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ON $\mathfrak{m}$-PARACOMPACT SPACES AND ω-MAPPINGS

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1. **Introduction.** The purpose of this note is to present a characterization of $\mathfrak{m}$-paracompact normal spaces in terms of ω-mappings into $\mathfrak{m}$-separable metric spaces; this result is almost contained in Morita's original paper [2] and is implicitly contained in Shapiro's thesis paper [5]. This result is a natural generalization of the well-known Katětov-Ponomarev characterization of paracompact spaces (see [4]), and a special case of it (when $\mathfrak{m}=\aleph_0$) was recently discovered by Pareek [3].

If $\mathfrak{m}$ is an infinite cardinal number, then $\mathfrak{m}$-separable metric spaces are defined in [5], and $\mathfrak{m}$-paracompact normal spaces are defined in [2]. For the concept of normal covers, see [7], and for the term ω-mapping, see [3] or [4].

2. **The theorem.** Specifically, we augment Morita's Theorem 1.2 of [2] by proving the following.

THEOREM. *Let $\mathfrak{m}$ be an infinite cardinal number and let ω be an open cover of the topological space X such that the power of ω is $\leq\mathfrak{m}$. Then ω is a normal cover of X if and only if there exists an $\mathfrak{m}$-separable metric space Y and an ω-mapping f of X onto Y.*

Proof. Suppose that ω is normal. The proof of Theorem 1 of Stone's paper [6] shows that ω has a locally finite σ-discrete open refinement ω_0 having the same power as ω. We need only the local finiteness of ω_0 and the fact that its power is $\leq\mathfrak{m}$. Let $\omega_0=\{U_\alpha\}_{\alpha\in I}$, where the power of I is $\leq\mathfrak{m}$. Then, using the first part of the proof of Michael's Proposition 1 in [1], there is a locally finite partition of unity $\{\varphi_\alpha\}_{\alpha\in I}$ such that, for each $\alpha\in I$, $\operatorname{coz}(\varphi_\alpha)=\{x:\varphi_\alpha(x)>0\}\subset U_\alpha$. The remainder of the proof follows the arguments of Morita [2]. Indeed, let $H^{\mathfrak{m}}$ be the generalized Hilbert cube of weight $\mathfrak{m}$, so that $H^{\mathfrak{m}}$ is $\mathfrak{m}$-separable (see [8, p. 170]). Define the map $\varphi:X\to H^{\mathfrak{m}}$ by $\varphi(x)=\{\varphi_\alpha(x)\}_{\alpha\in I}$; φ is continuous. Indeed, if $\varepsilon>0$ and $x\in X$, then we can first pick a neighborhood O_0x such that O_0x meets only a finite number of the sets $\operatorname{coz}(\varphi_\alpha)$, say $\operatorname{coz}(\varphi_{\alpha_1}),\ldots,\operatorname{coz}(\varphi_{\alpha_k})$; and x is clearly in one of them. Moreover, we can also find, for each $i=1,\ldots,k$, a neighborhood O_ix of x such that $y\in O_ix$ implies $(\varphi_{\alpha_i}(x)-\varphi_{\alpha_i}(y))^2<\varepsilon/k$. Now let $Ox=\bigcap_{i=0}^{k} O_ix$. If $y\in Ox$, then $d(\varphi(x),\varphi(y))<\varepsilon$, where d is the metric of $H^{\mathfrak{m}}$, and so φ is continuous at each

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$x \in X$. Next, for each $\beta \in I$, let V_β denote the set of all points $\{x_\alpha\}_{\alpha \in I}$ of $H^{\mathfrak{m}}$ such that $x_\beta > 0$; then $\varphi^{-1}(V_\beta) = \text{coz}(\varphi_\beta) \subset U_\beta$, so φ is an ω-mapping of X into $H^{\mathfrak{m}}$. φ, in turn, determines an ω-mapping f of X onto the $\mathfrak{m}$-separable metric space $Y = \varphi(X) \subset H^{\mathfrak{m}}$.

The converse easily follows from results out of [7]. Indeed, if $f: X \to Y$ is an ω-mapping of X onto any metric space Y, then there is an open cover η of Y such that $f^{-1}\eta$ refines ω. But Y is metric, so η is normal in Y, hence $f^{-1}\eta$ is normal in X. Therefore ω is normal in X.

Corollary. *A topological space X is normal and $\mathfrak{m}$-paracompact if and only if, for each open cover ω of X of power $\leq \mathfrak{m}$, there exists an $\mathfrak{m}$-separable metric space Y, depending on ω, and an ω-mapping f of X onto Y.*

This corollary follows at once from the above theorem together with Theorem 1.1 of [2].

Clearly the Katětov-Ponomarev Theorem, and the result obtained by Pareek, both follow from this corollary.

3. Remark. The paper [5], by Shapiro, implicitly contains the above results; the proofs in his paper depend on the delicate relationship between normal sequences of open covers and continuous pseudometrics. The proof becomes complicated in that one needs to pass from normal sequences to their associated pseudometrics several times. But, Lemma 2.6 of [5] shows that if the normal cover ω has power $\leq \mathfrak{m}$, then it has a normal sequence $\{\omega_n\}_{n=1}^{\infty}$ of open covers each of power $\leq \mathfrak{m}$. Then the pseudometric d associated with this normal sequence (see [5, 2.4]) is continuous relative to the topology τ of X and the pseudometric topology τ_d is $\mathfrak{m}$-separable. By the way that the d-open balls are related to the stars of the various covers ω_n about the points of X, it follows that the identity map $(X, \tau) \to (X, \tau_d)$ is continuous and an ω-mapping. To get the Y of the theorem, one then identifies all points of X that are at d-distance 0 from one another.

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