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Frobenius categories, Gorenstein algebras and rational surface singularities

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Dedicated to Ragnar-Olaf Buchweitz on the occasion of his 60th birthday

Abstract

We give sufficient conditions for a Frobenius category to be equivalent to the category of Gorenstein projective modules over an Iwanaga–Gorenstein ring. We then apply this result to the Frobenius category of special Cohen–Macaulay modules over a rational surface singularity, where we show that the associated stable category is triangle equivalent to the singularity category of a certain discrepant partial resolution of the given rational singularity. In particular, this produces uncountably many Iwanaga–Gorenstein rings of finite Gorenstein projective type. We also apply our method to representation theory, obtaining Auslander–Solberg and Kong type results.

Contents

1. Introduction

This paper is motivated by the study of certain triangulated categories associated to rational surface singularities, first constructed in [IW11]. The purpose is to develop both the algebraic and geometric techniques necessary to give precise information regarding these categories, and to put them into a more conceptual framework. It is only by developing both sides of the picture that we are able to prove the results that we want.

We explain the algebraic side first. Frobenius categories [Hap88, Kel90, Kel96] are now ubiquitous in algebra, since they give rise to many of the triangulated categories arising in algebraic and geometric contexts. One of the points of this paper is that we should treat Frobenius categories which admit a 'non-commutative resolution' as a special class of Frobenius categories.

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We show that such a Frobenius category is equivalent to the category $\mathrm{GP}(E)$ of Gorenstein projective modules over some Iwanaga–Gorenstein ring E (for definitions see §§ 2, especially 2.1). The precise statement is as follows. For a Frobenius category $\mathcal{E}$ we denote by $\operatorname{proj}\mathcal{E}$ the full subcategory of $\mathcal{E}$ consisting of projective objects and for an object P of $\mathcal{E}$ we denote by $\operatorname{add} P$ the full subcategory of $\mathcal{E}$ consisting of direct summands of finite direct sums of copies of P.

Theorem 1.1 (= Theorem 2.7). *Let $\mathcal{E}$ be a Frobenius category with $\operatorname{proj}\mathcal{E} = \operatorname{add} P$ for some $P \in \operatorname{proj}\mathcal{E}$. Assume that there exists $M \in \mathcal{E}$ such that $A := \operatorname{End}_{\mathcal{E}}(P \oplus M)$ is a noetherian ring of global dimension n. Then the following statements hold.*

(1) *$E := \operatorname{End}_{\mathcal{E}}(P)$ is an Iwanaga–Gorenstein ring of dimension at most n, that is, a noetherian ring with $\operatorname{inj.dim}_E E \leqslant n$ and $\operatorname{inj.dim} E_E \leqslant n$.*

(2) *We have an equivalence $\operatorname{Hom}_{\mathcal{E}}(P,-) : \mathcal{E} \to \mathrm{GP}(E)$ up to direct summands. It is an equivalence if $\mathcal{E}$ is idempotent complete. This induces a triangle equivalence*

$$\underline{\mathcal{E}} \xrightarrow{\simeq} \underline{\mathrm{GP}}(E) \simeq \mathrm{D}_{\mathrm{sg}}(E)$$

up to direct summands. It is an equivalence if $\mathcal{E}$ or $\underline{\mathcal{E}}$ is idempotent complete.

(3) *$\underline{\mathcal{E}} = \operatorname{thick}_{\underline{\mathcal{E}}}(M)$, i.e. the smallest full triangulated subcategory of $\underline{\mathcal{E}}$ containing M which is closed under direct summands is $\underline{\mathcal{E}}$.*

This abstract result has applications in, and is motivated by, problems in algebraic geometry. If R is a Gorenstein singularity, then the category $\mathrm{CM}(R)$ of maximal Cohen–Macaulay modules over R is a Frobenius category. Moreover, if R is a simple surface singularity, then the classical algebraic McKay correspondence can be formulated in terms of the associated stable category $\underline{\mathrm{CM}}(R)$; see [Aus86].

When R is not Gorenstein, $\mathrm{CM}(R)$ is no longer Frobenius. However, for a complete local rational surface singularity R over an algebraically closed field of characteristic zero (for details, see § 3 or § 4.4), there is a subcategory $\mathrm{SCM}(R) \subseteq \mathrm{CM}(R)$ of *special* CM modules (recalled in § 3). By Wunram's GL(2) McKay correspondence [Wun88], if we denote $Y \to \operatorname{Spec} R$ to be the minimal resolution, and let $\{E_i\}_{i\in I}$ denote the set of exceptional curves, then there is a natural bijection

$$\left\{\begin{matrix}\text{non-free indecomposable}\\ \text{special CM } R\text{-modules}\end{matrix}\right\}\Big/\cong \quad\longleftrightarrow\quad \{E_i \mid i \in I\}.$$

We let M_i denote the indecomposable special CM R-module corresponding to the exceptional curve E_i. We remark that the set of exceptional curves can be partitioned into two subsets, namely $I = \mathcal{C} \cup \mathcal{D}$ where $\mathcal{C}$ are all the (C)repant curves (i.e. the (-2)-curves), and $\mathcal{D}$ are all the (D)iscrepant curves (i.e. the non-(-2)-curves). In this paper, the following module plays a central role.

Definition 1.2. We define the module $D \in \mathrm{SCM}(R)$ by $D := R \oplus (\bigoplus_{d\in\mathcal{D}} M_d)$.

It was shown in [IW11] that the category $\mathrm{SCM}(R)$ has at least one natural Frobenius structure. Our first result in this setting is that there are often many different Frobenius structures on $\mathrm{SCM}(R)$, and so the one found in [IW11] is not unique.

Proposition 1.3 (= Proposition 3.7). *Let R be a complete local rational surface singularity over an algebraically closed field of characteristic zero and let $D \in \mathrm{SCM}(R)$ be defined as above. Choose $N \in \mathrm{SCM}(R)$ such that $\operatorname{add} D \subseteq \operatorname{add} N$. Then $\mathrm{SCM}(R)$ has the structure of a Frobenius category whose projective objects are exactly $\operatorname{add} N$. We denote the category $\mathrm{SCM}(R)$, equipped with this Frobenius structure, by $\mathrm{SCM}_N(R)$.*

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It then follows from Theorem 1.1 and Proposition 1.3 that $\mathrm{End}_R(N)$ is Iwanaga–Gorenstein (Theorem 3.8), using the fact that the reconstruction algebra, i.e. $\mathrm{End}_R(R \oplus \bigoplus_{i\in I} M_i)$, has finite global dimension [IW10, Wem11a].

We then interpret the stable category $\underline{\mathrm{SCM}}_N(R)$ of the Frobenius category $\mathrm{SCM}_N(R)$ geometrically. To do this, we remark that the condition $\mathrm{add}\, D \subseteq \mathrm{add}\, N$ implies (after passing to the basic module) that we can write

$$N = D \oplus \bigoplus_{j\in J} M_j$$

for some subset $J \subseteq \mathcal{C}$. Set $\mathcal{S} := \mathcal{C} \backslash J$, the complement of J in $\mathcal{C}$, so that

$$N = D \oplus \bigoplus_{j\in \mathcal{C}\backslash\mathcal{S}} M_j := N^{\mathcal{S}}.$$

Contracting all the curves in $\mathcal{S}$, we obtain a scheme $X^{\mathcal{S}}$ together with maps

$$Y \xrightarrow{f^{\mathcal{S}}} X^{\mathcal{S}} \xrightarrow{g^{\mathcal{S}}} \mathrm{Spec}\, R.$$

Knowledge of the derived category of $X^{\mathcal{S}}$ leads to our main result, which also explains geometrically why $\mathrm{End}_R(N^{\mathcal{S}})$ is Iwanaga–Gorenstein (Corollary 4.15).

Theorem 1.4 (= Theorems 4.6, 4.10). *With the assumptions as in Proposition 1.3, choose $\mathcal{S} \subseteq \mathcal{C}$ (i.e. $N^{\mathcal{S}} \in \mathrm{SCM}(R)$ such that $\mathrm{add}\, D \subseteq \mathrm{add}\, N^{\mathcal{S}}$). Then the following statements hold.*

(1) *There is a derived equivalence between $\mathrm{End}_R(N^{\mathcal{S}})$ and $X^{\mathcal{S}}$.*

(2) *As a consequence, we obtain triangle equivalences*

$$\underline{\mathrm{SCM}}_{N^{\mathcal{S}}}(R) \simeq \underline{\mathrm{GP}}(\mathrm{End}_R(N^{\mathcal{S}})) \simeq \mathrm{D_{sg}}(\mathrm{End}_R(N^{\mathcal{S}})) \simeq \mathrm{D_{sg}}(X^{\mathcal{S}}) \simeq \bigoplus_{x\in \mathrm{Sing}\, X^{\mathcal{S}}} \underline{\mathrm{CM}}(\widehat{\mathcal{O}}_{X^{\mathcal{S}},x})$$

where $\mathrm{Sing}\, X^{\mathcal{S}}$ denotes the set of singular points of $X^{\mathcal{S}}$.

