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LOCAL BOUNDEDNESS OF NONAUTONOMOUS SUPERPOSITION OPERATORS IN $BV[0,1]$

PIOTR KASPRZAK[✉] **and PIOTR MAĆKOWIAK**

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Abstract

The main goal of this paper is to give the answer to one of the main problems of the theory of nonautonomous superposition operators acting in the space of functions of bounded variation in the sense of Jordan. Namely, we prove that if the superposition operator maps the space $BV[0,1]$ into itself, then it is automatically locally bounded, provided its generator is a locally bounded function.

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1. Introduction

The notion of variation, introduced by Jordan in 1881 (see [12]), is one of the basic notions of mathematical analysis. Since the end of the 19th century the Jordan variation (as well as its generalisations and extensions) has been an object of interest for many mathematicians due to the fact that functions of bounded variation have found applications in many fields: for example, in geometric measure theory (see, for example, [1, 16]); in image processing, analysis and recovery (see, for example, [8–10, 14, 17]); in the theory of Fourier series (see [18]); in the theory of integration and integral equations (see [5–7]); and in economics (see [11]).

In the recently published monograph [2], whose aim is to give a thorough account of functions of bounded (generalised) variation and their relation to other important classes of functions as well as their applications to various problems arising in nonlinear analysis, the authors stated three basic problems regarding nonautonomous superposition operators acting in the space of functions of bounded variation in the sense of Jordan. The first two problems concern both necessary and sufficient conditions under which the nonautonomous superposition operator maps the space $BV[0,1]$ into itself and is (locally) bounded, respectively. For completeness, let us add that the third problem concerns the continuity of the superposition operator, but it will not be addressed in this paper.

In the case of autonomous superposition operators the problems concerning the so-called *acting conditions* and local boundedness were solved by Josephy who in 1981 established the following result.

THEOREM 1.1 ([3, Theorem 6.13], [13]). *Suppose that F is an autonomous superposition operator generated by a function $f : \mathbb{R} \to \mathbb{R}$. The superposition operator F maps the space $BV[0, 1]$ into itself if and only if the function f satisfies a local Lipschitz condition, that is, for every $r > 0$ there exists a number $L_r \geq 0$ such that $|f(u) - f(w)| \leq L_r|u - w|$, whenever $u, w \in [-r, r]$.*

Until very recently is seemed that the counterpart of Josephy's theorem for nonautonomous superposition operators had been established and 'quite' general sufficient conditions for a nonautonomous superposition operator to act in $BV[0, 1]$ had been found. However, in 2014 Maćkowiak, motivated by the doubts concerning the so-called 'Ljamin's theorem' raised by Bugajewska in [4], constructed a counterexample showing the falsity of the result (for the details see [15]).

In this paper we continue the study of superposition operators in the space of functions of bounded Jordan variation and prove that if the superposition operator F maps the space $BV[0, 1]$ into itself, then F is automatically locally bounded, provided that its generator is a locally bounded function. Let us add that the local boundedness of the generator is a necessary condition for the local boundedness of the superposition operator.

Before we proceed further, let us briefly explain the idea behind our approach. The key observation is as follows: if the superposition operator maps the space $BV[0, 1]$ into itself but is not locally bounded, then it is possible to 'transfer' (and then 'localise') its undesired properties to the generator f, that is, it is possible to find a point $(t^*, u^*) \in [0, 1] \times \mathbb{R}$ and a sequence $(x_q)_{q\in\mathbb{N}}$ of functions with uniformly bounded Jordan variation such that the graphs of the functions are eventually contained in an arbitrary open neighbourhood of the point (t^*, u^*), and the corresponding variation sums of the superposition[1] of f and x_q grow to infinity (see Theorem 3.4). Next, we show that the functions x_q can be 'redefined' to make their Jordan variation on a certain interval around t^* arbitrarily small (see Lemmas 3.6 and 3.7). In the final step it 'suffices' to glue the modified functions together in order to get a function of bounded variation which after superposition with f does not belong to $BV[0, 1]$ (see Theorem 4.1).

The paper is organised as follows: In Section 2 we fix the notation used in this paper and recall basic definitions concerning the Jordan variation and superposition operators. In Section 3 we establish several auxiliary results which will be required in Section 4 to prove the main result of the paper, Theorem 4.1.

[1]By the superposition of $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $x_q : [0, 1] \to \mathbb{R}$ we mean the function $t \mapsto f(t, x_q(t))$.

2. Preliminaries

We begin this section by recalling the notion of the Jordan variation. For simplicity, we will restrict our considerations to the interval $[0, 1]$.

DEFINITION 2.1. Let x be a real-valued function defined on the interval $[0, 1]$. The number

$$\bigvee_0^1 x = \sup \sum_{i=1}^n |x(t_i) - x(t_{i-1})|,$$

where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$, is called the *variation (in the sense of Jordan)* of the function x over $[0, 1]$.

It is well known that the space of all functions of bounded Jordan variation,

$$BV[0, 1] = \Big\{ x : [0, 1] \to \mathbb{R} : \bigvee_0^1 x < +\infty \Big\},$$

endowed with the norm $\|x\|_{BV} = |x(0)| + \bigvee_0^1 x$, is a Banach space. (For a thorough treatment of various classical and nonclassical spaces of functions of bounded variation we refer the reader to [2].) The closed ball in the space $BV[0, 1]$ with centre at x and radius $r \in (0, +\infty)$ will be denoted by $B_{BV}(x, r)$.

Now, let us pass onto superposition operators. Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and let us consider the nonlinear operator F which to each function $x : [0, 1] \to \mathbb{R}$ (belonging to a given set X) assigns the function $t \mapsto f(t, x(t))$. The operator F is known as the *superposition operator* (or *Nemytskii operator*) and the function f is said to be its *generator*. The superposition operator F is called *autonomous* if its generator f does not depend on the 'time' variable $t \in [0, 1]$; otherwise F is said to be *nonautonomous*. For a deeper discussion of superposition operators we refer the reader to [3].

In this paper, we understand locally bounded mappings acting in $BV[0, 1]$ to be the mappings defined as follows.

DEFINITION 2.2. A mapping $G : BV[0, 1] \to BV[0, 1]$ is said to be *locally bounded* if for each $r > 0$ there exists $R > 0$ such that $G(B_{BV}(0, r)) \subset B_{BV}(0, R)$.

Finally, throughout the paper, for $t \in [0, 1]$ and $\varepsilon \in (0, +\infty)$ we will write $l_\varepsilon(t) := \max\{0, t - \varepsilon\}$ and $r_\varepsilon(t) := \min\{1, t + \varepsilon\}$.

3. Auxiliary results

The aim of this section is to prove several technical results which will be needed in the sequel to 'translate' the properties of a nonautonomous superposition operator to the properties of its generator. Throughout the section we assume that $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a function which maps bounded sets into bounded sets and that F is a nonautonomous superposition operator generated by f.

