})|M_1M_2\\
&\quad = v(S^m(M_1)_{(M_2)_\gamma\gamma M_2^{-1}}(M_2)_\gamma\gamma M_2^{-1}M_1^{-1})|M_1M_2\\
&\quad = ((v||M_1)||M_2)(\gamma).
\end{aligned}$$

To obtain an explicit formula for $(b_f||M)([m,n])$, with $(m,n) = 1$ and $M \in \mathfrak{M}_p$, we observe that, if $\left(\begin{smallmatrix} * & * \\ m & n \end{smallmatrix}\right) \in \Gamma$, then

$$M_\gamma\begin{pmatrix} * & * \\ m & n \end{pmatrix}M^{-1} = \begin{pmatrix} * & * \\ \frac{d_{M_\gamma}}{p}(dm-cn) & \frac{d_{M_\gamma}}{p}(-bm+na) \end{pmatrix}.$$

Hence,

$$\begin{aligned}
(b_f\|M)([m,n]) &= (b_f||M)\left(\begin{pmatrix} * & * \\ m & n \end{pmatrix}\right)\\
&= \pi_f\left(\sigma_f\left(T, \begin{pmatrix} * & * \\ \frac{d_{M_\gamma}}{p}(dm-cn) & \frac{d_{M_\gamma}}{p}(-bm+na) \end{pmatrix}\right)\right)|TM
\end{aligned}$$

or

$$(b_f\|M)([m,n]) = b_f([m,n]\mathrm{adj}(M))|M,$$

where $\mathrm{adj}(M) := \det(M)M^{-1}$ and the (right) action of $M_2'(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ is the obvious one.

Finally, the image of b_f under the projection map from $(P_{k-2}(\mathbb{C})/M_f)^{\mathbb{P}^1(\mathbb{Q})}$ to the quotient $(P_{k-2}(\mathbb{C})/M_f)^{\mathbb{P}^1(\mathbb{Q})}/(P_{k-2}(\mathbb{C})/M_f)$ (with the obvious embedding of $P_{k-2}(\mathbb{C})/M_f$ into $(P_{k-2}(\mathbb{C})/M_f)^{\mathbb{P}^1(\mathbb{Q})}$) is denoted by $\bar{b}_f$. Again, since $P_{k-2}(\mathbb{C})/M_f$ is invariant under the action of $M_2'(\mathbb{Z})$ $\|$ is well-defined on the quotient. With this notation we can prove the following Proposition which was inspired by a discussion with D. Zagier.

PROPOSITION 2 (Eichler–Shimura Relations). *For* $U := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$*, we have:*

$$\bar{b}_f\|(1+T) = \bar{b}_f\|(1+U+U^2) = 0.$$

Proof. The 2-cocycle condition gives:

$$\sigma_f(T, \gamma T)|T = -\sigma_f(T, \gamma) + \sigma_f(T, T), \quad \text{for } \gamma \in \Gamma.$$

For $\gamma = \begin{pmatrix} * & * \\ m & n \end{pmatrix} \in \Gamma$, this implies:

$$(b_f\|(1+T))([m, n]) = \pi_f(\sigma_f(T, T)), \quad \text{for all } [m, n] \in \mathbb{P}^1(\mathbb{Q}).$$

For the proof of the second equality, we apply the 2-cocycle condition

$$\sigma_f(g_2, g_3)|g_1 = \sigma_f(g_2 g_1, g_3) - \sigma_f(g_1, g_3 g_2) + \sigma_f(g_1, g_2)$$

to the triplets

$$(g_1, g_2, g_3) = (UT, U, \delta), (UT, U^{-1}, \delta), (T, U, \delta), (T, U^{-1}, \delta), (\delta \in \Gamma)$$

to obtain, respectively,

$$\begin{aligned}
&\sigma_f(U, \delta)|U^{-1} = \sigma_f(U^{-1}T, \delta)|TU, \\
&\sigma_f(U^{-1}, \delta)|U = \sigma_f(T, \delta)|T, \\
&\sigma_f(U, \delta)|U^{-1} = -\sigma_f(T, \delta U)|TU^{-1} + \sigma_f(T, U)|TU^{-1}, \\
&\sigma_f(U^{-1}, \delta)|U = \sigma_f(U^{-1}T, \delta)|TU - \sigma_f(T, \delta U^{-1})|TU + \sigma_f(T, U^{-1})|TU.
\end{aligned}$$