(3) *In particular, $\underline{\mathrm{SCM}}_{N^{\mathcal{S}}}(R)$ is 1-Calabi–Yau, and its shift functor satisfies $[2] = \mathrm{id}$.*

Thus Theorem 1.4 shows that $\underline{\mathrm{SCM}}_{N^{\mathcal{S}}}(R)$ is nothing other than the usual singularity category of some partial resolution of $\mathrm{Spec}\, R$. We remark that it is the geometry that determines the last few statements in Theorem 1.4, as we are unable to prove them using algebra alone. In § 5 we give a relative singularity category version of the last two equivalences in Theorem 1.4.

The following corollary to Theorem 1.4 extends [IW11, Corollary 4.11] and gives a 'relative' version of Auslander's algebraic McKay correspondence for all rational surface singularities.

Corollary 1.5 (= Corollary 4.14). *With the assumptions as in Proposition 1.3, choose $N^{\mathcal{S}} \in \mathrm{SCM}(R)$ such that $\mathrm{add}\, D \subseteq \mathrm{add}\, N^{\mathcal{S}}$. Then the Auslander–Reiten (AR) quiver of the category $\underline{\mathrm{SCM}}_{N^{\mathcal{S}}}(R)$ is the double of the dual graph with respect to the morphism $Y \to X^{\mathcal{S}}$.*

Using the geometry, we are also able to improve Theorem 1.1(1) in the situation of rational surface singularities, since we are able to give the precise value of the injective dimension. The following is a generalization of a known result, Proposition 3.2, for the case $\mathrm{add}\, N = \mathrm{SCM}(R)$.

Theorem 1.6 (= Theorem 4.18). *With the assumptions as in Proposition 1.3, choose $N \in \mathrm{SCM}(R)$ such that $\mathrm{add}\, D \subseteq \mathrm{add}\, N$ and put $\Gamma = \mathrm{End}_R(N)$. Then*

$$\mathrm{inj.dim}_\Gamma\, \Gamma = \begin{cases} 2 & \text{if } R \text{ is Gorenstein,} \\ 3 & \text{otherwise.} \end{cases}$$

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This gives many new examples of Iwanaga–Gorenstein rings Γ, of injective dimension three, for which there are only finitely many Gorenstein projective modules up to isomorphism. In contrast to the commutative situation, we also show the following result. Some explicit examples are given in § 6.1.

Theorem 1.7 (= Theorem 4.19). *Let $G \leqslant \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup, with $G \ncong E_8$. Then there are uncountably many non-isomorphic Iwanaga–Gorenstein rings Λ with $\mathrm{inj.dim}_\Lambda \Lambda = 3$, such that $\underline{\mathrm{GP}}(\Lambda) \simeq \underline{\mathrm{CM}}(\mathbb{C}[[x,y]]^G)$.*

Conventions and notation. We use the convention that the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a category is denoted by fg. By a module over a ring A we mean a left module, and we denote by $\mathrm{Mod}\, A$ ($\mathrm{mod}\, A$) the category of A-modules (finitely generated A-modules). We denote by $\mathrm{proj}\, A$ the category of finitely generated projective A-modules. If M is an object of an additive category $\mathcal{C}$, we denote by $\mathrm{add}\, M$ all those objects of $\mathcal{C}$ which are direct summands of finite direct sums of M. We say that M is an *additive generator* of $\mathcal{C}$ if $\mathcal{C} = \mathrm{add}\, M$. If $\mathcal{T}$ is a triangulated category and $M \in \mathcal{T}$, we denote by $\mathrm{thick}(M)$ the smallest full triangulated subcategory containing M which is closed under taking direct summands.

2. A Morita type theorem for Frobenius categories

Throughout this section let $\mathcal{E}$ denote a Frobenius category, and denote by $\mathrm{proj}\, \mathcal{E} \subseteq \mathcal{E}$ the full subcategory of projective objects. We denote the stable category of $\mathcal{E}$ by $\underline{\mathcal{E}}$. It has the same objects as $\mathcal{E}$, but the morphism spaces are defined as $\underline{\mathrm{Hom}}_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{E}}(X,Y)/\mathcal{P}(X,Y)$, where $\mathcal{P}(X,Y)$ is the subspace of morphisms factoring through $\mathrm{proj}\, \mathcal{E}$. We refer to Keller's overview article for definitions and unexplained terminologies [Kel90, Kel96].

2.1 Frobenius categories as categories of Gorenstein projective modules

Recall that a noetherian ring E is called *Iwanaga–Gorenstein of dimension at most n* if $\mathrm{inj.dim}_E E \leqslant n$ and $\mathrm{inj.dim}\, E_E \leqslant n$ [EJ00]. For an Iwanaga–Gorenstein ring E of dimension at most n, we denote by

$$\mathrm{GP}(E) := \{X \in \mathrm{mod}\, E \mid \mathrm{Ext}^i_E(X,E) = 0 \text{ for any } i > 0\} = \Omega^n(\mathrm{mod}\, E)$$

the category of *Gorenstein projective* E-modules [AB69, EJ00]. Here Ω is the syzygy functor of $\mathrm{mod}\, E$. This is a Frobenius category with $\mathrm{proj}\, E$ the subcategory of projective objects.

Remark 2.1. The objects of $\mathrm{GP}(E)$ are sometimes called Cohen–Macaulay modules, but there are reasons why we do not do this; see Remark 3.3 below. They are sometimes also called totally reflexive modules.

Definition 2.2 ([Buch86, Orl04]). Let R be a left noetherian ring. The triangulated category $\mathrm{D_{sg}}(R) := \mathrm{D^b}(\mathrm{mod}\, R)/\,\mathrm{K^b}(\mathrm{proj}\, R)$ is called the *singularity category* of R.

Remark 2.3. Let E be an Iwanaga–Gorenstein ring. By a result of Buchweitz [Buch86, Theorem 4.4.1(2)], we have an equivalence of triangulated categories

$$\underline{\mathrm{GP}}(E) \simeq \mathrm{D_{sg}}(E).$$

The purpose of this section is to show that the existence of a non-commutative resolution of a Frobenius category $\mathcal{E}$ puts strong restrictions on $\mathcal{E}$ (Theorem 2.7).

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Definition 2.4. Let $\mathcal{E}$ be a Frobenius category with $\operatorname{proj}\mathcal{E} = \operatorname{add} P$ for some $P \in \operatorname{proj}\mathcal{E}$. By a *non-commutative resolution* of $\mathcal{E}$, we mean $A := \operatorname{End}_{\mathcal{E}}(M)$ for some $M \in \mathcal{E}$ with $P \in \operatorname{add} M$, such that A is noetherian with $\operatorname{gl.dim} A < \infty$.

At the level of generality of abstract Frobenius categories, the above definition is new. We remark that when $\mathcal{E} = \operatorname{CM} R$ (with R a Gorenstein ring), our definition of non-commutative resolution is much weaker than Van den Bergh's notion of a non-commutative crepant resolution (= NCCR) [vdB04b], and especially in higher dimension, examples occur much more often.

Remark 2.5. Not every Frobenius category with a projective generator admits a non-commutative resolution. Indeed, let R be a normal Gorenstein surface singularity over $\mathbb{C}$, and consider $\mathcal{E} := \operatorname{CM}(R)$. Then any non-commutative resolution in the above sense is automatically an NCCR, and the existence of an NCCR is well known to imply that R must have rational singularities [SV08].

Our strategy to prove Theorem 2.7 is based on [AIR11, Theorem 2.2(a)], but the set-up here is somewhat different. We need the following technical observation.

Lemma 2.6. *Let $\mathcal{E}$ be a Frobenius category with $\operatorname{proj}\mathcal{E} = \operatorname{add} P$ for some $P \in \operatorname{proj}\mathcal{E}$. If $f : X \to Y$ is a morphism in $\mathcal{E}$ such that $\operatorname{Hom}_{\mathcal{E}}(f, P)$ is surjective, then there exists an exact sequence*

$$0 \to X \xrightarrow{(f\ 0)} Y \oplus P' \to Z \to 0$$

in $\mathcal{E}$ with $P' \in \operatorname{proj}\mathcal{E}$.

Proof. This follows, for example, from [Kal13, Lemma 2.10]. □

Theorem 2.7. *Let $\mathcal{E}$ be a Frobenius category with $\operatorname{proj}\mathcal{E} = \operatorname{add} P$ for some $P \in \operatorname{proj}\mathcal{E}$. Assume that there exists a non-commutative resolution $\operatorname{End}_{\mathcal{E}}(M)$ of $\mathcal{E}$ with $\operatorname{gl.dim}\operatorname{End}_{\mathcal{E}}(M) = n$. Then the following statements hold.*

(1) *$E := \operatorname{End}_{\mathcal{E}}(P)$ is an Iwanaga–Gorenstein ring of dimension at most n.*

(2) *We have an equivalence $\operatorname{Hom}_{\mathcal{E}}(P, -) : \mathcal{E} \to \operatorname{GP}(E)$ up to direct summands. It is an equivalence if $\mathcal{E}$ is idempotent complete. This induces a triangle equivalence*

$$\underline{\mathcal{E}} \xrightarrow{\simeq} \underline{\operatorname{GP}}(E) \simeq \mathrm{D}_{\mathrm{sg}}(E)$$

up to direct summands. It is an equivalence if $\mathcal{E}$ or $\underline{\mathcal{E}}$ is idempotent complete.