LEMMA 3.1. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and suppose that a sequence $(x_n)_{n\in\mathbb{N}}$ of real-valued functions defined on $[0, 1]$ is uniformly bounded and such that $\|F(x_n)\|_{BV} \geq n$ for $n \in \mathbb{N}$.*

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Then there exists $t_0 \in [0,1]$ *such that for each* $\varepsilon > 0$,

$$\sup_{n\in\mathbb{N}} \bigvee_{l_\varepsilon(t_0)}^{r_\varepsilon(t_0)} F(x_n) = +\infty.$$

Proof. Assume that the assertion is false. Then we deduce that for any $t \in [0,1]$ there exists $\varepsilon_t > 0$ such that

$$\sup_{n\in\mathbb{N}} \bigvee_{l_{\varepsilon_t}(t)}^{r_{\varepsilon_t}(t)} F(x_n) =: M_t < +\infty.$$

Since $[0,1]$ is compact, we conclude that there exists $M > 0$ such that

$$\sup_{n\in\mathbb{N}} \|F(x_n)\|_{BV} \le \sup_{n\in\mathbb{N}} |f(0, x_n(0))| + \sup_{n\in\mathbb{N}} \bigvee_0^1 F(x_n) \le \sup_{n\in\mathbb{N}} |f(0, x_n(0))| + M < +\infty,$$

which is impossible. □

Lemma 3.2. *Let* $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *and suppose that a sequence* $(x_n)_{n\in\mathbb{N}}$ *of real-valued functions defined on* $[0,1]$ *is uniformly bounded and such that* $\|F(x_n)\|_{BV} \ge n$ *for* $n \in \mathbb{N}$. *Then there exist* $t_0 \in [0,1]$ *and a subsequence* $(x_{n_k})_{k\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$ *such that for each* $\varepsilon > 0$ *a positive integer* k_0 *can be found with*

$$\bigvee_{l_\varepsilon(t_0)}^{r_\varepsilon(t_0)} F(x_{n_k}) \ge k \quad \text{for } k \ge k_0.$$

Proof. In view of Lemma 3.1, we infer that there exist a point $t^1 \in [0,1]$ and a subsequence $(x_n^1)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \bigvee_{l_1(t^1)}^{r_1(t^1)} F(x_n^1) = +\infty.$$

Let us note that, passing to a subsequence if necessary, we may assume that

$$\|F(x_n^1)\|_{BV} \ge \bigvee_{l_1(t^1)}^{r_1(t^1)} F(x_n^1) \ge n \quad \text{for } n \in \mathbb{N}.$$

Invoking Lemma 3.1 once again, we deduce that there is a $t^2 \in [0,1]$ and a subsequence $(x_n^2)_{n\in\mathbb{N}}$ of $(x_n^1)_{n\in\mathbb{N}}$ such that

$$\|F(x_n^2)\|_{BV} \ge \bigvee_{l_{1/2}(t^2)}^{r_{1/2}(t^2)} F(x_n^2) \ge n \quad \text{for } n \in \mathbb{N}.$$

This procedure yields a family of sequences $(x_n^q)_{n\in\mathbb{N}}$ and points $t^q \in [0,1]$ such that $(x_n^{q+1})_{n\in\mathbb{N}}$ is a subsequence of $(x_n^q)_{n\in\mathbb{N}}$ and $\|F(x_n^q)\|_{BV} \ge \bigvee_{l_{1/q}(t^q)}^{r_{1/q}(t^q)} F(x_n^q) \ge n$ for $n \in \mathbb{N}$, where $q \in \mathbb{N}$.

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Now, define $y_q := x^q_q$ for $q \in \mathbb{N}$. Since the sequence $(t^q)_{q\in\mathbb{N}}$ is bounded, it contains a subsequence $(t^{q_k})_{k\in\mathbb{N}}$ which converges to a point $t_0 \in [0, 1]$. The point t_0 and the sequence $(y_{q_k})_{k\in\mathbb{N}}$ satisfy the claim. To see this, notice that for a given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $[l_{1/q_k}(t^{q_k}), r_{1/q_k}(t^{q_k})] \subset [l_\varepsilon(t_0), r_\varepsilon(t_0)]$ for all $k \geq k_0$. Hence,

$$\bigvee_{l_\varepsilon(t_0)}^{r_\varepsilon(t_0)} F(y_{q_k}) \geq \bigvee_{l_{1/q_k}(t^{q_k})}^{r_{1/q_k}(t^{q_k})} F(x^{q_k}_{q_k}) \geq q_k \quad \text{for } k \geq k_0,$$

which ends the proof. □

Our further considerations are based on the following result.

Lemma 3.3. *Let $r > 0$ and suppose that $x_n \in B_{BV}(0, r)$ and $\|F(x_n)\|_{BV} \geq n$ for $n \in \mathbb{N}$. Then there exists $t_0 \in [0, 1]$ such that for any $\delta > 0$ and $\varepsilon > 0$ there exist a subsequence $(x_{n_q})_{q\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, a point $u_0 \in [-r, r]$ and a sequence of finite collections of points $l_\varepsilon(t_0) \leq t^q_0 < t^q_1 < \cdots < t^q_{d_q} \leq r_\varepsilon(t_0)$, where $q \in \mathbb{N}$, for which the following properties hold: $x_{n_q}(t^q_i) \in [u_0 - \delta, u_0 + \delta]$ for $i = 0, \ldots, d_q$ and*

$$\lim_{q\to\infty} \sum_{i=1}^{d_q} |f(t^q_i, x_{n_q}(t^q_i)) - f(t^q_{i-1}, x_{n_q}(t^q_{i-1}))| = +\infty.$$

Proof. By Lemma 3.2 there exist a point $t_0 \in [0, 1]$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which for every $\varepsilon > 0$ satisfy the condition

$$\bigvee_{l_\varepsilon(t_0)}^{r_\varepsilon(t_0)} F(x_{n_k}) \geq k \quad \text{for } k \geq k_0(\varepsilon).$$

For the sake of simplicity, we may assume that $(x_{n_k})_{k\in\mathbb{N}}$ coincides with the initial sequence $(x_n)_{n\in\mathbb{N}}$.

Fix $\delta > 0$ and $\varepsilon > 0$. (Without loss of generality, we may assume that $6r/\delta$ is a positive integer greater than or equal to 2 and that $k_0(\varepsilon) = 1$.) For each $n \in \mathbb{N}$, choose a number $k_n \in \mathbb{N}$ and a finite partition $l_\varepsilon(t_0) = \tau^n_0 < \cdots < \tau^n_{k_n} = r_\varepsilon(t_0)$ of the interval $[l_\varepsilon(t_0), r_\varepsilon(t_0)]$ such that

$$\sum_{i=1}^{k_n} |f(\tau^n_i, x_n(\tau^n_i)) - f(\tau^n_{i-1}, x_n(\tau^n_{i-1}))| \geq n - M_r,$$

where

$$0 < M_r := \sup\{|f(t, u)| : t \in [0, 1] \text{ and } u \in [-r, r]\} < +\infty.$$

This, in particular, shows that $k_n \to +\infty$ as $n \to +\infty$.