These equalities imply that

$$\begin{aligned}
&\sigma_f(T, \delta U)|TU^{-1} + \sigma_f(T, \delta U^{-1})|TU + \sigma_f(T, \delta)|T - \sigma_f(T, U)|TU^{-1} - \\
&\quad - \sigma_f(T, U^{-1})|TU \\
&\quad = (\sigma_f(T, \delta U)|TU^{-1} - \sigma_f(T, U)|TU^{-1} + \sigma_f(U, \delta)|U^{-1}) + \\
&\qquad + (\sigma_f(T, \delta U^{-1})|TU - \sigma_f(T, U^{-1})|TU - \sigma_f(U^{-1}T, \delta)|TU + \\
&\qquad + \sigma_f(U^{-1}, \delta)|U) + (-\sigma_f(U, \delta)|U^{-1} + \sigma_f(U^{-1}T, \delta)|TU) \\
&\qquad + (-\sigma_f(U^{-1}, \delta)|U + \sigma_f(T, \delta)|T) = 0
\end{aligned}$$

For $\delta = \begin{pmatrix} * & * \\ m & n \end{pmatrix}$, this implies that for all $[m, n] \in \mathbb{P}^1(\mathbb{Q})$, $(b_f\|(1+U+U^2))([m, n])$ equals $\pi_f(\sigma_f(T, U))|TU^{-1} + \pi_f(\sigma_f(T, U^{-1}))|TU$, which is what we wanted to prove.

This is equivalent to saying that the map

$$\beta_f : \Gamma \to \left(\frac{P_{k-2}(\mathbb{C})}{M_f}\right)^{\mathbb{P}^1(\mathbb{Q})} \Big/ \left(\frac{P_{k-2}(\mathbb{C})}{M_f}\right),$$

such that $\beta_f(T) = \bar{b}_f$ and $\beta_f(S) = 0$ is an element of the compactly supported 1-cohomology group

$$H_c^1\left(\Gamma, \left(\frac{P_{k-2}(\mathbb{C})}{M_f}\right)^{\mathbb{P}^1(\mathbb{Q})} \Big/ \left(\frac{P_{k-2}(\mathbb{C})}{M_f}\right)\right).$$

(If $C^*(\Gamma, -)$ is the standard complex for Γ, then we denote by $H_c^*(\Gamma, -)$ the cohomology functor induced by the kernel of the restriction $C^*(\Gamma, -) \to C^*(\Gamma_\infty, -)$ (cf. [H], Kap.2). In particular, we have: $H_c^1(\Gamma, V) = \mathrm{Ker}(Z^1(\Gamma, V) \to Z^1(\Gamma_\infty, V))$ where V is a Γ-module.) The first compactly supported cohomology group has a geometrical interpretation which we would like to apply in the above context and to make as explicit as possible. We hope to complete this in a future paper.

6. The Action of Hecke Operators

Let n be an integer $\geqslant 2$ and let v_f be the map associated to $f \in S_k(\Gamma)$ by Lemma 2. Then, since S fixes ∞, we have $v_f \| S = v_f$ and thus, (4), together with Lemma 1, imply

$$v_f || T_p^\infty \| (T-1) = v_f \| (T-1) \| \left(\sum_{M \in Man_p} M\right). \tag{8}$$

We now evaluate both sides of this equality at $g \in \Gamma$. From (3) we obtain

$$(v_f \| T_p^\infty \| (T-1))(g) = (T_p v_f)(gT^{-1})|T - (T_p v_f)(g).$$

For the right-hand side, by applying directly the definitions we take

$$\begin{aligned}
&\left(v_f \| (T-1) \| \left(\sum_{M \in Man_p} M\right)\right)(g) \\
&\quad = \sum_{M \in Man_p} (v_f \| (T-1))(M_g g M^{-1})|M \\
&\quad = \sum_{M \in Man_p} v_f(M_g g M^{-1} T^{-1})|TM - \sum_{M \in Man_p} v_f(M_g g M^{-1})|M \\
&\quad = \sum_{M \in Man_p} (v_f(M_g g M^{-1} T^{-1}) - v_f(M_g g M^{-1})|T^{-1})|TM.
\end{aligned}$$

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Lemma 2, then (with $h = T^{-1}, g = M_g g M^{-1}$) implies that the last sum equals:

$$\sum_{M \in Man_p} (\sigma_f(T^{-1}, M_g g M^{-1}) + v_f(T^{-1}))|TM$$

and, hence (because of (8)),

$$\begin{aligned}&(T_p v_f)(gT^{-1})|T - (T_p v_f)(g)\\ &\quad = \sum_{M \in Man_p} \sigma_f(T^{-1}, M_g g M^{-1})|TM + v_f(T^{-1})|T\left(\sum_{M \in Man_p} M\right).\end{aligned}$$

Subtracting this equality evaluated at g from the same equality evaluated at $g = T$, we obtain:

$$\begin{aligned}&(T_p v_f)(g) - (T_p v_f)(gT)|T - (T_p v_f)(T)\\ &\quad = \sum_{M \in Man_p} (\sigma_f(T, M_T T M^{-1}) - \sigma_f(T, M_g g M^{-1}))|TM.\end{aligned} \tag{9}$$