(3) *$\underline{\mathcal{E}} = \operatorname{thick}_{\underline{\mathcal{E}}}(M)$.*

Proof. Since $P \in \operatorname{add} M$, $\operatorname{End}_{\mathcal{E}}(M)$ is Morita equivalent to $A := \operatorname{End}_{\mathcal{E}}(P \oplus M)$ and so $\operatorname{gl.dim} A = n$. Since A is noetherian, so is E (see, for example, [MR87, Proposition 1.1.7]). It follows from a standard argument that the functor $\operatorname{Hom}_{\mathcal{E}}(P, -) : \mathcal{E} \to \operatorname{mod} E$ is fully faithful, restricting to an equivalence $\operatorname{Hom}_{\mathcal{E}}(P, -) : \operatorname{add} P \to \operatorname{proj} E$ up to direct summands. We can drop the 'up to direct summands' assumption if $\mathcal{E}$ is idempotent complete. We establish (1) in three steps:

(i) We first show that $\operatorname{Ext}^i_E(\operatorname{Hom}_{\mathcal{E}}(P, X), E) = 0$ for any $X \in \mathcal{E}$ and $i > 0$. Let

$$0 \to Y \to P' \to X \to 0 \tag{2.A}$$

be an exact sequence in $\mathcal{E}$ with P' projective. Applying $\operatorname{Hom}_{\mathcal{E}}(P, -)$, we have an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{E}}(P, Y) \to \operatorname{Hom}_{\mathcal{E}}(P, P') \to \operatorname{Hom}_{\mathcal{E}}(P, X) \to 0 \tag{2.B}$$

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with a projective E-module $\mathrm{Hom}_{\mathcal{E}}(P,P')$. Applying $\mathrm{Hom}_{\mathcal{E}}(-,P)$ to (2.A) and $\mathrm{Hom}_E(-,E)$ to (2.B) respectively and comparing them, we have a commutative diagram of exact sequences.

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{E}}(P',P) & \longrightarrow & \mathrm{Hom}_{\mathcal{E}}(Y,P) & \longrightarrow & 0 & & \\
\downarrow\wr & & \downarrow\wr & & & & \\
\mathrm{Hom}_E(\mathrm{Hom}_{\mathcal{E}}(P,P'),E) & \longrightarrow & \mathrm{Hom}_E(\mathrm{Hom}_{\mathcal{E}}(P,Y),E) & \longrightarrow & \mathrm{Ext}^1_E(\mathrm{Hom}_{\mathcal{E}}(P,X),E) & \longrightarrow & 0
\end{array}$$

Thus we have $\mathrm{Ext}^1_E(\mathrm{Hom}_{\mathcal{E}}(P,X),E)=0$. Since the syzygy of $\mathrm{Hom}_{\mathcal{E}}(P,X)$ has the same form $\mathrm{Hom}_{\mathcal{E}}(P,Y)$, we have $\mathrm{Ext}^i_E(\mathrm{Hom}_{\mathcal{E}}(P,X),E)=0$ for any $i>0$.

(ii) We show that, for any $X\in \mathrm{mod}\,E$, there exists an exact sequence

$$0\to Q_n\to\cdots\to Q_0\to X\to 0 \tag{2.C}$$

of E-modules with $Q_i\in \mathrm{add}\,\mathrm{Hom}_{\mathcal{E}}(P,P\oplus M)$.

Define an A-module by $\widetilde{X}:=\mathrm{Hom}_{\mathcal{E}}(P\oplus M,P)\otimes_E X$. Let e be the idempotent of $A=\mathrm{End}_{\mathcal{E}}(P\oplus M)$ corresponding to the direct summand P of $P\oplus M$. Then we have $eAe=E$ and $e\widetilde{X}=X$. Since the global dimension of A is at most n, there exists a projective resolution

$$0\to P_n\to\cdots\to P_0\to\widetilde{X}\to 0.$$

Applying $e(-)$ and using $eA=\mathrm{Hom}_{\mathcal{E}}(P,P\oplus M)$, we have the assertion.

(iii) By (i) and (ii), we have that $\mathrm{Ext}^{n+1}_E(X,E)=0$ for any $X\in\mathrm{mod}\,E$, and so the injective dimension of the E-module E is at most n. The dual argument shows that the injective dimension of the E^{op}-module E is at most n. Thus E is Iwanaga–Gorenstein, which shows (1).

(2) By (i) again, we have a functor $\mathrm{Hom}_{\mathcal{E}}(P,-):\mathcal{E}\to\mathrm{GP}(E)$, and it is fully faithful. We will now show that it is dense up to direct summands.

For any $X\in\mathrm{GP}(E)$, we take an exact sequence (2.C). Since $Q_i\in\mathrm{add}\,\mathrm{Hom}_{\mathcal{E}}(P,P\oplus M)$, we have a complex

$$M_n\xrightarrow{f_n}\cdots\xrightarrow{f_0}M_0 \tag{2.D}$$

in $\mathcal{E}$ with $M_i\in\mathrm{add}(P\oplus M)$ such that

$$0\to\mathrm{Hom}_{\mathcal{E}}(P,M_n)\xrightarrow{\cdot f_n}\cdots\xrightarrow{\cdot f_0}\mathrm{Hom}_{\mathcal{E}}(P,M_0)\to X\oplus Y\to 0 \tag{2.E}$$

is exact for some $Y\in\mathrm{GP}(E)$. (Note that due to the possible lack of direct summands in $\mathcal{E}$ it is not always possible to choose M_i such that $\mathrm{Hom}_{\mathcal{E}}(P,M_i)=Q_i$.) Applying $\mathrm{Hom}_{\mathcal{E}}(-,P)$ to (2.D) and $\mathrm{Hom}_E(-,E)$ to (2.E) and comparing them, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{E}}(M_0,P) & \longrightarrow\cdots\longrightarrow & \mathrm{Hom}_{\mathcal{E}}(M_n,P) & \longrightarrow & 0 \\
\downarrow\wr & & \downarrow\wr & & \\
\mathrm{Hom}_E(\mathrm{Hom}_{\mathcal{E}}(P,M_0),E) & \longrightarrow\cdots\longrightarrow & \mathrm{Hom}_E(\mathrm{Hom}_{\mathcal{E}}(P,M_n),E) & \longrightarrow & 0
\end{array}$$

where the lower sequence is exact since $X\oplus Y\in\mathrm{GP}(E)$. Thus the upper sequence is also exact. But applying Lemma 2.6 repeatedly to (2.D), we have a complex

$$0\to M_n\xrightarrow{(f_n\ 0)}M_{n-1}\oplus P_{n-1}\xrightarrow{\left(\begin{smallmatrix}f_{n-1}&0&0\\0&1&0\end{smallmatrix}\right)}M_{n-2}\oplus P_{n-1}\oplus P_{n-2}\to\cdots\to M_0\oplus P_1\oplus P_0\to N\to 0$$

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with projective objects P_i which is a splicing of exact sequences in $\mathcal{E}$. Then we have $X \oplus Y \oplus \mathrm{Hom}_{\mathcal{E}}(P, P_0) \simeq \mathrm{Hom}_{\mathcal{E}}(P, N)$, and we have the assertion. The final statement follows by Remark 2.3.

(3) The existence of (2.C) implies that $\mathrm{Hom}_{\mathcal{E}}(P, -)$ gives a triangle equivalence $\mathrm{thick}_{\underline{\mathcal{E}}}(M) \to \underline{\mathrm{GP}}(E)$ up to direct summands. Thus the natural inclusion $\mathrm{thick}_{\underline{\mathcal{E}}}(M) \to \underline{\mathcal{E}}$ is also a triangle equivalence up to direct summands. This must be an isomorphism since $\mathrm{thick}_{\underline{\mathcal{E}}}(M)$ is closed under direct summands in $\underline{\mathcal{E}}$. □

We note the following more general version stated in terms of functor categories [Aus66]. For an additive category $\mathcal{P}$ we denote by $\mathrm{Mod}\,\mathcal{P}$ the category of contravariant additive functors from $\mathcal{P}$ to the category of abelian groups. For $X \in \mathcal{E}$, we have a $\mathcal{P}$-module $H_X := \mathrm{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{P}}$. We denote by $\mathrm{mod}\,\mathcal{P}$ the full subcategory of $\mathrm{Mod}\,\mathcal{P}$ consisting of finitely presented objects. Similarly, we define $\mathrm{Mod}\,\mathcal{P}^{\mathrm{op}}$, H^X and $\mathrm{mod}\,\mathcal{P}^{\mathrm{op}}$. If $\mathcal{P}$ has pseudokernels (pseudocokernels), then $\mathrm{mod}\,\mathcal{P}$ ($\mathrm{mod}\,\mathcal{P}^{\mathrm{op}}$) is an abelian category.