Now, let us define

$$u^0 := -r \quad \text{and} \quad u^{j+1} := u^j + \frac{\delta}{3} \quad \text{for } j = 0, 1, \ldots, s - 1,$$

where $s = 6r/\delta$. Then, the intervals $[u^j - \delta/6, u^j + \delta/6]$, where $j = 0, 1, \ldots, s$, cover the interval $[-r, r]$. Moreover,

$$[u^j - \delta/6, u^j + \delta/6] \cap [u^{j+1} - \delta/6, u^{j+1} + \delta/6] = \{u^j + \delta/6\} \quad \text{for } j = 0, \ldots, s-1,$$

and we have $|u - w| \geq \delta/3$ for $u \in [u^i - \delta/6, u^i + \delta/6]$, $w \in [u^j - \delta/6, u^j + \delta/6]$ and i, j with $|i - j| > 1$.

For every $n \in \mathbb{N}$,

$$\{x_n(\tau_i^n) : i = 0, 1, \ldots, k_n\} \subset \bigcup_{j=0}^{s} [u^j - \delta/6, u^j + \delta/6].$$

Then to each $i \in \{0, \ldots, k_n\}$ we can assign a number $j^n(i) \in \{0, \ldots, s\}$ by

$$j^n(i) := \min\{j : x_n(\tau_i^n) \in [u^j - \delta/6, u^j + \delta/6]\}.$$

For every $n \in \mathbb{N}$, either:

- $|j^n(i) - j^n(0)| < 2$ for all $i \in \{0, \ldots, k_n\}$, which means that $x_n(\tau_i^n)$ is in the interval $[u^{j^n(0)} - \delta/2, u^{j^n(0)} + \delta/2]$ for all $i \in \{0, \ldots, k_n\}$; or
- there exists a δ-jump, that is, there exists the first index $i^* \in \{1, \ldots, k_n\}$ such that $|j^n(i^*) - j^n(0)| \geq 2$, and then $|x_n(\tau_{i^*}^n) - x_n(\tau_0^n)| > \delta/3$.

Taking i^* in place of 0 and repeating the above reasoning, we see that either there is a δ-jump for some $i^{**} > i^*$ or there is not, and so on. Let us note that since $\|x_n\|_{BV} \leq r$, the number of consecutive δ-jumps for the function x_n, which will be denoted by m_n, is at most $3r/\delta$ (observe that the upper bound for the number of δ-jumps does not depend on n). Indeed, if the function x_n has consecutive δ-jumps at the points[1] $\tau_0^n, \tau_1^n, \ldots, \tau_{m_n}^n$, then

$$r \geq \sum_{i=0}^{m_n - 1} |x_n(\tau_i^n) - x_n(\tau_{i+1}^n)| \geq \frac{1}{3}\delta m_n.$$

This procedure leads to the following definition of the sets I_l^n, where $l = 0, 1, \ldots, m_n$:

$$I_l^n = \begin{cases} \{i \in \mathbb{N} : i_l^n + 1 \leq i \leq i_{l+1}^n - 1\} & \text{if } i_{l+1}^n < k_n + 1, \\ \{i \in \mathbb{N} : i_l^n + 1 \leq i \leq k_n\} & \text{if } i_{l+1}^n = k_n + 1, \end{cases}$$

where

$$i_0^n := 0 \quad \text{and} \quad i_{l+1}^n := \min(\{i \in \{i_l^n, \ldots, k_n\} : |j^n(i_l^n) - j^n(i)| \geq 2\} \cup \{k_n + 1\}).$$

Let us note that[2]

$$\begin{aligned} \sum_{i=1}^{k_n} |f(\tau_i^n, x_n(\tau_i^n)) - f(\tau_{i-1}^n, x_n(\tau_{i-1}^n)) &= \sum_{l=0}^{m_n} \sum_{i \in I_l^n} |f(\tau_i^n, x_n(\tau_i^n)) - f(\tau_{i-1}^n, x_n(\tau_{i-1}^n))\| \\ &\quad + \sum_{l=1}^{m_n} |f(\tau_{i_l^n}^n, x_n(\tau_{i_l^n}^n)) - f(\tau_{i_l^n - 1}^n, x_n(\tau_{i_l^n - 1}^n))|, \end{aligned}$$

[1]For the sake of simplicity, we assume that the consecutive δ-jumps of the function x_n appear at the first $m_n + 1$ points of the partition.

[2]If $I_l^n = \emptyset$, then by definition the sum corresponding to the set I_l^n equals zero.

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and since

$$\sum_{l=1}^{m_n} |f(\tau^n_{i^n_l}, x_n(\tau^n_{i^n_l})) - f(\tau^n_{i^n_l-1}, x_n(\tau^n_{i^n_l-1}))| \leq 2m_n M_r \leq sM_r,$$

we infer that for n sufficiently large there exists $l_n \in \{0, \ldots, m_n\}$ such that

$$\sum_{i \in I^n_{l_n}} |f(\tau^n_i, x_n(\tau^n_i)) - f(\tau^n_{i-1}, x_n(\tau^n_{i-1}))| \geq \frac{\delta}{\delta + 3r}[n - (s+1)M_r].$$

Since $m_n \leq 3r/\delta$ for every $n \in \mathbb{N}$, there exist a strictly increasing sequence $(n_q)_{q \in \mathbb{N}}$ of positive integers which diverges to $+\infty$ and a number $l \in \{0, 1, \ldots, \max_{n \in \mathbb{N}} m_n\}$ independent of n for which

$$\lim_{q \to \infty} \sum_{i \in I^{n_q}_l} |f(\tau^{n_q}_i, x_{n_q}(\tau^{n_q}_i)) - f(\tau^{n_q}_{i-1}, x_{n_q}(\tau^{n_q}_{i-1}))| = +\infty. \tag{3.1}$$

Similarly, there is $j \in \{0, \ldots, s\}$, for which $x_{n_q}(\tau^{n_q}_{i^{n_q}_l}) \in [u^j - \delta/6, u^j + \delta/6]$ for infinitely many $q \in \mathbb{N}$. Therefore, passing to a subsequence if necessary, we may assume that $x_{n_q}(\tau^{n_q}_{i^{n_q}_l}) \in [u^j - \delta/6, u^j + \delta/6]$ for every $q \in \mathbb{N}$. So, by the definition of the set $I^{n_q}_l$, we see that $|u^j - x_{n_q}(\tau^{n_q}_i)| \leq \delta$ for $q \in \mathbb{N}$ and $i \in I^{n_q}_l$.

Finally, set

$$u_0 := u^j, \quad d_q := |I^{n_q}_l|, \quad t^q_h := \tau^{n_q}_{i^{n_q}_l + h} \quad \text{for } h = 0, \ldots, d_q;$$

here $|A|$ denotes the cardinality of the set A. Together with the condition (3.1), this proves our assertion. □

The following result gives a necessary and sufficient condition for F to map $BV[0, 1]$ into itself and not be locally bounded.

