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ON COMMUTATIVE REDUCED FILIAL RINGS

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Abstract

A ring in which every accessible subring is an ideal is called filial. We continue the study of commutative reduced filial rings started in [R. R. Andruszkiewicz and K. Pryszczepko, 'A classification of commutative reduced filial rings', *Comm. Algebra* to appear]. In particular we describe the Noetherian commutative reduced rings and construct nontrivial examples of commutative reduced filial rings without ideals which are domains.

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1. Introduction and preliminaries

Throughout this paper we assume that all rings are associative not necessarily with unity. We denote by $\mathbb{Z}$ the ring of integers, and by $\mathbb{P}$ the set of all prime integers. If $p \in \mathbb{P}$ then we write $\mathbb{Z}_p$ and $\mathbb{Q}_p$ for the ring of p-adic integers and the quotient field of p-adic integers, respectively. For arbitrary $\Pi \subseteq \mathbb{P}$ we denote $\mathbb{Q}_\Pi = \prod_{p\in\Pi} \mathbb{Q}_p$, $\mathbb{Z}_\Pi = \prod_{p\in\Pi} \mathbb{Z}_p$.

An associative ring R is called filial if $A \lhd B \lhd R$ implies $A \lhd R$ for all subrings A, B of R. The problem of describing filial rings was raised by Szász in [12]. The problem has been studied by various authors, namely, Ehrlich [5], Filipowicz and Puczyłowski [6, 7] Sands [10] and Veldsman [13].

A ring R is strongly regular if $a \in Ra^2$ for every $a \in R$. It is well known that all strongly regular rings are von Neumann regular and for commutative rings the two properties coincide. The class of all strongly regular rings $\mathbb{S}$ forms a radical in the sense of Kurosh and Amitsur [8]. A ring is reduced if it has no nontrivial nilpotent elements.

For a torsion-free ring R let $\Pi(R) = \{p \in \mathbb{P} \mid pR \neq R\}$. A ring R is called a CRF-ring if R is commutative, reduced and filial. Theorem 4.4 in [2] gives the following description of the $\mathbb{S}$-semisimple CRF-rings with an identity.

THEOREM 1.1. *Let Π be an arbitrary nonempty subset of $\mathbb{P}$. Then a ring R is an $\mathbb{S}$-semisimple CRF-ring with an identity, such that $\Pi(R) = \Pi$ if and only if R is isomorphic to a subring of $\mathbb{Q}_\Pi$ of the form $K \cap \mathbb{Z}_\Pi$ where K is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$.*

The above theorem is important because every CRF-ring is an extension of a commutative strongly regular ring by an $\mathbb{S}$-semisimple CRF-ring (see [2, Proposition 4.1]).

In the present paper we study some nontrivial consequences of Theorem 1.1. In particular, using some techniques from Boolean algebra theory we characterize Noetherian CRF-rings. We also prove a structure theorem for finitely generated CRF-rings. Finally, we describe CRF-rings without ideals which are domains, and we give some nontrivial examples of such rings.

We shall need the following result proved in [2].

THEOREM 1.2. *If R is an $\mathbb{S}$-semisimple torsion-free CRF-ring without an identity, then R is isomorphic to some essential ideal of a ring S, where S is a torsion-free CRF-subring of the ring $\mathrm{End}_R(R)$ with an identity and $\Pi(R) = \Pi(S)$.*

Let K be a subring of $\mathbb{Q}_\Pi$ with the same identity. Take any $a \in K$. Let us denote by $\mathrm{supp}(a)$ the set $\{p \in \Pi \mid a_p \neq 0\}$. Then $\mathcal{B}_K = \{\mathrm{supp}(a) \mid a \in K\}$ is a Boolean algebra.