Theorem 2.8. *Let $\mathcal{E}$ be a Frobenius category with the category $\mathcal{P}$ of projective objects. Assume that there exists a full subcategory $\mathcal{M}$ of $\mathcal{E}$ such that $\mathcal{M}$ contains $\mathcal{P}$, $\mathcal{M}$ has pseudokernels and pseudocokernels, and* $\mathrm{mod}\,\mathcal{M}$ *and* $\mathrm{mod}\,\mathcal{M}^{\mathrm{op}}$ *have global dimension at most n. Then the following statements hold.*

(1) *$\mathcal{P}$ is an Iwanaga–Gorenstein category of dimension at most n, i.e.* $\mathrm{Ext}^i_{\mathrm{mod}\,\mathcal{P}}(-, H_P) = 0$ *and* $\mathrm{Ext}^i_{\mathrm{mod}\,\mathcal{P}^{\mathrm{op}}}(-, H^P) = 0$ *for all $P \in \mathcal{P}$, $i > n$.*

(2) *For the category*

$$\mathrm{GP}(\mathcal{P}) := \{X \in \mathrm{mod}\,\mathcal{P} \mid \mathrm{Ext}^i_{\mathcal{P}}(X, H_P) = 0 \textit{ for any } i > 0 \textit{ and } P \in \mathcal{P}\}$$

of Gorenstein projective *$\mathcal{P}$-modules, we have an equivalence $\mathcal{E} \to \mathrm{GP}(\mathcal{P})$, $X \mapsto H_X$ up to summands. It is an equivalence if $\mathcal{E}$ is idempotent complete. This induces a triangle equivalence*

$$\underline{\mathcal{E}} \to \underline{\mathrm{GP}}(\mathcal{P}) \simeq \mathrm{D_{sg}}(\mathcal{P})$$

up to summands. It is an equivalence if $\mathcal{E}$ or $\underline{\mathcal{E}}$ is idempotent complete.

(3) $\underline{\mathcal{E}} = \mathrm{thick}_{\underline{\mathcal{E}}}(\mathcal{M})$.

Remark 2.9. In the setting of Theorem 2.8, we remark that [Che12, Theorem 4.2] also gives an embedding $\mathcal{E} \to \mathrm{GP}(\mathcal{P})$.

2.2 Alternative approach

We now give an alternative proof of Theorem 2.7 by using certain quotients of derived categories. This will be necessary to interpret some results in § 5 later. We retain the set-up from the previous subsection; in particular, $\mathcal{E}$ always denotes a Frobenius category. Recall the following definition.

Definition 2.10. Let $N \in \mathbb{Z}$. A complex P^* of projective objects in $\mathcal{E}$ is called *acyclic in degrees* $\leqslant N$ if there exist exact sequences in $\mathcal{E}$,

$$Z^n(P^*) \xrightarrow{i_n} P^n \xrightarrow{p_n} Z^{n+1}(P^*)$$

such that $d^n_{P^*} = p_n i_{n+1}$ holds for all $n \leqslant N$. Let $\mathrm{K}^{-,\mathrm{b}}(\mathrm{proj}\,\mathcal{E}) \subseteq \mathrm{K}^-(\mathrm{proj}\,\mathcal{E})$ be the full subcategory consisting of those complexes which are acyclic in degrees $\leqslant d$ for some $d \in \mathbb{Z}$. This defines a triangulated subcategory of $\mathrm{K}^-(\mathrm{proj}\,\mathcal{E})$ (cf. [KV87]).

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Taking projective resolutions yields a functor $\mathcal{E} \to \mathrm{K}^{-,\mathrm{b}}(\operatorname{proj}\mathcal{E})$. We need the following dual version of [KV87, Example 2.3]; see also [Kal13, Proposition 2.36].

Proposition 2.11. *This functor induces an equivalence of triangulated categories*

$$\mathbb{P} : \underline{\mathcal{E}} \longrightarrow \mathrm{K}^{-,\mathrm{b}}(\operatorname{proj}\mathcal{E})/\mathrm{K}^{\mathrm{b}}(\operatorname{proj}\mathcal{E}).$$

Corollary 2.12. *If there exists $P \in \operatorname{proj}\mathcal{E}$ such that $\operatorname{proj}\mathcal{E} = \operatorname{add} P$ and, moreover, $E = \operatorname{End}_{\mathcal{E}}(P)$ is left noetherian, then there is a fully faithful triangle functor*

$$\widetilde{\mathbb{P}} : \underline{\mathcal{E}} \longrightarrow \mathrm{D}_{\mathrm{sg}}(E). \tag{2.F}$$

Proof. The fully faithful functor $\operatorname{Hom}_{\mathcal{E}}(P, -) : \operatorname{proj}\mathcal{E} \to \operatorname{proj} E$ induces a fully faithful triangle functor $\mathrm{K}^{-}(\operatorname{proj}\mathcal{E}) \to \mathrm{K}^{-}(\operatorname{proj} E)$. Its restriction $\mathrm{K}^{-,\mathrm{b}}(\operatorname{proj}\mathcal{E}) \to \mathrm{K}^{-,\mathrm{b}}(\operatorname{proj} E)$ is well defined since P is projective. Define $\widetilde{\mathbb{P}}$ as the composition

$$\underline{\mathcal{E}} \xrightarrow{\ \mathbb{P}\ } \frac{\mathrm{K}^{-,\mathrm{b}}(\operatorname{proj}\mathcal{E})}{\mathrm{K}^{\mathrm{b}}(\operatorname{proj}\mathcal{E})} \longrightarrow \frac{\mathrm{K}^{-,\mathrm{b}}(\operatorname{proj} E)}{\mathrm{K}^{\mathrm{b}}(\operatorname{proj} E)} \xrightarrow{\ \sim\ } \frac{\mathrm{D}^{\mathrm{b}}(\operatorname{mod} E)}{\mathrm{K}^{\mathrm{b}}(\operatorname{proj} E)},$$

where $\mathbb{P}$ is the equivalence from Proposition 2.11 and the last functor is induced by the well-known triangle equivalence $\mathrm{K}^{-,\mathrm{b}}(\operatorname{proj} E) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\operatorname{mod} E)$. □

Remark 2.13. In the special case when E is an Iwanaga–Gorenstein ring and $\mathcal{E} := \mathrm{GP}(E)$, the functor $\widetilde{\mathbb{P}}$ in (2.F) was shown to be an equivalence in [Buch86, Theorem 4.4.1(2)] (see Remark 2.3). For general Frobenius categories, $\widetilde{\mathbb{P}}$ is far from being an equivalence. For example, let $\mathcal{E} = \operatorname{proj} R$ be the category of finitely generated projective modules over a left noetherian algebra R equipped with the split exact structure. Then always $\underline{\mathcal{E}} = 0$ but $\mathrm{D}_{\mathrm{sg}}(E) = \mathrm{D}_{\mathrm{sg}}(R) \neq 0$ if $\operatorname{gl.dim}(R) = \infty$. We refer the reader to [Kal13, Remark 2.41] for a detailed discussion.

Below in Theorem 2.15, we give a sufficient criterion for $\widetilde{\mathbb{P}}$ to be an equivalence. To do this requires the following result.

Proposition 2.14. *Let A be a left noetherian ring and let $e \in A$ be an idempotent. The exact functor $\operatorname{Hom}_A(Ae, -)$ induces a triangle equivalence*

$$\mathbb{G} : \frac{\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)/\operatorname{thick}(Ae)}{\operatorname{thick}\left(q(\operatorname{mod} A/AeA)\right)} \longrightarrow \frac{\mathrm{D}^{\mathrm{b}}(\operatorname{mod} eAe)}{\operatorname{thick}(eAe)}, \tag{2.G}$$

where $q : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)/\operatorname{thick}(Ae)$ denotes the canonical projection.