THEOREM 3.4. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function mapping bounded sets into bounded sets which generates a nonautonomous superposition operator F that maps the space $BV[0, 1]$ into itself. The superposition operator F is not locally bounded in $BV[0, 1]$ if and only if there exist a number $r > 0$ and a point $(t_0, u_0) \in [0, 1] \times [-r, r]$, together with a sequence of functions $(x_q)_{q \in \mathbb{N}} \subset B_{BV}(0, r)$, such that for any $\varepsilon > 0$ and $\delta > 0$ there exists a sequence of finite collections of points $l_\varepsilon(t_0) < t^q_0 < t^q_1 < \cdots < t^q_{d_q} < r_\varepsilon(t_0)$ for which the following properties hold: $x_q(t^q_i) \in [u_0 - \delta, u_0 + \delta]$, for $i = 0, 1, \ldots, d_q$ and all q sufficiently large, and*

$$\lim_{q \to \infty} \sum_{i=1}^{d_q} |f(t^q_i, x_q(t^q_i)) - f(t^q_{i-1}, x_q(t^q_{i-1}))| = +\infty.$$

PROOF. Suppose that the superposition operator F is not locally bounded. Then there exist a number $r > 0$ and a sequence $(x_n)_{n \in \mathbb{N}} \subset B_{BV}(0, r)$ such that $\|F(x_n)\|_{BV} \geq n$ for every $n \in \mathbb{N}$. Thus, in view of Lemma 3.3 there exist $t^1 \in [0, 1]$ and a subsequence

$(x_n^1)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, together with a point $u^1 \in [-r, r]$ and a finite collection of points $l_1(t^1) \le t_0^1 < \cdots < t_{d_1}^1 \le r_1(t^1)$, such that $x_1^1(t_i^1) \in [u^1 - 1, u^1 + 1]$ for $i = 0, 1, \ldots, d_1$ and

$$\sum_{i=1}^{d_1} |f(t_i^1, x_1^1(t_i^1)) - f(t_{i-1}^1, x_1^1(t_{i-1}^1))| \ge 1.$$

Furthermore, note that we may assume that $\|F(x_n^1)\|_{BV} \ge n$ for $n \in \mathbb{N}$, which implies that Lemma 3.3 can be applied to the sequence $(x_n^1)_{n\in\mathbb{N}}$. Therefore, we obtain a family of sequences $(x_n^q)_{n\in\mathbb{N}}$, where $q \in \mathbb{N}$, together with points $(t^q, u^q) \in [0, 1] \times [-r, r]$ and finite collections of points $l_{1/q}(t^q) \le t_0^q < \cdots < t_{d_q}^q \le r_{1/q}(t^q)$, which for $q \in \mathbb{N}$ satisfy the following properties:

- $(x_n^{q+1})_{n\in\mathbb{N}}$ is a subsequence of $(x_n^q)_{n\in\mathbb{N}}$;
- $x_q^q(t_i^q) \in [u^q - 1/q, u^q + 1/q]$ for $i = 0, 1, \ldots, d_q$;
- $\sum_{i=1}^{d_q} |f(t_i^q, x_q^q(t_i^q)) - f(t_{i-1}^q, x_q^q(t_{i-1}^q))| \ge q$; and
- $\|F(x_n^q)\|_{BV} \ge n$ for $n \in \mathbb{N}$.

Now, define $y_q := x_q^q$ for $q \in \mathbb{N}$. Since the sequence $(t^q, u^q)_{q\in\mathbb{N}}$ is bounded, it contains a subsequence $(t^{q_k}, u^{q_k})_{k\in\mathbb{N}}$ which converges to a point $(t_0, u_0) \in [0, 1] \times [-r, r]$. For the sake of simplicity, we assume that $(t^{q_k}, u^{q_k})_{k\in\mathbb{N}}$ coincides with $(t^q, u^q)_{q\in\mathbb{N}}$.

We are going to show that the point (t_0, u_0) and the sequence $(y_q)_{q\in\mathbb{N}}$ satisfy the claim. Fix $\delta > 0$ and $\varepsilon > 0$. Then there exists $q_0 \in \mathbb{N}$ such that

$$l_\varepsilon(t_0) < l_{1/q}(t^q) < r_{1/q}(t^q) < r_\varepsilon(t_0) \quad \text{and} \quad u_0 - \delta < u^q - \frac{1}{q} < u^q + \frac{1}{q} < u_0 + \delta$$

for all $q \ge q_0$. Therefore, for $q \ge q_0$, we take $l_{1/q}(t^q) \le t_0^q < \cdots < t_{d_q}^q \le r_{1/q}(t^q)$ as the required collections of points, whereas for $1 \le q \le q_0 - 1$, one can take an arbitrary partition of $(l_\varepsilon(t_0), r_\varepsilon(t_0))$. Hence, from the first part of the proof, it follows that for $q \ge q_0$ we have $y_q(t_i^q) \in [u_0 - \delta, u_0 + \delta]$ and

$$\sum_{i=1}^{d_q} |f(t_i^q, y_q(t_i^q)) - f(t_{i-1}^q, y_q(t_{i-1}^q))| \ge q.$$

This completes the proof, because the other implication is obvious. □

Let us now introduce some auxiliary notation. For a given $r > 0$, $\delta > 0$ and a point $(t, u) \in [0, 1] \times \mathbb{R}$, let

$$S_\delta^r(t, u) := \Big\{x \in BV[0, 1] : x(\tau) \in [u - \delta, u + \delta] \text{ for } \tau \in [l_\delta(t), r_\delta(t)],$$
$$x(\tau) = u \text{ for } \tau \notin [l_\delta(t), r_\delta(t)] \text{ and } \bigvee_0^1 x \le r\Big\}$$

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and[1]

$$V_\delta^r(t,u) := \sup\Big\{\bigvee_0^1 F(x) : x \in S_\delta^r(t,u)\Big\} \quad \text{as well as } R_\delta(t,u) := \inf\{r > 0 : V_\delta^r(t,u) = +\infty\},$$

where F is the nonautonomous superposition operator generated by the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$. If no confusion can arise, we simply write S_δ^r, V_δ^r and R_δ.

Lemma 3.5. *If $r \le r'$ and $\delta \le \delta'$, then $S_\delta^r(t,u) \subset S_{\delta'}^{r'}(t,u)$.*

Proof. The claim follows from the obvious inclusions $[u-\delta, u+\delta] \subset [u-\delta', u+\delta']$ and $[l_\delta(t), r_\delta(t)] \subset [l_{\delta'}(t), r_{\delta'}(t)]$ and the definition of the sets $S_\delta^r(t,u)$. □

The following two lemmas are crucial in the construction of a function of bounded variation in the sense of Jordan which after superposition with f does not belong to $BV[0,1]$.