The left-hand side equals $d(T_p v_f)(T, gT) = (T_p \sigma_f)(T, gT)$ or, according to Lemma 3,

$$\sigma_{T_p f}(T, gT) + \sum_{M \in \mathfrak{M}_p} C_M\left(\int_\infty^0 f(M_T z)(z - X)^{k-2} \mathrm{d}z\right)|T,$$

with

$$C_M = \frac{p^{k-1}}{(d_{M_T})^k}\left(\kappa_{M_g g T M_T^{-1}} - \kappa_{gT} + \log\left(\frac{d_{M_g}}{d_{M_T}}\right)\right).$$

By a change of variables in each of the integrals at the last sum ($w = M_T z$) we can see that each is a multiple of a polynomial of the form $(\int_\alpha^\beta f(z)(z - X)^{k-2} \mathrm{d}z)|M'$ for some $M' \in M_2'(\mathbb{Z})$ and $\alpha, \beta \in \mathbb{Q}$. However (cf. [M]), every integral of the form $\int_\alpha^\beta f(z) z^j \mathrm{d}z$, $(j = 0, \dots, k-2)$ can be expressed as a integral linear combination of the periods $\rho_i(f)$ $(i = 0, , k-2)$. Therefore the left-hand side of (9) is a sum of $\sigma_{T_p f}(T, gT)$ plus an element of M_f. Hence,

$$\begin{aligned}&\pi_f(\sigma_{T_p f}(T, gT))\\ &\quad = \sum_{M \in Man_p} (\pi_f(\sigma_f(T, M_T T M^{-1}))|TM - \pi_f(\sigma_f(T, M_g g M^{-1}))|TM).\end{aligned}$$

If we furthermore use the identity

$$\sigma_{T_p f}(T, g)|T = -\sigma_{T_p f}(T, gT) + \sigma_{T_p f}(T, T)$$

which is implied by the 2-cocycle condition, and set $g = \begin{pmatrix} * & * \\ m & n \end{pmatrix}$ this equality can be written in the form:

$$\pi_f(\sigma_{T_pf}(T,g)|T) - \sum_{M\in Man_p} b_f([m,n]\mathrm{adj}(M))|M$$
$$= \pi_f(\sigma_{T_pf}(T,T)) - \sum_{M\in Man_p} \pi_f(\sigma_f(T, M_T TM^{-1}))|TM,$$

for all $[m,n] \in \mathbb{P}^1(\mathbb{Q})$. If we then denote again by $\bar{b}_{T_pf}$ the image of $\bar{b}_{T_pf}$ under the map from

$$\left(\frac{P_{k-2}(\mathbb{C})}{M_{T_pf}}\right)^{\mathbb{P}^1(\mathbb{Q})} \Big/ \left(\frac{P_{k-2}(\mathbb{C})}{M_{T_pf}}\right) \quad \text{to} \quad \left(\frac{P_{k-2}(\mathbb{C})}{M_f}\right)^{\mathbb{P}^1(\mathbb{Q})} \Big/ \left(\frac{P_{k-2}(\mathbb{C})}{M_f}\right)$$

induced by the inclusion $M_{T_pf} \subset M_f$ (proved above), we obtain:

THEOREM 2. *For each $f \in S_k(\Gamma)$, and positive integer $p \geqslant 2$ we have $\bar{b}_{T_pf} = \bar{b}_f \| X_p$, where the maps and linear combinations of operators involved have been defined in Sections 3, 4 and 5.*

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References

[A] Antoniadis, J. A.: Modulformen auf $\Gamma_0(N)$ mit rationalen Perioden, *Manuscripta. Math.* **74** (1992), 359-384.

[CZ] Choie, Yj. and Zagier, D.: Rational period functions for PSL(2, **Z**), *Contemp. Math.* **143** (1993), 89–108.

[D] Diamantis, N.: Special values of higher derivatives of L-functions, *Forum Math.* **11** (1999), 229–252.

[G] Goldfeld, D.: Special values of derivatives of L-functions, In: *CMS Conf. Proc.* 15 Amer. Math. Soc., Providence, 1995, pp. 245–282.

[H] Haberland, K.: Perioden von Modulformen einer Variabler and Gruppencohomologie. I, *Math. Nachr.* **112** (1983), 159–173.

[K] Knopp, M.: Rational period functions of the modular group, *Duke Math. J.* **45** (1978), 47–62.

[M] Manin, Y. T.: Periods of parabolic points and p-adic Hecke series, *Math. Sb.* (1973), 371–393.

[S] Shimura, G.: *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, 1971.

[Sk] Skoruppa, N. P.: Binary quadratic forms and the Fourier coefficients of elliptic and Jacobi modular forms, *J. Reine angew. Math.* **411** (1990), 66–95.

[Z] Zagier, D.: Hecke operators and periods of modular forms, In: *Israel Math. Conf. Proc.* 3, Bar-Ilan Univ., Ramat gan, 1990, pp. 321–336.