Proof. Taking $e = f$ in [KY12, Proposition 3.3] yields the triangle equivalence (2.G). □

Theorem 2.15. *Let $\mathcal{E}$ be a Frobenius category with $\operatorname{proj}\mathcal{E} = \operatorname{add} P$ for some $P \in \operatorname{proj}\mathcal{E}$. Assume that there exists $M \in \mathcal{E}$ such that $A := \operatorname{End}_{\mathcal{E}}(P \oplus M)$ is a left noetherian ring of finite global dimension, and denote $E := \operatorname{End}_{\mathcal{E}}(P)$. Then the following statements hold.*

(1) *$\widetilde{\mathbb{P}} : \underline{\mathcal{E}} \longrightarrow \mathrm{D}_{\mathrm{sg}}(E)$ is a triangle equivalence up to direct summands. If $\underline{\mathcal{E}}$ is idempotent complete, then $\widetilde{\mathbb{P}}$ is an equivalence.*

(2) *$\underline{\mathcal{E}} = \operatorname{thick}_{\underline{\mathcal{E}}}(M)$.*

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Proof. Let $e \in A$ be the idempotent corresponding to the identity endomorphism 1_P of P; then $eAe = E$. We have the following commutative diagram of categories and functors.

$$\begin{array}{ccccc}
\dfrac{\big(\mathrm{D^b}(\mathrm{mod}\,A)/\mathrm{thick}(Ae)\big)}{\mathrm{thick}\big(q(\mathrm{mod}\,A/AeA)\big)} & \xrightarrow[\sim]{\mathbb{G}} & \dfrac{\mathrm{D^b}(\mathrm{mod}\,eAe)}{\mathrm{K^b}(\mathrm{proj}\,eAe)} & \xleftarrow{\widetilde{\mathbb{P}}} & \underline{\mathcal{E}} \\
\uparrow{\scriptstyle \mathbb{I}_1} & & \uparrow{\scriptstyle \mathbb{I}_2} & & \uparrow{\scriptstyle \mathbb{I}_3} \\
\dfrac{\big(\mathrm{K^b}(\mathrm{proj}\,A)/\mathrm{thick}(Ae)\big)}{\mathrm{thick}\big(q(\mathrm{mod}\,A/AeA)\big)} & \xrightarrow[\sim]{\mathbb{G}^{\mathrm{restr.}}} & \dfrac{\mathrm{thick}(eA)}{\mathrm{K^b}(\mathrm{proj}\,eAe)} & \xleftarrow[\sim]{\widetilde{\mathbb{P}}^{\mathrm{restr.}}} & \mathrm{thick}_{\underline{\mathcal{E}}}(M)
\end{array}$$

where $\mathbb{I}_i$ are the natural inclusions. Since A has finite global dimension the inclusion $\mathrm{K^b}(\mathrm{proj}\,A) \to \mathrm{D^b}(\mathrm{mod}\,A)$ is an equivalence and so $\mathbb{I}_1$ is an equivalence. But $\mathbb{G}$ is an equivalence from Proposition 2.14, and $\mathbb{G}^{\mathrm{restr.}}$ denotes its restriction. It is also an equivalence since $\mathbb{G}$ maps the generator A to eA. Thus, by commutativity of the left square we deduce that $\mathbb{I}_2$ is an equivalence. Now $\widetilde{\mathbb{P}}$ denotes the fully faithful functor from Corollary 2.12, so since $\widetilde{\mathbb{P}}$ maps $P \oplus M$ to $\mathrm{Hom}_{\mathcal{E}}(P, P \oplus M)$, which is isomorphic to eA as left eAe-modules, the restriction $\widetilde{\mathbb{P}}^{\mathrm{restr.}}$ is a triangle equivalence up to summands. Hence the fully faithful functors $\widetilde{\mathbb{P}}$ and $\mathbb{I}_3$ are also equivalences, up to summands. In particular, $\mathbb{I}_3$ is an equivalence. If $\underline{\mathcal{E}}$ is idempotent complete then $\mathrm{thick}_{\underline{\mathcal{E}}}(M)$ is idempotent complete and $\widetilde{\mathbb{P}}^{\mathrm{restr.}}$ is an equivalence. It follows that $\widetilde{\mathbb{P}}$ is an equivalence in this case. □

2.3 A result of Auslander and Solberg

Let K be a field and denote $\mathbb{D} := \mathrm{Hom}_K(-, K)$. The following is implicitly included in Auslander and Solberg's relative homological algebra [AS93b] (compare [Che12, Theorem 5.1]), and will be required later (in §§ 3 and 6) to produce examples of Frobenius categories on which we can apply our previous results.

Proposition 2.16. *Let $\mathcal{E}$ be a K-linear exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. Assume that there exist an equivalence $\tau : \underline{\mathcal{E}} \to \overline{\mathcal{E}}$ and a functorial isomorphism $\mathrm{Ext}^1_{\mathcal{E}}(X, Y) \simeq \mathbb{D}\overline{\mathrm{Hom}}_{\mathcal{E}}(Y, \tau X)$ for any $X, Y \in \mathcal{E}$. Let $\mathcal{M}$ be a functorially finite subcategory of $\mathcal{E}$ containing $\mathcal{P}$ and $\mathcal{I}$, which satisfies $\tau\underline{\mathcal{M}} = \overline{\mathcal{M}}$. Then the following statements hold.*

(1) *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathcal{E}$. Then $\mathrm{Hom}_{\mathcal{E}}(\mathcal{M}, g)$ is surjective if and only if $\mathrm{Hom}_{\mathcal{E}}(f, \mathcal{M})$ is surjective.*

(2) *$\mathcal{E}$ has the structure of a Frobenius category whose projective objects are exactly $\mathrm{add}\,\mathcal{M}$. More precisely, the short exact sequences of this Frobenius structure are the short exact sequences $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ of $\mathcal{E}$ such that $\mathrm{Hom}_{\mathcal{E}}(f, \mathcal{M})$ is surjective.*

Proof. (1) Applying $\mathrm{Hom}_{\mathcal{E}}(\mathcal{M}, -)$ to $0 \to X \to Y \to Z \to 0$, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{E}}(\mathcal{M}, Y) \xrightarrow{g} \mathrm{Hom}_{\mathcal{E}}(\mathcal{M}, Z) \to \mathrm{Ext}^1_{\mathcal{E}}(\mathcal{M}, X) \xrightarrow{f} \mathrm{Ext}^1_{\mathcal{E}}(\mathcal{M}, Y). \tag{2.H}$$

Thus we know that $\mathrm{Hom}_{\mathcal{E}}(\mathcal{M}, g)$ is surjective if and only if $\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{M}, f)$ is injective. Using AR duality, this holds if and only if $\overline{\mathrm{Hom}}_{\mathcal{E}}(f, \tau\mathcal{M})$ is surjective, which holds if and only if $\overline{\mathrm{Hom}}_{\mathcal{E}}(f, \mathcal{M})$ is surjective. This holds if and only if $\mathrm{Hom}_{\mathcal{E}}(f, \mathcal{M})$ is surjective.

(2) One can easily check (e.g. by using [DRSS99, Propositions 1.4 and 1.7]) that exact sequences fulfilling the equivalent conditions in (1) satisfy the axioms of exact categories in which any object of $\mathrm{add}\,\mathcal{M}$ is a projective and an injective object (see [Kal13, Remark 2.28] for details).

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We will show that $\mathcal{E}$ has enough projectives with respect to this exact structure. For any $X \in \mathcal{E}$, we take a right $\mathcal{M}$-approximation $f : N' \to X$ of X. Since $\mathcal{M}$ contains $\mathcal{P}$, any morphism from $\mathcal{P}$ to X factors through f. By a version of Lemma 2.6 for exact categories, we have an exact sequence

$$0 \to Y \to N' \oplus P \xrightarrow{\binom{f}{0}} X \to 0$$

in $\mathcal{E}$ with $P \in \mathcal{P}$. This sequence shows that $\mathcal{E}$ has enough projectives with respect to this exact structure.

Dually, we have that $\mathcal{E}$ has enough injectives. Moreover, both projective objects and injective objects are $\operatorname{add} \mathcal{M}$. Thus the assertion holds. □

3. Frobenius structures on special Cohen–Macaulay modules

Throughout this section we let R denote a complete local rational surface singularity over an algebraically closed field of characteristic zero. Because of the characteristic zero assumption, rational singularities are always Cohen–Macaulay. We refer the reader to § 4.4 for more details regarding rational surface singularities.

We denote by $\mathrm{CM}(R)$ the category of maximal Cohen–Macaulay (CM) R-modules. Since R is normal and two-dimensional, a module is CM if and only if it is reflexive. The category $\mathrm{CM}(R)$, and all subcategories thereof, are Krull–Schmidt categories since R is complete local. One such subcategory is the category of *special* CM modules, denoted $\mathrm{SCM}(R)$, which consists of all those CM R-modules X satisfying $\mathrm{Ext}^1_R(X, R) = 0$.

The category $\mathrm{SCM}(R)$ is intimately related to the geometry of $\operatorname{Spec} R$. If we denote the minimal resolution of $\operatorname{Spec} R$ by

$$Y \xrightarrow{\pi} \operatorname{Spec} R,$$

and define $\{E_i\}_{i \in I}$ to be the set of exceptional curves, then the following is well known.

Proposition 3.1. (1) *There are only finitely many indecomposable objects in* $\mathrm{SCM}(R)$.

(2) *Indecomposable non-free objects in* $\mathrm{SCM}(R)$ *correspond bijectively to* $\{E_i\}_{i \in I}$.

Proof. (2) is Wunram [Wun88, Theorem 1.2] (using [IW10, Theorem 2.7] to show that definition of special in [Wun88] is the same as the one used here), and (1) is a consequence of (2). □

Thus, by Proposition 3.1(2), $\mathrm{SCM}(R)$ has an additive generator $M := R \oplus \bigoplus_{i \in I} M_i$, where as in the introduction by convention M_i is the indecomposable special CM module corresponding to E_i. The corresponding endomorphism ring $\Lambda := \mathrm{End}_R(M)$ is called the *reconstruction algebra* of R; see [Wem11b, IW10]. The following is also well known.