Lemma 3.6. *Suppose that the nonautonomous superposition operator F that maps the space $BV[0,1]$ into itself is not locally bounded. If (t_0,u_0) is a point for which the assertion of Theorem 3.4 holds, then $R_\delta(t_0,u_0) = 0$ for every $\delta > 0$.*

Proof. We start by showing that $R_\delta = R_\delta(t_0,u_0)$ is finite for every $\delta > 0$. In view of Theorem 3.4, there exists a sequence $(x_q)_{q\in\mathbb{N}} \subset B_{BV}(0,\bar{r})$, together with finite collections of points $l_\delta(t_0) < t_0^q < \cdots < t_{d_q}^q < r_\delta(t_0)$, such that $x_q(t_i^q) \in [u_0-\delta, u_0+\delta]$ for $i = 0,1,\ldots,d_q$ and all $q \ge q_0$ and

$$\lim_{q\to\infty} \sum_{i=1}^{d_q} |f(t_i^q, x_q(t_i^q)) - f(t_{i-1}^q, x_q(t_{i-1}^q))| = +\infty.$$

Now, define $y_q : [0,1] \to \mathbb{R}$ as a piecewise linear function[2] whose graph is a simple polygonal curve specified by the points

$$(0,u_0), (l_\delta(t_0), u_0), (t_0^q, x_q(t_0^q)), \ldots, (t_{d_q}^q, x_q(t_{d_q}^q)), (r_\delta(t_0), u_0), (1, u_0).$$

Then $y_q \in S_\delta^{\bar{r}+2\delta}$ for $q \ge q_0$, since

$$\bigvee_0^1 y_q = \bigvee_{l_\delta(t_0)}^{r_\delta(t_0)} y_q \le |u_0 - x_q(t_0^q)| + \sum_{i=1}^{d_q} |x_q(t_i^q) - x_q(t_{i-1}^q)| + |x_q(t_{d_q}^q) - u_0| \le 2\delta + \bar{r}.$$

Furthermore,

$$\bigvee_0^1 F(y_q) \ge \sum_{i=1}^{d_q} |f(t_i^q, x_q(t_i^q)) - f(t_{i-1}^q, x_q(t_{i-1}^q))| \to +\infty \quad \text{as } q \to +\infty,$$

which shows that $R_\delta \le 2\delta + \bar{r}$.

[1] Let us recall that if $A = \emptyset$, then, by definition, $\inf A = +\infty$.

[2] Throughout the paper, we understand a *piecewise linear function* to be a continuous function whose graph is a simple polygonal curve which can be specified by *finitely* many vertices.

Now we proceed to the main part of the proof. Suppose on the contrary that $R_\delta > 0$ for some $\delta > 0$. For a given $\delta' \in (0, \frac{1}{16}R_\delta] \cap (0, \delta]$, in view of the first part of the proof, there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset S_{\delta'}^{\bar{r}+2\delta'}$, together with finite collections of points $l_{\delta'}(t_0) < t_0^n < \cdots < t_{d_n}^n < r_{\delta'}(t_0)$ such that for every $n \in \mathbb{N}$, $z_n(t_i^n) \in [u_0 - \delta', u_0 + \delta']$ for $i = 0, 1, \ldots, d_n$ and

$$\bigvee_0^1 F(z_n) \geq \sum_{i=1}^{d_n} |f(t_i^n, z_n(t_i^n)) - f(t_{i-1}^n, z_n(t_{i-1}^n))| \geq n.$$

Observe that $(z_n)_{n\in\mathbb{N}}$ is just a certain subsequence of the sequence $(y_q^{\delta'})_{q\in\mathbb{N}}$ constructed analogously to $(y_q)_{q\in\mathbb{N}}$ from the first part of the proof with δ' in place of δ. Thus,

$$\bigvee_0^1 z_n \leq |u_0 - z_n(t_0^n)| + \sum_{i=1}^{d_n} |z_n(t_i^n) - z_n(t_{i-1}^n)| + |z_n(t_{d_n}^n) - u_0|.$$

Therefore, by the definition of R_δ, without loss of generality, we may assume that

$$\sum_{i=1}^{d_n} |z_n(t_i^n) - z_n(t_{i-1}^n)| \geq \frac{3}{4}R_\delta \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Indeed, if

$$\sum_{i=1}^{d_n} |z_n(t_i^n) - z_n(t_{i-1}^n)| < \frac{3}{4}R_\delta \quad \text{for some } n \in A \subset \mathbb{N},$$

then if $|A| \in \mathbb{N}$, we could omit finitely many terms of the sequence $(z_n)_{n\in\mathbb{N}}$ in order to obtain the required property; on the other hand, if $|A| = \aleph_0$, we would have

$$\bigvee_0^1 z_n \leq 2\delta' + \frac{3}{4}R_\delta \leq \frac{7}{8}R_\delta \quad \text{for } n \in A.$$

This, in turn, would imply that $(z_n)_{n\in A} \subset S_{\delta'}^{(7/8)R_\delta} \subset S_\delta^{(7/8)R_\delta}$, and hence $V_\delta^{(7/8)R_\delta} = +\infty$, from which it would follow that

$$0 < R_\delta := \inf\{r > 0 : V_\delta^r = +\infty\} \leq \tfrac{7}{8}R_\delta.$$

Therefore, as stated before, we may assume that the sequence $(z_n)_{n\in\mathbb{N}}$ satisfies (3.2).

Now, we are going to decompose the expressions $\sum_{i=1}^{d_n} |z_n(t_i^n) - z_n(t_{i-1}^n)|$ into blocks whose sums are greater than or equal to $\frac{1}{2}R_\delta$ and which share the following property: the sum of each block diminished by the last term is either void, or smaller than $\frac{1}{2}R_\delta$. So, let $l_0^n := 0$ and assume that we have already defined l_i^n for some $i \in \mathbb{N} \cup \{0\}$. If $l_i^n = d_n$, then put $l_{i+1}^n := d_n$. On the other hand, if $l_i^n < d_n$, then define l_{i+1}^n to be the smallest index $k \in \{l_i^n + 1, \ldots, d_n\}$ for which

$$\sum_{j=l_i^n+1}^{k} |z_n(t_j^n) - z_n(t_{j-1}^n)| \geq \frac{1}{2}R_\delta,$$

if such an index exists, or $l_{i+1}^n := d_n$, otherwise.

Since

$$\frac{3}{4}R_\delta \le \sum_{j=1}^{d_n} |z_n(t_j^n) - z_n(t_{j-1}^n)| \le \overline{r},$$

we infer that for every $n \in \mathbb{N}$ the sequence $(l_i^n)_{i\in\mathbb{N}\cup\{0\}}$ is eventually constant, that is, there exists a smallest index $q^n \ge 1$ such that $l_i^n = d_n$ for $i \ge q^n$. Furthermore, note that the upper bound for the number q^n is independent of n and does not exceed $m_\delta := (2\overline{r} + R_\delta)/R_\delta$. Indeed, for every $n \in \mathbb{N}$,

$$\begin{aligned}\overline{r} \ge \sum_{j=1}^{d_n} |z_n(t_j^n) - z_n(t_{j-1}^n)| &= \sum_{i=0}^{q^n-1} \sum_{j=l_i^n+1}^{l_{i+1}^n} |z_n(t_j^n) - z_n(t_{j-1}^n)| \\ &\ge \frac{1}{2}(q^n - 1)R_\delta + \sum_{j=l_{q^n-1}^n+1}^{l_{q^n}^n} |z_n(t_j^n) - z_n(t_{j-1}^n)| \ge \frac{1}{2}(q^n-1)R_\delta,\end{aligned}$$

whence the claim follows. Therefore, for each $n \in \mathbb{N}$, we get[1]