For every $Y \subseteq \Pi$, we define $\chi_Y = (a_p)_{p\in\Pi} \in \mathbb{Z}_\Pi$ to be

$$a_p = \begin{cases} 0 & \text{if } p \notin Y \\ 1 & \text{if } p \in Y. \end{cases} \tag{1.1}$$

LEMMA 1.3. *Let Π be an arbitrary nonempty subset of $\mathbb{P}$. Let K be a subring of $\mathbb{Q}_\Pi$ with the same identity. Then K is a strongly regular ring if and only if for every $a \in K$ there exists $b \in K$ such that $ab = \chi_{\mathrm{supp}(a)}$. In particular, if K is a strongly regular ring, then $\chi_Y \in K$ for every $Y \in \mathcal{B}_K$.*

LEMMA 1.4. *Let Π be an arbitrary nonempty subset of $\mathbb{P}$. Let K be a strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$. Put $S = K \cap \mathbb{Z}_\Pi$. Then:*

(1) *every ideal J of K is of the form $J = \{(1/n)i \mid i \in J \cap S, n \in \mathbb{N}\}$;*
(2) *if S is Noetherian, then K is also Noetherian;*
(3) *S contains a nonzero ideal which is a domain, if and only K contains a nonzero ideal which is a domain.*

PROOF. (1) According to the proof of Theorem 4.4 of [2], $K = \{(1/n)a \mid a \in S, n \in \mathbb{N}\}$. Let us first observe that $J \lhd K$ implies that $J \cap S \lhd S$. We claim that $J = \{(1/n)i \mid i \in J \cap S, n \in \mathbb{N}\}$. Indeed, $(1/n)i = ((1/n) \cdot 1)i \in J$ for $i \in J \cap S$. If $j \in J$, there exists $n \in \mathbb{N}$ such that $n \cdot j \in S$. Then obviously $j = (1/n)(nj)$.

Parts (2) and (3) are direct consequences of (1). □

2. Finiteness conditions for $\mathbb{S}$-semisimple CRF-rings

For a nonempty subset X of a ring R, $\langle X\rangle$ will denote the additive subgroup by X, and $[X]$ will denote the subring generated generated by$X$. Let (a, b) denote the greatest common divisor of given integers a and b.

THEOREM 2.1. *Given a ring R with an identity element, the following conditions are equivalent.*

(1) *R is a Noetherian $\mathbb{S}$-semisimple CRF-ring.*
(2) *$R \cong \bigoplus_{i=1}^{n} D_i$, where D_i is a filial integral domain of characteristic 0, which is not a field for every $i \in \{1, 2, \ldots, n\}$ and $\Pi(D_i) \cap \Pi(D_j) = \emptyset$ for $i \neq j$.*

PROOF. Suppose a ring R with an identity satisfies (1). We first note that by Theorem 1.1 there exist a nonempty subset $\Pi \subseteq \mathbb{P}$ and a unique strongly regular subring K of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and $R \cong K \cap \mathbb{Z}_\Pi$. Lemma 1.4 yields that K is Noetherian. Applying Lemma 1.3, we get that $\mathcal{B}_K$ is an Artinian Boolean algebra ($\mathcal{B}_K$ satisfies the descending chain condition).

Next, we can take pairwise disjoint atoms $\Pi_1, \ldots, \Pi_k \in \mathcal{B}_K$ such that $\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$. This is possible thanks to some standard results in Boolean algebra theory (see [9]). A trivial verification and Lemma 1.3 show that $\chi_{\Pi_1}, \chi_{\Pi_2}, \ldots, \chi_{\Pi_k} \in K$ are pairwise orthogonal idempotents and $1 = \chi_{\Pi_1} + \chi_{\Pi_2} + \cdots + \chi_{\Pi_k}$. Since Π_i is an atom, $\chi_{\Pi_i} K$ is an integral domain. But $\chi_{\Pi_i} K$ is an ideal in a strongly regular ring K, hence $\chi_{\Pi_i} K \in \mathbb{S}$. From this we conclude that $\chi_{\Pi_i} K$ is a field. It follows that $K = \bigoplus_{i=1}^{k} \chi_{\Pi_i} K$ and consequently $R \cong \bigoplus_{i=1}^{k}[(\chi_{\Pi_i} K) \cap \mathbb{Z}_{\Pi_i}]$. Moreover, [1, Theorem 8.8] gives that $D_i = (\chi_{\Pi_i} K) \cap \mathbb{Z}_{\Pi_i}$ is a filial integral domain of characteristic 0 and $\Pi(D_i) = \Pi_i$ for $i \in \{1, 2, \ldots, k\}$.