Proposition 3.2. *Consider the reconstruction algebra* Λ. *Then*

$$\operatorname{gl.dim} \Lambda = \begin{cases} 2 & \text{if } R \text{ is Gorenstein}, \\ 3 & \text{otherwise}. \end{cases}$$

Proof. An algebraic proof can be found in [IW10, Theorem 2.10] or [IW11, Theorem 2.6]. A geometric proof can be found in [Wem11a]. □

Remark 3.3. The reconstruction algebra Λ, and some of the $e\Lambda e$ below, will turn out to be Iwanaga–Gorenstein in § 4. However, we remark here that Λ is usually *not* Gorenstein in the stronger sense that $\omega_\Lambda := \mathrm{Hom}_R(\Lambda, \omega_R)$ is a projective Λ-module. Thus, unfortunately the objects

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of $\mathrm{GP}(\Lambda)$ are not simply those Λ-modules that are CM as R-modules, i.e. $\mathrm{GP}(\Lambda) \subsetneq \{X \in \mathrm{mod}(\Lambda) \mid X \in \mathrm{CM}(R)\}$ in general. In this paper we will always reserve 'CM' to mean CM as an R-module, and this is why we use the terminology 'Gorenstein projective' (GP) for non-commutative Iwanaga–Gorenstein rings.

We will be considering many different factor categories of $\mathrm{SCM}(R)$, so in order to avoid confusion we now fix some notation.

Definition 3.4. Let $X \in \mathrm{SCM}(R)$. We define the factor category $\underline{\mathrm{SCM}}_X(R)$ to be the category consisting of the same objects as $\mathrm{SCM}(R)$, but where

$$\mathrm{Hom}_{\underline{\mathrm{SCM}}_X(R)}(a,b) := \frac{\mathrm{Hom}_{\mathrm{SCM}(R)}(a,b)}{\mathcal{X}(a,b)},$$

where $\mathcal{X}(a,b)$ is the subgroup of morphisms $a \to b$ which factor through an element in $\mathrm{add}\, X$.

As in the introduction, we consider the module $D := R \oplus (\bigoplus_{d\in\mathcal{D}} M_d)$. Algebraically the following is known; the geometric properties of D will be established in Corollary 4.8 and 4.9 below.

Proposition 3.5. (1) *The category* $\mathrm{SCM}(R)$ *has the natural structure of a Frobenius category, whose projective objects are precisely the objects of* $\mathrm{add}\, D$*. Consequently* $\underline{\mathrm{SCM}}_D(R)$ *is a triangulated category.*

(2) *For any indecomposable object* X *in* $\underline{\mathrm{SCM}}_D(R)$*, there exists an AR triangle of the form*

$$X \to E \to X \to X[1].$$

(3) *The stable category* $\underline{\mathrm{SCM}}_D(R)$ *has a Serre functor* $\mathbb{S}$ *such that* $\mathbb{S}X \simeq X[1]$ *for any* $X \in \underline{\mathrm{SCM}}_D(R)$.

Proof. (1) The exact sequences are defined using the embedding $\mathrm{SCM}(R) \subseteq \mathrm{mod}\, R$, and the result follows from [IW11, Theorem 4.2].

(2) This is [IW11, Proposition 4.9].

(3) $\underline{\mathrm{SCM}}_D(R)$ has AR triangles by (2), so there exists a Serre functor by [RV02, Proposition I.2.3] such that

$$\tau X \to E \to X \to \mathbb{S}X$$

is the AR triangle. By inspection of (2), we see that $\mathbb{S}X[-1] \simeq X$.

□

Remark 3.6. The above proposition almost asserts that the category $\underline{\mathrm{SCM}}_D(R)$ is 1-Calabi–Yau, but it does not show that the isomorphism in Proposition 3.5(3) is functorial. We prove that it is functorial in Theorem 4.10, using geometric arguments.

The following important observation, which generalizes Proposition 3.5(1), is obtained by applying Proposition 2.16 to $(\mathcal{E}, \mathcal{M}, \tau) = (\mathrm{SCM}(R), \mathrm{add}\, N, \mathbb{S}[-1])$.

Proposition 3.7. *Let* $N \in \mathrm{SCM}(R)$ *such that* $\mathrm{add}\, D \subseteq \mathrm{add}\, N$*. Then the following statements hold.*

(1) *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence of* R*-modules with* $X, Y, Z \in \mathrm{SCM}(R)$*. Then* $\mathrm{Hom}_R(N, g)$ *is surjective if and only if* $\mathrm{Hom}_R(f, N)$ *is surjective.*

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(2) $\mathrm{SCM}(R)$ *has the structure of a Frobenius category whose projective objects are exactly* $\operatorname{add} N$. *We denote it by* $\mathrm{SCM}_N(R)$. *More precisely, the short exact sequences of* $\mathrm{SCM}_N(R)$ *are the short exact sequences* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *of* R*-modules such that* $\operatorname{Hom}_R(f, N)$ *is surjective.*

We maintain the notation from above, in particular $\Lambda := \operatorname{End}_R(M)$ is the reconstruction algebra, where $M := R \oplus (\bigoplus_{i \in I} M_i)$, and $D := R \oplus (\bigoplus_{d \in \mathcal{D}} M_d)$. For any summand N of M, we denote by e_N the idempotent in Λ corresponding to the summand N. The following is the main result of this section.

Theorem 3.8. *Let* $N \in \mathrm{SCM}(R)$ *such that* $\operatorname{add} D \subseteq \operatorname{add} N$. *Then the following statements hold.*

(1) $e_N \Lambda e_N = \operatorname{End}_R(N)$ *is an Iwanaga–Gorenstein ring of dimension at most three.*

(2) *There is an equivalence* $\operatorname{Hom}_R(N, -) : \mathrm{SCM}(R) \to \mathrm{GP}(\operatorname{End}_R(N))$ *that induces a triangle equivalence*

$$\underline{\mathrm{SCM}}_N(R) \simeq \underline{\mathrm{GP}}(\operatorname{End}_R(N)).$$

Proof. By Proposition 3.7(2), $\mathrm{SCM}(R)$ has the structure of a Frobenius category in which $\operatorname{proj} \mathrm{SCM}(R) = \operatorname{add} N$. Since $\mathrm{SCM}(R)$ has finite type, there is some $X \in \mathrm{SCM}(R)$ such that $\operatorname{add}(N \oplus X) = \mathrm{SCM}(R)$, in which case $\operatorname{End}_R(N \oplus X)$ is Morita equivalent to the reconstruction algebra, so $\operatorname{gl.dim} \operatorname{End}_R(N \oplus X) \leqslant 3$ by Proposition 3.2. Hence (1) follows from Theorem 2.7(1), and since $\mathrm{SCM}(R)$ is idempotent complete, (2) follows from Theorem 2.7(2). □

Remark 3.9. We will give an entirely geometric proof of Theorem 3.8(1) in § 4, which also holds in greater generality.

The following corollary will be strengthened in Remark 4.7 below.

Corollary 3.10. *Let* $N \in \mathrm{SCM}(R)$ *such that* $\operatorname{add} D \subseteq \operatorname{add} N \subsetneq \operatorname{add} M$. *Then* $e_N \Lambda e_N = \operatorname{End}_R(N)$ *has infinite global dimension.*

Proof. By Theorem 3.8(2), we know that $\underline{\mathrm{SCM}}_N(R) \simeq \underline{\mathrm{GP}}(\operatorname{End}_R(N)) \simeq \mathrm{D_{sg}}(\operatorname{End}_R(N))$, where the last equivalence holds by Buchweitz [Buch86, Theorem 4.4.1(2)]. It is clear that $\underline{\mathrm{SCM}}_N(R) \neq 0$ since $\operatorname{add} N \subsetneq \operatorname{add} M$. Hence $\mathrm{D_{sg}}(\operatorname{End}_R(N)) \neq 0$, which is well known to imply that $\operatorname{gl.dim} \operatorname{End}_R(N) = \infty$. □

4. Relationship to partial resolutions of rational surface singularities

We show in § 4.1 that if an algebra Γ is derived equivalent to a Gorenstein scheme that is projective birational over a CM ring, then Γ is Iwanaga–Gorenstein. In § 4.2 we then exhibit algebras derived equivalent to partial resolutions of rational surface singularities, and we use this information to strengthen many of our previous results.

In this section we will assume that all schemes Y are noetherian, separated, normal CM, of pure Krull dimension $d < \infty$, and of finite type over a field k. This implies that $\mathrm{D}(\operatorname{Qcoh} Y)$ is compactly generated, with compact objects precisely the perfect complexes $\operatorname{Perf}(Y)$ [Nee96, Proposition 2.5 and Lemma 2.3], and $\omega_Y = g^! k[-\dim Y]$ where $g : Y \to \operatorname{Spec} k$ is the structure morphism.

4.1 Gorenstein schemes and Iwanaga–Gorenstein rings

Serre functors are somewhat more subtle in the singular setting. Recall from [Gin06, Definition 7.2.6] the following definition.