$$\begin{aligned}n &\le \sum_{j=1}^{d_n} |f(t_j^n, z_n(t_j^n)) - f(t_{j-1}^n, z_n(t_{j-1}^n))| \\ &= \sum_{i=0}^{q^n-1} \sum_{j=l_i^n+1}^{l_{i+1}^n-1} |f(t_j^n, z_n(t_j^n)) - f(t_{j-1}^n, z_n(t_{j-1}^n))| + \sum_{i=1}^{q^n} |f(t_{l_i^n}^n, z_n(t_{l_i^n}^n)) - f(t_{l_i^n-1}^n, z_n(t_{l_i^n-1}^n))| \\ &\le \sum_{i=0}^{q^n-1} \sum_{j=l_i^n+1}^{l_{i+1}^n-1} |f(t_j^n, z_n(t_j^n)) - f(t_{j-1}^n, z_n(t_{j-1}^n))| + 2Mm_\delta,\end{aligned}$$

where $M := \sup\{|f(t,u)| : t \in [0,1], u \in [u_0 - \delta, u_0 + \delta]\}$. Thus, for any $n \in \mathbb{N}$ sufficiently large,

$$\sum_{i=0}^{q^n-1} \sum_{j=l_i^n+1}^{l_{i+1}^n-1} |f(t_j^n, z_n(t_j^n)) - f(t_{j-1}^n, z_n(t_{j-1}^n))| \ge n - 2Mm_\delta > 0, \tag{3.3}$$

and so there exists $k_n \in \{0, 1, \ldots, q^n - 1\}$ such that

$$\sum_{j=l_{k_n}^n+1}^{l_{k_n+1}^n-1} |f(t_j^n, z_n(t_j^n)) - f(t_{j-1}^n, z_n(t_{j-1}^n))| \ge \frac{n}{m_\delta} - 2M. \tag{3.4}$$

In particular, the above sum is not void, and therefore $l_{k_n+1}^n \ge l_{k_n}^n + 2$.

Observe that, so far, we have constructed finite collections of vertices $(t_j^n, z_n(t_j^n))$ with 'variation' bounded by a constant independent of n that turns out to be smaller

[1] If the upper summation limit is smaller than the lower one, then, by definition, the sum is equal to zero.

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than R_δ and which, after the superposition with f, generate vertices whose 'variation' grows unboundedly. Therefore, to end the proof of Lemma 3.6, it suffices to translate this information into the language of functions. For a fixed n sufficiently large (cf. (3.3)), define $\zeta_n : [0,1] \to \mathbb{R}$ as a piecewise linear function whose graph is a simple polygonal curve specified by the points

$$(0,u_0), (l_{\delta'}(t_0),u_0), (t^n_{l^n_{k_n}}, z_n(t^n_{l^n_{k_n}})), (t^n_{l^n_{k_n}+1}, z_n(t^n_{l^n_{k_n}+1})),$$
$$\dots, (t^n_{l^n_{k_n+1}-2}, z_n(t^n_{l^n_{k_n+1}-2})), (t^n_{l^n_{k_n+1}-1}, z_n(t^n_{l^n_{k_n+1}-1})), (r_{\delta'}(t_0),u_0), (1,u_0).$$

In view of the definition of the numbers l^n_i and the choice of the functions z_n,

$$\bigvee_0^1 \zeta_n \le |u_0 - z_n(t^n_{l^n_{k_n}})| + \sum_{j=l^n_{k_n}+1}^{l^n_{k_n+1}-1} |z_n(t^n_j) - z_n(t^n_{j-1})| + |z_n(t^n_{l^n_{k_n+1}-1}) - u_0| \le \frac{5}{8} R_\delta,$$

and thus $\zeta_n \in S^{(5/8)R_\delta}_{\delta'} \subset S^{(5/8)R_\delta}_{\delta}$. Furthermore, in view of the inequality (3.4),

$$\bigvee_0^1 F(\zeta_n) \ge \frac{n}{m_\delta} - 2M,$$

which proves that $V^{(5/8)R_\delta}_\delta = +\infty$ and contradicts the fact that $R_\delta = \inf\{r > 0 : V^r_\delta = +\infty\}$. Thus, $R_\delta = 0$. □

Lemma 3.7. *Suppose that the nonautonomous superposition operator F that maps the space $BV[0,1]$ into itself is not locally bounded. Moreover, let us assume that $R_\delta(t^*,u^*) = 0$ for every $\delta > 0$, where $t^* \in (0,1)$. Then for any given real numbers $r > 0$ and $\gamma > 0$ and any positive integer $n \in \mathbb{N}$, there exists a piecewise linear function $x \in S^r_\gamma(t^*,u^*)$ whose graph is specified by the points (τ_i,μ_i), where $i = 0,\dots,d$, which share the following properties:*

- $\tau_0 := 0 \le \tau_1 := l_\gamma(t^*) < \tau_2 < \cdots < \tau_{c-1} < \tau_c := t^* < \tau_{c+1} < \cdots < \tau_{d-2} < \tau_{d-1} := r_\gamma(t^*) \le \tau_d := 1$;
- $\mu_0 := u^* \le \mu_1 := u^* < \mu_2 < \cdots < \mu_{c-1} < \mu_c := u^* < \mu_{c+1} < \cdots < \mu_{d-2} < \mu_{d-1} := u^* \le \mu_d := u^*$;
- $3 \le c \le d-3$; *and*
- $\sum_{i=2}^{c-1} |f(\tau_i,\mu_i) - f(\tau_{i-1},\mu_{i-1})| + \sum_{i=c+2}^{d-1} |f(\tau_i,\mu_i) - f(\tau_{i-1},\mu_{i-1})| \ge n.$

Proof. For a fixed $r > 0$ and $\gamma > 0$, in view of the fact that $R_\delta(t^*,u^*) = 0$ for every $\delta > 0$, we infer that $V^{(1/2)r}_{\gamma'} = +\infty$, where $\gamma' := \frac{1}{2}\min\{2\gamma, t^*, 1-t^*, \frac{1}{4}r\}$. Thus, there exists a function $y \in S^{(1/2)r}_{\gamma'}$ with $\bigvee_0^1 F(y) \ge \bigvee_0^1 F(x_{u^*}) + n + 12M + 1$, where $M := \sup\{|f(t,u)| : t \in [0,1], u \in [u^*-\gamma, u^*+\gamma]\}$ and x_{u^*} is the constant function which at every point in the interval $[0,1]$ attains the value u^*. Therefore, there is a finite partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of the interval $[0,1]$ such that

$$\sum_{i=1}^m |f(t_i,y(t_i)) - f(t_{i-1},y(t_{i-1}))| \ge \bigvee_0^1 F(x_{u^*}) + n + 12M =: \sigma.$$

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Let us note that, without loss of generality, we may assume that the partition contains the points $l_{\gamma'}(t^*)$, t^* and $r_{\gamma'}(t^*)$, say, $t_j = l_{\gamma'}(t^*)$, $t_k = t^*$ and $t_l = r_{\gamma'}(t^*)$. Furthermore, we can assume that $j + 2 \le k$ and $k + 2 \le l$.