Finally, suppose that (2) holds. Note that [4, Corollary 3] implies that R is an $\mathbb{S}$-semisimple CRF-ring. From [1, Theorem 3.3] it follows that D_i is a Noetherian ring as a principal ideal domain. Obviously R is a Noetherian ring. □

We have been working under the assumption that a ring has an identity element. This condition was essential for the above proof. We will now show how to dispense with this assumption.

THEOREM 2.2. *The following conditions on a ring R are equivalent.*

(1) *R is a Noetherian $\mathbb{S}$-semisimple CRF-ring.*
(2) *$R \cong \bigoplus_{i=1}^{n} m_i D_i$, where D_i is a filial integral domain of characteristic 0, which is not a field, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$ and $\Pi(D_i) \cap \Pi(D_j) = \emptyset$ for $i \neq j$.*

PROOF. Let R be a Noetherian $\mathbb{S}$-semisimple CRF-ring. Let us first observe that Theorem 1.2 shows that there exists a torsion-free CRF-ring S with an identity such that R is an essential ideal in S. Since R is a Noetherian ring, $\mathrm{End}_R(R)$ is a

Noetherian R-module. But S is an R-submodule of $\mathrm{End}_R(R)$, so S is a Noetherian R-module. Consequently S is a Noetherian ring. According to Theorem 2.1 we have $S \cong \bigoplus_{i=1}^n D_i$, where D_i is a filial integral domain of characteristic 0, which is not a field for every $i \in \{1, 2, \ldots, n\}$ and $\Pi(D_i) \cap \Pi(D_j) = \emptyset$ for $i \neq j$. Since R is an essential ideal of S it is easy to see that $R \cong \bigoplus_{i=1}^n J_i$, where J_i is an nonzero ideal of D_i. Applying [1, Theorem 3.3], we get $J_i \cong m_i D_i$, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$. Finally, $R \cong \bigoplus_{i=1}^n m_i D_i$. This shows that (1) implies (2).

Suppose that (2) holds. From [1, Theorem 3.3] we get that D_i is a Noetherian ring. By filiality of D_i it follows that $m_i D_i$ is a Noetherian ring for every $i \in \{1, 2, \ldots, n\}$. Consequently, R is a Noetherian ring. Moreover, from [4, Corollary 3] it may be concluded that R is an $\mathbb{S}$-semisimple CRF-ring. □

Our next goal is to determine the structure of Noetherian CRF-rings. Suppose now that R is a Noetherian CRF-ring such that $\mathbb{S}(R) \neq 0$. It is easy to verify that $\mathbb{S}(R)$ is a Noetherian ring with an identity. So $\mathbb{S}(R)$ is a direct summand of R. Let $R = \mathbb{S}(R) \oplus T$. Since T satisfies conditions of Theorem 2.2 so we need only consider $\mathbb{S}(R)$. But the standard computation shows that every strongly regular, Noetherian CRF-ring is a finite direct sum of fields (see, for instance, [11]).

Applying the above observation and Theorem 2.2, one can immediately obtain the following structure theorem.

THEOREM 2.3. *The following conditions on a ring R are equivalent.*

(1) *R is a Noetherian CRF-ring.*
(2) *$R \cong (\bigoplus_{j=1}^k F_j) \oplus (\bigoplus_{i=1}^n m_i D_i)$, where D_i is a filial integral domain of characteristic 0, which is not a field, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$, $\Pi(D_i) \cap \Pi(D_t) = \emptyset$ for $i \neq t$ and F_j is a field for every $j \in \{1, 2, \ldots, k\}$.*

As a final result in this section, we prove an analogue of Theorem 2.3 for finitely generated $\mathbb{S}$-semisimple CRF-rings.

THEOREM 2.4. *The following conditions on a ring R are equivalent.*

(1) *R is a finitely generated CRF-ring.*
(2) *$R \cong (\bigoplus_{j=1}^k F_j) \oplus (\bigoplus_{i=1}^n m_i D_i)$ where D_i is a finitely generated subring of $\mathbb{Q}$ with identity, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$, $\Pi(D_i) \cap \Pi(D_t) = \emptyset$ for $i \neq t$ and F_j is a finite field for every $j \in \{1, 2, \ldots, k\}$.*