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Definition 4.1. Suppose that $Y \to \operatorname{Spec} S$ is a projective birational map where S is a CM ring with canonical module ω_S. We say that a functor $\mathbb{S} : \operatorname{Perf}(Y) \to \operatorname{Perf}(Y)$ is a *Serre functor relative to* ω_S if there are functorial isomorphisms

$$\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_Y(\mathcal{F},\mathcal{G}),\omega_S) \cong \mathbf{R}\mathrm{Hom}_Y(\mathcal{G},\mathbb{S}(\mathcal{F}))$$

in $\mathrm{D}(\operatorname{Mod} S)$ for all $\mathcal{F},\mathcal{G} \in \operatorname{Perf}(Y)$.

Lemma 4.2. *Let Γ be a module finite S-algebra, where S is a CM ring with canonical module ω_S, and suppose that there exists a functor $\mathbb{T} : \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Gamma) \to \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$ such that*

$$\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_\Gamma(a,b),\omega_S) \cong \mathbf{R}\mathrm{Hom}_\Gamma(b,\mathbb{T}(a))$$

for all $a,b \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$. Then $\operatorname{inj.dim}\Gamma_\Gamma < \infty$.

Proof. Denote $(-)^\dagger := \mathbf{R}\mathrm{Hom}_S(-,\omega_S)$. We first claim that $\Gamma^\dagger \in \mathrm{K}^{\mathrm{b}}(\operatorname{Inj}\Gamma^{\mathrm{op}})$. By taking an injective resolution of ω_S,

$$0 \to \omega_S \to I_0 \to \cdots \to I_d \to 0,$$

and applying $\mathrm{Hom}_S(\Gamma,-)$ we see that $\Gamma^\dagger$ is given as the complex

$$\cdots \to 0 \to \mathrm{Hom}_S(\Gamma,I_0) \to \cdots \to \mathrm{Hom}_S(\Gamma,I_d) \to 0 \to \cdots .$$

Since $\mathrm{Hom}_\Gamma(-,\mathrm{Hom}_S(\Gamma,I_i)) = \mathrm{Hom}_S(\Gamma \otimes_\Gamma -, I_i)$ is an exact functor, each $\mathrm{Hom}_S(\Gamma,I_i)$ is an injective Γ^{op}-module. Hence $\Gamma^\dagger \in \mathrm{K}^{\mathrm{b}}(\operatorname{Inj}\Gamma^{\mathrm{op}})$, as claimed.

Now $\mathbb{T}(\Gamma) \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Gamma)$, and further

$$\mathbb{T}(\Gamma) \cong \mathbf{R}\mathrm{Hom}_\Gamma(\Gamma,\mathbb{T}(\Gamma)) \cong \mathbf{R}\mathrm{Hom}_\Gamma(\Gamma,\Gamma)^\dagger = \Gamma^\dagger.$$

Hence $\Gamma^\dagger \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\Gamma) = \operatorname{thick}(\Gamma)$ and so $\Gamma = \Gamma^{\dagger\dagger} \in \operatorname{thick}(\Gamma^\dagger) \subseteq \mathrm{K}^{\mathrm{b}}(\operatorname{Inj}\Gamma^{\mathrm{op}})$. This shows that Γ has finite injective dimension as a Γ^{op}-module, i.e. as a right Γ-module. □

Grothendieck duality gives us the following theorem.

Theorem 4.3. *Let $Y \to \operatorname{Spec} S$ be a projective birational map where S a CM ring with canonical module ω_S. Suppose that Y is Gorenstein. Then the functor $\mathbb{S} := \omega_Y \otimes - : \operatorname{Perf}(Y) \to \operatorname{Perf}(Y)$ is a Serre functor relative to ω_S.*

Proof. Since Y is Gorenstein, the canonical sheaf ω_Y is locally free, and hence $\mathbb{S} := \omega_Y \otimes - = \omega_Y \otimes^{\mathbf{L}} -$ does indeed take $\operatorname{Perf}(Y)$ to $\operatorname{Perf}(Y)$. Also, $\omega_Y = f^! \omega_S$ and so

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_Y(\mathcal{G},\mathbb{S}(\mathcal{F})) = \mathbf{R}\mathrm{Hom}_Y(\mathcal{G},\mathcal{F}\otimes^{\mathbf{L}}\omega_Y) &\cong \mathbf{R}\mathrm{Hom}_Y(\mathbf{R}\mathcal{H}om_Y(\mathcal{F},\mathcal{G}),\omega_Y)\\ &\cong \mathbf{R}\mathrm{Hom}_Y(\mathbf{R}\mathcal{H}om_Y(\mathcal{F},\mathcal{G}),f^!\omega_S)\\ &\cong \mathbf{R}\mathrm{Hom}_S(\mathbf{R}f_*\,\mathbf{R}\mathcal{H}om_Y(\mathcal{F},\mathcal{G}),\omega_S)\\ &\cong \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_Y(\mathcal{F},\mathcal{G}),\omega_S)\end{aligned}$$

for all $\mathcal{F},\mathcal{G} \in \operatorname{Perf}(Y)$, where the second-last isomorphism is Grothendieck duality. □

The last two results combine to give the following corollary, which is the main result of this subsection.

Corollary 4.4. *Let $Y \to \operatorname{Spec} S$ be a projective birational map where S is a CM ring with canonical module ω_S. Suppose that Y is derived equivalent to Γ. Then if Y is a Gorenstein scheme, Γ is an Iwanaga–Gorenstein ring.*

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Proof. By Theorem 4.3 there is a Serre functor $\mathbb{S} : \mathrm{Perf}(Y) \to \mathrm{Perf}(Y)$ relative to ω_S. By [IW14, Lemma 4.12] this induces a Serre functor relative to ω_S on $\mathrm{K^b}(\mathrm{proj}\,\Gamma)$. Hence Lemma 4.2 shows that $\mathrm{inj.dim}\,\Gamma_\Gamma < \infty$.

Repeating the argument with $\mathcal{V}^\vee := \mathbf{R}\mathcal{H}om_Y(\mathcal{V}, \mathcal{O}_Y)$, which is well known to give an equivalence between Y and Γ^{op} (see, for example, [BH13, Proposition 2.6]), we obtain an induced Serre functor relative to ω_S on $\mathrm{K^b}(\mathrm{proj}\,\Gamma^{\mathrm{op}})$. Applying Lemma 4.2 to Γ^{op} shows that $\mathrm{inj.dim}_\Gamma\,\Gamma < \infty$. □

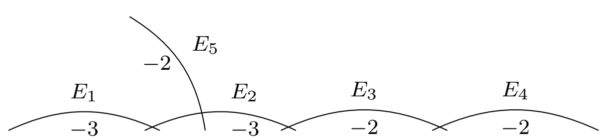
4.2 Tilting bundles on partial resolutions

We now return to the set-up in § 3, namely R denotes a complete local rational surface singularity over an algebraically closed field of characteristic zero. We inspect the exceptional divisors in Y, the minimal resolution of $\mathrm{Spec}\,R$. Recall from the introduction that we have $I = \mathcal{C} \cup \mathcal{D}$ where $\mathcal{C}$ are the crepant curves and $\mathcal{D}$ are the discrepant curves. We choose a subset $\mathcal{S} \subseteq I$, and contract all curves in $\mathcal{S}$. In this way we obtain a scheme which we will denote $X^{\mathcal{S}}$ (see, for example, [Rei93, § 4.15]). In fact, the minimal resolution $\pi : Y \to \mathrm{Spec}\,R$ factors as

$$Y \xrightarrow{f^{\mathcal{S}}} X^{\mathcal{S}} \xrightarrow{g^{\mathcal{S}}} \mathrm{Spec}\,R.$$

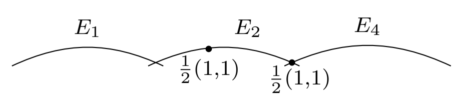
When $\mathcal{S} \subseteq \mathcal{C}$, $f^{\mathcal{S}}$ is crepant and, furthermore, $X^{\mathcal{S}}$ has only isolated ADE singularities since we have contracted only (-2)-curves. It is well known that in the dual graph of the minimal resolution, all maximal (-2)-curves must lie in ADE configurations (see, for example, [TT04, Proposition 3.2]).

Example 4.5. To make this concrete, consider the $\mathbb{T}_9$ singularity [Rie77, p47] $\mathrm{Spec}\,R = \mathbb{C}^2/\mathbb{T}_9$, which has minimal resolution

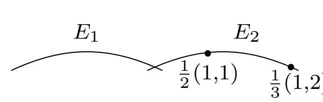
$Y :=$ 

$\longrightarrow \mathrm{Spec}\,R$

so $\mathcal{C} = \{E_3, E_4, E_5\}$. Choosing $\mathcal{S} = \{E_3, E_5\}$ gives

$X^{\mathcal{S}} :=$ 

where $\frac{1}{2}(1,1)$ is complete locally the A_1 surface singularity. On the other hand, choosing $\mathcal{S} = \mathcal{C} = \{E_3, E_4, E_5\}$ gives

$X^{\mathcal{C}} :=$ 

Note, in particular, that in these cases $\mathcal{S} \subseteq \mathcal{C}$ so $\mathrm{Sing}\,X^{\mathcal{S}}$ always has only finitely many points, and each is Gorenstein ADE.