Define $x : [0, 1] \to \mathbb{R}$ as a piecewise linear function whose graph is determined by the points

$$(0, u^*), (l_\gamma(t^*), u^*), (t_{j+1}, y(t_{j+1})), \dots, (t_{k-1}, y(t_{k-1})), (t_k, u^*),$$
$$(t_{k+1}, y(t_{k+1})), \dots, (t_{l-1}, y(t_{l-1})), (r_\gamma(t^*), u^*), (1, u^*).$$

Since

$$\begin{aligned}\bigvee_0^1 x &= |u^* - y(t_{j+1})| + \sum_{i=j+2}^{k-1} |y(t_i) - y(t_{i-1})| + |y(t_{k-1}) - u^*|\\ &\quad + |u^* - y(t_{k+1})| + \sum_{i=k+2}^{l-1} |y(t_i) - y(t_{i-1})| + |y(t_{l-1}) - u^*| \le \frac{1}{2} r + 4\gamma' \le r,\end{aligned}$$

we see that $x \in S_\gamma^r$. Furthermore,

$$\begin{aligned}\sigma &\le \sum_{i=1}^{m} |f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))|\\
&= \sum_{i=1}^{j-1} |f(t_i, u^*) - f(t_{i-1}, u^*)| + \sum_{i=j}^{l+1} |f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))|\\
&\qquad + \sum_{i=l+2}^{m} |f(t_i, u^*) - f(t_{i-1}, u^*)|\\
&\le \bigvee_0^1 F(x_{u^*}) + \sum_{i=j}^{l+1} |f(t_i, y(t_i)) - f(t_{i-1}, y(t_{i-1}))|\\
&= \bigvee_0^1 F(x_{u^*}) + |f(t_j, y(t_j)) - f(t_{j-1}, y(t_{j-1}))| + |f(t_{j+1}, y(t_{j+1})) - f(t_j, y(t_j))|\\
&\qquad + \sum_{i=j+2}^{k-1} |f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))| + |f(t_k, y(t_k)) - f(t_{k-1}, y(t_{k-1}))|\\
&\qquad + |f(t_{k+1}, y(t_{k+1})) - f(t_k, y(t_k))| + \sum_{i=k+2}^{l-1} |f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))|\\
&\qquad + |f(t_l, y(t_l)) - f(t_{l-1}, y(t_{l-1}))| + |f(t_{l+1}, y(t_{l+1})) - f(t_l, y(t_l))|.\end{aligned}$$

Hence, using the triangle inequality and the local boundedness of the function f,

$$\bigvee_0^1 F(x_{u^*}) + 12M + \sum_{i\in I} |f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))| \ge \sigma = \bigvee_0^1 F(x_{u^*}) + n + 12M,$$

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where $I := \{j+2, \dots, k-2, k-1, k+2, k+3, \dots, l-1\}$, which shows that

$$\sum_{i=j+2}^{k-1} |f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))| + \sum_{i=k+2}^{l-1} |f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))| \geq n.$$

To end the proof it suffices to observe that the function x, together with the collection of points

$$\begin{gathered}(\tau_0, \mu_0) := (0, u^*), \quad (\tau_1, \mu_1) := (l_\gamma(t^*), u^*), \quad (\tau_2, \mu_2) := (t_{j+1}, y(t_{j+1})), \dots, \\ (\tau_{c-1}, \mu_{c-1}) := (t_{k-1}, y(t_{k-1})), \quad (\tau_c, \mu_c) := (t^*, u^*), \quad (\tau_{c+1}, \mu_{c+1}) := (t_{k+1}, y(t_{k+1})), \dots, \\ (\tau_{d-2}, \mu_{d-2}) := (t_{l-1}, y(t_{l-1})), \quad (\tau_{d-1}, \mu_{d-1}) := (r_\gamma(t^*), u^*), \quad (\tau_d, \mu_d) := (1, u^*)\end{gathered}$$

which span the graph of x, satisfies the claim. □

Remark 3.8. Let us observe that a result analogous to Lemma 3.7 with $t^* = 0$ or $t^* = 1$ is also valid.

4. The main results

Now we are going to prove the main result of our paper concerning the local boundedness of the nonautonomous superposition operator acting in the space $BV[0, 1]$.

Theorem 4.1. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function mapping bounded sets into bounded sets which generates the nonautonomous superposition operator F. If the superposition operator F maps the space $BV[0, 1]$ into itself, then it is automatically locally bounded.*

Proof. On the contrary, suppose that F maps the space $BV[0, 1]$ into itself but is not locally bounded. Given $\delta_1 = \frac{1}{4}$, $r_1 = \frac{1}{2}$ and $n = 1$, in view of Theorem 3.4 and Lemmas 3.6 and 3.7, there exists a piecewise linear function $x_1 \in S_{\delta_1}^{r_1}(t_0, u_0)$ satisfying the properties described in Lemma 3.7. For the sake of simplicity, as in Lemma 3.7, assume that $t_0 \in (0, 1)$. In particular, t_0 is contained in the interior of the interval $[\tau^1_{c_1-1}, \tau^1_{c_1+1}]$. Hence $\delta_2 := \frac{1}{4} \min\{\delta_1, t_0 - \tau^1_{c_1-1}, \tau^1_{c_1+1} - t_0\} > 0$, and so Lemma 3.7 is applicable with $n = 2$, $r_2 = \frac{1}{2^2}$ and δ_2 as defined above.

The continuation of this procedure yields a sequence $(x_n)_{n\in\mathbb{N}}$ of piecewise linear functions whose graphs are specified by the points (τ^n_i, μ^n_i), $i = 0, \dots, d_n$, satisfying the properties described in Lemma 3.7. In particular, for $n \in \mathbb{N}$:

- $x_n \in S_{\delta_n}^{r_n}(t_0, u_0)$, where $r_n = 1/2^n$ and $0 < \delta_n \leq 1/4^n$;
- t_0 is contained in the interior of the interval $[\tau^n_{c_n-1}, \tau^n_{c_n+1}]$;
- $l_{\delta_n}(t_0) < \tau^n_{c_n-1} < l_{\delta_{n+1}}(t_0) < r_{\delta_{n+1}}(t_0) < \tau^n_{c_n+1} < r_{\delta_n}(t_0)$; and
- $\sum_{i=2}^{c_n-1} |f(\tau^n_i, x_n(\tau^n_i)) - f(\tau^n_{i-1}, x_n(\tau^n_{i-1}))| + \sum_{i=c_n+2}^{d_n-1} |f(\tau^n_i, x_n(\tau^n_i)) - f(\tau^n_{i-1}, x_n(\tau^n_{i-1}))| \geq n$.