PROOF. Suppose that R satisfies condition (1). It is clear that R is Noetherian, so by Theorem 2.3 we obtain that $R \cong (\bigoplus_{j=1}^k F_j) \oplus (\bigoplus_{i=1}^n m_i D_i)$, where D_i is a filial integral domain of characteristic 0, which is not a field, $m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, n\}$, $\Pi(D_i) \cap \Pi(D_t) = \emptyset$ for $i \neq t$ and F_j is a field for every $j \in \{1, 2, \ldots, k\}$. Moreover, every $m_i D_i$ is a homomorphic image of the ring R. So $m_i D_i$ is finitely generated, but by filiality of D_i we have $D_i = m_i D_i + \mathbb{Z} \cdot 1$, so consequently D_i is finitely generated. Applying [1, Theorem 5.1], we see at once that D_i is a finitely generated subring of $\mathbb{Q}$. Every F_j is also a homomorphic image of R. Hence every F_j is finitely generated. But every finitely generated field is finite.

Suppose that (2) holds. Since D_i is a finitely generated subring of $\mathbb{Q}$ with identity, there exists $M \in \mathbb{N}$ such that $D_i = [1/M]$. Hence there exists $k \in \mathbb{N}$ such that $(k, M) = 1$ and $m_i D_i = k[1/M]$ (where $k = m_i/(m_i, M)$). We will show that $m_i D_i = [k/M]$. Clearly $[k/M] \subseteq k[1/M]$. Let $a \in [1/M]$. Then there exist $l \in \mathbb{Z}$ and $t \in \mathbb{N}$ such that $a = l/M^t$. But $(k, M) = 1$, so there are integers u, v such that $k^{t-1}u + M^{t-1}v = 1$. Thus $ka = (k/M)^t lu + (k/M)lv \in [k/M]$. Consequently, $m_i D_i$ is finitely generated for every $i = 1, \ldots, n$. It is obvious that every F_j is finitely generated. Hence R is a finitely generated. Moreover, $\bigoplus_{i=1}^{n} m_i D_i$ is a CRF-ring by [4, Corollary 3] and $\bigoplus_{j=1}^{k} F_j$ is clearly a subidempotent ring. Proposition 3 of [3] implies that R is filial. □

3. CRF-rings without ideals which are domains

THEOREM 3.1. *Let Π be an arbitrary nonempty subset of $\mathbb{P}$. Then R is an $\mathbb{S}$-semisimple CRF-ring with an identity without ideals which are domains, such that $\Pi(R) = \Pi$ if and only if R is isomorphic to a subring of $\mathbb{Q}_\Pi$ of the form $K \cap \mathbb{Z}_\Pi$ where K is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and the Boolean algebra $\mathcal{B}_K$ is atom-free.*

PROOF. Let R be an $\mathbb{S}$-semisimple CRF-ring with an identity without ideals which are domains, such that $\Pi(R) = \Pi$. From Theorem 1.1 we have that R is isomorphic to a subring of $\mathbb{Q}_\Pi$ of the form $K \cap \mathbb{Z}_\Pi$ where K is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$. Lemma 1.4 implies that a ring K does not contain an ideal which is a domain. Take any nonempty $Y \in B_K$. By Lemma 1.3, $a = \chi_Y \in K$. But $I = Ka$ is not a domain so there exist $c, d \in I$ such that $cd = 0$. Obviously $\emptyset \neq \mathrm{supp}(c) \subseteq Y$ and $\emptyset \neq \mathrm{supp}(d) \subseteq Y$. Moreover, $\mathrm{supp}(c) \cap \mathrm{supp}(d) = \emptyset$ because $cd = 0$. Hence $\mathrm{supp}(c) \subsetneq Y$ or $\mathrm{supp}(d) \subsetneq Y$ and $\mathcal{B}_K$ is atom-free.

Conversely, according to Lemma 1.4 it is sufficient to prove that a ring K does not contain an ideal which is a domain. Let $\{0\} \neq I \lhd K$. Take any nonzero $a \in I$. B_K is atom-free so exists $Y \in B_K$ such that $\emptyset \neq Y \subsetneq \mathrm{supp}(a)$. Lemma 1.3 implies that $\chi_Y, \chi_{\mathrm{supp}(a)\setminus Y} \in K$ and $a\chi_Y, a\chi_{\mathrm{supp}(a)\setminus Y}$ are nonzero elements of I. Finally, I is not a domain and the proof is complete. □

From Theorems 1.2 and 3.1 we can easy obtain following structure theorem.