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Let us now define the function $x : [0,1] \to \mathbb{R}$ in the following way:

$$x(t) = \begin{cases} u_0 & \text{if } t = t_0 \text{ or } t \in [0,1]\setminus[l_{\delta_1}(t_0), r_{\delta_1}(t_0)], \\ x_n(t) & \text{if } t \in [l_{\delta_n}(t_0), r_{\delta_n}(t_0)]\setminus(\tau^n_{c_n-1}, \tau^n_{c_n+1}) \text{ for some } n \in \mathbb{N}, \\ \xi_n(t) & \text{if } t \in (\tau^n_{c_n-1}, l_{\delta_{n+1}}(t_0)) \text{ for some } n \in \mathbb{N}, \\ \eta_n(t) & \text{if } t \in (r_{\delta_{n+1}}(t_0), \tau^n_{c_n+1}) \text{ for some } n \in \mathbb{N}, \end{cases}$$

where

$$\xi_n(t) = \frac{l_{\delta_{n+1}}(t_0) - t}{l_{\delta_{n+1}}(t_0) - \tau^n_{c_n-1}} x_n(\tau^n_{c_n-1}) + \frac{t - \tau^n_{c_n-1}}{l_{\delta_{n+1}}(t_0) - \tau^n_{c_n-1}} u_0 \quad \text{for } t \in (\tau^n_{c_n-1}, l_{\delta_{n+1}}(t_0))$$

and

$$\eta_n(t) = \frac{t - r_{\delta_{n+1}}(t_0)}{\tau^n_{c_n+1} - r_{\delta_{n+1}}(t_0)} x_n(\tau^n_{c_n+1}) + \frac{\tau^n_{c_n+1} - t}{\tau^n_{c_n+1} - r_{\delta_{n+1}}(t_0)} u_0 \quad \text{for } t \in (r_{\delta_{n+1}}(t_0), \tau^n_{c_n+1}).$$

The above definition is correct. Indeed, defining

$$\begin{aligned} \mathcal{A} := \{&[l_{\delta_n}(t_0), r_{\delta_n}(t_0)]\setminus(\tau^n_{c_n-1}, \tau^n_{c_n+1}) : n \in \mathbb{N}\} \\ &\cup \{(\tau^n_{c_n-1}, l_{\delta_{n+1}}(t_0)) : n \in \mathbb{N}\} \cup \{(r_{\delta_{n+1}}(t_0), \tau^n_{c_n+1}) : n \in \mathbb{N}\} \end{aligned}$$

and $J_n := [l_{\delta_n}(t_0), l_{\delta_{n+1}}(t_0)) \cup (r_{\delta_{n+1}}(t_0), r_{\delta_n}(t_0)]$ for $n \in \mathbb{N}$, we infer that the family $\mathcal{A}$ consists of pairwise disjoint sets, and, moreover,

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{n=1}^{\infty} J_n = [l_{\delta_1}(t_0), r_{\delta_1}(t_0)]\setminus\{t_0\}.$$

Now, we are going to show that $x \in BV[0,1]$. Let us consider an arbitrary finite partition $0 = s_0 < s_1 < \cdots < s_k = 1$ of the interval $[0,1]$. Clearly, we may assume that $s_m = t_0$ for some $m \in \{1, \ldots, k-1\}$ and, moreover, that $s_{m-1} \geq l_{\delta_1}(t_0)$ and $s_{m+1} \leq r_{\delta_1}(t_0)$. Then, adding new points to the partition and reindexing, if necessary, we may assume that $s_{m-1}, s_{m+1} \in J_n$ for some $n \in \mathbb{N}$ and that $s_{m-1} = \tau^n_{c_n-1}$, $s_{m+1} = \tau^n_{c_n+1}$. Then

$$\begin{aligned} \sum_{i=1}^{k} |x(s_i) - x(s_{i-1})| &= \sum_{i=1}^{m-1} |x(s_i) - x(s_{i-1})| + \sum_{i=m+2}^{k} |x(s_i) - x(s_{i-1})| \\ &\qquad + |x(s_m) - x(s_{m-1})| + |x(s_{m+1}) - x(s_m)| \\ &\leq \sum_{l=1}^{n} \Big(\bigvee_0^1 x_l + |x_l(\tau^l_{c_l-1}) - u_0| + |x_l(\tau^l_{c_l+1}) - u_0| \Big) \\ &\leq \sum_{l=1}^{\infty} \frac{1}{2^l} + \sum_{l=1}^{\infty} \frac{2}{4^l} = \frac{5}{3}. \end{aligned}$$

Thus, $x \in BV[0,1]$.

To complete the proof we need to show that $F(x) \notin BV[0,1]$. To this end, for every $n \in \mathbb{N}$ let us consider a finite partition of the interval $[0,1]$ defined as follows:

$$\tau^n_0 = 0 \leq \tau^n_1 < \tau^n_2 < \cdots < \tau^n_{c_n-1} < \tau^n_{c_n} < \tau^n_{c_n+1} < \cdots < \tau^n_{d_n-1} \leq \tau^n_{d_n} = 1.$$

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Then

$$\bigvee_0^1 F(x) \geq \sum_{i=2}^{c_n-1} |f(\tau_i^n, x_n(\tau_i^n)) - f(\tau_{i-1}^n, x_n(\tau_{i-1}^n))| + \sum_{i=c_n+2}^{d_n-1} |f(\tau_i^n, x_n(\tau_i^n)) - f(\tau_{i-1}^n, x_n(\tau_{i-1}^n))|,$$

which is greater than or equal to n, contradicting the fact that the superposition operator maps the space $BV[0, 1]$ into itself. □

From Theorem 4.1 we get the following result.

Corollary 4.2. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function mapping bounded sets into bounded sets which generates the nonautonomous superposition operator F. If there exists a pointwise bounded sequence $(x_n)_{n\in\mathbb{N}}$ of BV-functions with $\sup_{n\in\mathbb{N}} \bigvee_0^1 x_n < +\infty$ such that $\lim_{n\to\infty} \bigvee_0^1 F(x_n) = +\infty$, then the superposition operator does not map the space $BV[0, 1]$ into itself.*

Thanks to Theorem 4.1, we can refine the main result of the paper by Bugajewska *et al.* (Nonautonomous superposition operators in the spaces of functions of bounded variation, submitted for publication) concerning the necessary and sufficient conditions for the inclusion $F(BV[0, 1]) \subset BV[0, 1]$.

Theorem 4.3. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function mapping bounded sets into bounded sets which generates the nonautonomous superposition operator F. Then the following conditions are equivalent:*

(i) *the nonautonomous superposition operator F maps the space $BV[0, 1]$ into itself;*

(ii) *for every $r > 0$ there exists a constant $M_r > 0$ such that for every $k \in \mathbb{N}$, every finite partition $0 = t_0 < \cdots < t_k = 1$ of the interval $[0, 1]$ and every finite sequence $u_0, u_1, \ldots, u_k \in [-r, r]$ with $\sum_{i=1}^k |u_i - u_{i-1}| \leq r$, the following inequalities hold:*

$$\sum_{i=1}^k |f(t_i, u_i) - f(t_{i-1}, u_i)| \leq M_r \quad \text{and} \quad \sum_{i=1}^k |f(t_{i-1}, u_i) - f(t_{i-1}, u_{i-1})| \leq M_r.$$

Proof. We omit the proof of Theorem 4.3 since it follows directly from Theorem 4.1 and Theorem 6 in the paper by Bugajeswski *et al.* ('Nonautonomous superposition operators in the spaces of functions of bounded variation', submitted for publication). □

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PIOTR KASPRZAK, Optimization and Control Theory Department,
Faculty of Mathematics and Computer Science,
Adam Mickiewicz University, ul. Umultowska 87,
61-614 Poznań, Poland
e-mail: kasp@amu.edu.pl

PIOTR MAĆKOWIAK, Department of Mathematical Economics,
Poznań University of Economics, Al. Niepodległości 10,
61-875 Poznań, Poland
e-mail: p.mackowiak@ue.poznan.pl