THEOREM 3.2. *R is an $\mathbb{S}$-semisimple CRF-ring without ideals which are domains if and only if R is isomorphic to some essential ideal of a ring of the form $K \cap \mathbb{Z}_\Pi$, where K is the unique strongly regular subring of $\mathbb{Q}_\Pi$ with the same identity, such that for every $a \in K$, $a = (a_p)_{p\in\Pi}$, we have $a_p \in \mathbb{Z}_p$ for almost all $p \in \Pi$ and the Boolean algebra $\mathcal{B}_K$ is atom-free.*

4. Example

EXAMPLE 4.1. Let p be any prime number. Let $A_{i,k} = \{p^i t + k \mid t \in \mathbb{N}\}$ for $i \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, p^i - 1\}$. Let

$$\mathfrak{D} = \left\{ \bigcup_{j=1}^{n} X_j \,\middle|\, n \in \mathbb{N}\, \forall_j\, \exists_{i \in \mathbb{N}_0}\, \exists_{k \in \{0,1,\ldots,p^i-1\}}\, X_j = A_{i,k} \right\}.$$

It is easy to see that for $i_1 \leq i_2$,

$$A_{i_1,k_1} \cap A_{i_2,k_2} = \begin{cases} A_{i_2,k_2} & \text{if } k_1 \equiv k_2 \bmod p^{i_1} \\ \emptyset & \text{if } k_1 \not\equiv k_2 \bmod p^{i_1}. \end{cases}$$

So every element of $\mathfrak{D}$ can be written as a disjoint sum of sets $A_{i,k}$. This means that if $X, Y \in \mathfrak{D}$ then $X \cap Y \in \mathfrak{D}$. Next, it is also clear that $A'_{i,k} = \mathbb{N} \setminus A_{i,k} = \bigcup_{j \in \{0,1,\ldots,p^i-1\}, j \neq k} A_{i,j} \in \mathfrak{D}$. So $\mathfrak{D}$ is a field of sets. Of course, for every $A_{i,k}$ and for every $j > i$, $A_{i,k} \supsetneq A_{i,j}$.

EXAMPLE 4.2. Let $\Pi = \{p_1, p_2, \ldots\}$ be any infinite subset of prime numbers. Let $\mathfrak{D}$ be any atom-free Boolean algebra of subsets of Π. Such an algebra does exist, by Example 4.1. In $\mathbb{Q}_\Pi$ we define

$$K = [a\chi_Y : Y \in \mathfrak{D}, 0 \neq a \in \mathbb{Q}]. \tag{4.1}$$

It is easy to see that

$$K = \langle a\chi_Y : Y \in \mathfrak{D}, 0 \neq a \in \mathbb{Q} \rangle. \tag{4.2}$$

Hence every nonzero $d \in K$ can be written in the form

$$d = a_1\chi_{Y_1} + a_2\chi_{Y_2} + \cdots + a_k\chi_{Y_k} \tag{4.3}$$

where $0 \neq a_i \in \mathbb{Q}$, $\emptyset \neq Y_i \in \mathfrak{D}$ for every $i \in \{1, 2, \ldots, k\}$, $Y_i \cap Y_j = \emptyset$ for $i \neq j$ and $\operatorname{supp}(d) = Y_1 \cup Y_2 \cup \cdots \cup Y_k$. We claim that K is strongly regular. Let d be as in (4.3). Put $d' = a_1^{-1}\chi_{Y_1} + a_2^{-1}\chi_{Y_2} + \cdots + a_k^{-1}\chi_{Y_k}$. Obviously $d' \in K$; moreover, $d \cdot d' = \chi_{\operatorname{supp}(d)} \in K$. So by Lemma 1.3, K is strongly regular subring of $\mathbb{Q}_\Pi$. Clearly $K \cap \mathbb{Z}_\Pi \neq \{0\}$. It is easy to see that $\mathcal{B}_K$ is atom-free, so Theorem 3.1 implies that $K \cap \mathbb{Z}_\Pi$ is a nonzero $\mathbb{S}$-semisimple CRF-ring, without an ideal which is a domain. Moreover, $\Pi(K \cap \mathbb{Z}_\Pi) = \Pi$.

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