The following theorem is well known to experts and is somewhat implicit in the literature. For lack of any reference, we provide a proof here. As before, Λ denotes the reconstruction algebra.

THEOREM 4.6. *Let $\mathcal{S} \subseteq I$, set $N^{\mathcal{S}} := R \oplus (\bigoplus_{i \in I \setminus \mathcal{S}} M_i)$ and let e be the idempotent in Λ corresponding to $N^{\mathcal{S}}$. Then $e\Lambda e = \mathrm{End}_R(N^{\mathcal{S}})$ is derived equivalent to $X^{\mathcal{S}}$ via a tilting bundle*

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$\mathcal{V}_{\mathcal{S}}$ in such a way that

$$\begin{array}{ccc} \mathrm{D^b}(\mathrm{mod}\,\Lambda) & \xleftarrow{\mathbf{R}\mathrm{Hom}_Y(\mathcal{V}_\emptyset,-)} & \mathrm{D^b}(\mathrm{coh}\,Y) \\ {\scriptstyle e(-)}\downarrow & & \downarrow{\scriptstyle \mathbf{R}f^{\mathcal{S}}_*} \\ \mathrm{D^b}(\mathrm{mod}\,e\Lambda e) & \xleftarrow{\mathbf{R}\mathrm{Hom}_{X^{\mathcal{S}}}(\mathcal{V}_{\mathcal{S}},-)} & \mathrm{D^b}(\mathrm{coh}\,X^{\mathcal{S}}) \end{array}$$

commutes.

Proof. Since all the fibres are at most one-dimensional and R has rational singularities, by [vdB04a, Theorem B] there is a tilting bundle on Y given as follows. Let $E = \pi^{-1}(\mathfrak{m})$ where $\mathfrak{m}$ is the unique closed point of Spec R. Giving E the reduced scheme structure, write $E_{\mathrm{red}} = \bigcup_{i\in I} E_i$, and let $\mathcal{L}_i^Y$ denote the line bundle on Y such that $\mathcal{L}_i^Y \cdot E_j = \delta_{ij}$. If the multiplicity of E_i in E is equal to one, set $\mathcal{M}_i^Y := \mathcal{L}_i^Y$ [vdB04a, Proposition 3.5.4], otherwise define $\mathcal{M}_i^Y$ to be given by the maximal extension

$$0 \to \mathcal{O}_Y^{\oplus(r_i-1)} \to \mathcal{M}_i^Y \to \mathcal{L}_i^Y \to 0$$

associated to a minimal set of $r_i - 1$ generators of $H^1(Y, (\mathcal{L}_i^Y)^{-1})$. Then $\mathcal{V}_\emptyset := \mathcal{O}_Y \oplus (\bigoplus_{i\in I} \mathcal{M}_i^Y)$ is a tilting bundle on Y [vdB04a, Theorem 3.5.5].

For ease of notation denote $X := X^{\mathcal{S}}$, and further denote $Y \xrightarrow{f^{\mathcal{S}}} X^{\mathcal{S}} \xrightarrow{g^{\mathcal{S}}} \operatorname{Spec} R$ by

$$Y \xrightarrow{f} X \xrightarrow{g} \operatorname{Spec} R.$$

Then, in an identical manner to the above, $\mathcal{V}_{\mathcal{S}} := \mathcal{O}_X \oplus (\bigoplus_{i\in I\backslash\mathcal{S}} \mathcal{M}_i^X)$ is a tilting bundle on X.

We claim that $f^*(\mathcal{V}_{\mathcal{S}}) = \mathcal{O}_Y \oplus (\bigoplus_{i\in I\backslash\mathcal{S}} \mathcal{M}_i^Y)$. Certainly $f^*\mathcal{L}_i^X = \mathcal{L}_i^Y$ for all $i \in I\backslash\mathcal{S}$, and pulling back

$$0 \to \mathcal{O}_X^{\oplus(r_i-1)} \to \mathcal{M}_i^X \to \mathcal{L}_i^X \to 0$$

gives an exact sequence

$$0 \to \mathcal{O}_Y^{\oplus(r_i-1)} \to f^*\mathcal{M}_i^X \to \mathcal{L}_i^Y \to 0. \tag{4.A}$$

But

$$\mathrm{Ext}^1_Y(f^*\mathcal{M}_i^X, \mathcal{O}_Y) = \mathrm{Ext}^1_Y(\mathbf{L}f^*\,\mathcal{M}_i^X, \mathcal{O}_Y) = \mathrm{Ext}^1_X(\mathcal{M}_i^X, \mathbf{R}f_*\,\mathcal{O}_Y) = \mathrm{Ext}^1_X(\mathcal{M}_i^X, \mathcal{O}_X),$$

which equals zero since $\mathcal{V}_{\mathcal{S}}$ is tilting. Hence (4.A) is a maximal extension, so it follows (by construction) that $\mathcal{M}_i^Y \cong f^*\mathcal{M}_i^X$ for all $i \in I\backslash\mathcal{S}$, so $f^*(\mathcal{V}_{\mathcal{S}}) = \mathcal{O}_Y \oplus (\bigoplus_{i\in I\backslash\mathcal{S}} \mathcal{M}_i^Y)$ as claimed.

Now by the projection formula

$$\mathbf{R}f_*(f^*\mathcal{V}_{\mathcal{S}}) \cong \mathbf{R}f_*(\mathcal{O}_Y \otimes f^*\mathcal{V}_{\mathcal{S}}) \cong \mathbf{R}f_*(\mathcal{O}_Y) \otimes \mathcal{V}_{\mathcal{S}} \cong \mathcal{O}_X \otimes \mathcal{V}_{\mathcal{S}} = \mathcal{V}_{\mathcal{S}},$$

and so it follows that

$$\mathrm{End}_X(\mathcal{V}_{\mathcal{S}}) \cong \mathrm{Hom}_X(\mathcal{V}_{\mathcal{S}}, \mathbf{R}f_*(f^*\mathcal{V}_{\mathcal{S}})) \cong \mathrm{Hom}_Y(\mathbf{L}f^*\,\mathcal{V}_{\mathcal{S}}, f^*\mathcal{V}_{\mathcal{S}}) \cong \mathrm{End}_Y(f^*\mathcal{V}_{\mathcal{S}}),$$

i.e. $\mathrm{End}_X(\mathcal{V}_{\mathcal{S}}) \cong \mathrm{End}_Y(\mathcal{O}_Y \oplus (\bigoplus_{i\in I\backslash\mathcal{S}} \mathcal{M}_i^Y))$. But it is very well known (see, for example, [Wem11a, Lemma 3.2]) that $\mathrm{End}_Y(\mathcal{O}_Y \oplus (\bigoplus_{i\in I\backslash\mathcal{S}} \mathcal{M}_i^Y)) \cong \mathrm{End}_R(R \oplus_{i\in I\backslash\mathcal{S}} M_i) = \mathrm{End}_R(N^{\mathcal{S}})$.

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Hence we have shown that $\mathcal{V}_\mathcal{S}$ is a tilting bundle on $X^\mathcal{S}$ with endomorphism ring isomorphic to $\mathrm{End}_R(N^\mathcal{S})$, so the first statement follows. For the last statement, simply observe that we have functorial isomorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{X^\mathcal{S}}(\mathcal{V}_\mathcal{S}, \mathbf{R}f_*(-)) &= \mathbf{R}\mathrm{Hom}_Y(\mathbf{L}f^*\,\mathcal{V}_\mathcal{S}, -)\\ &= \mathbf{R}\mathrm{Hom}_Y(\mathcal{O}_Y \oplus_{i\in I\backslash\mathcal{S}} \mathcal{M}_i^Y, -)\\ &= e\,\mathbf{R}\mathrm{Hom}_Y(\mathcal{O}_Y \oplus_{i\in I} \mathcal{M}_i^Y, -)\\ &= e\,\mathbf{R}\mathrm{Hom}_Y(\mathcal{V}_\emptyset, -).\end{aligned}$$ □

Remark 4.7. Theorem 4.6 shows that if Λ is the reconstruction algebra and $e \neq 1$ is a non-zero idempotent containing the idempotent corresponding to R, then $e\Lambda e$ always has infinite global dimension, since it is derived equivalent to a singular variety. This greatly generalizes Corollary 3.10, which only deals with idempotents corresponding to partial resolutions 'above' $X^\mathcal{C}$; these generically do not exist. It would be useful to have a purely algebraic proof of the fact $\mathrm{gl.dim}\, e\Lambda e = \infty$, since this is related to many problems in higher dimensions.

Now recall from Definition 1.2 that $D := R \oplus (\bigoplus_{d\in\mathcal{D}} M_d)$. This is just $N^\mathcal{C}$, so as the special case of Theorem 4.6 when $\mathcal{S} = \mathcal{C}$ we obtain the following corollary.

COROLLARY 4.8. *$\mathrm{End}_R(D)$ is derived equivalent to $X^\mathcal{C}$.*

Remark 4.9. It follows from Corollary 4.8 that the module D corresponds to the largest totally discrepant partial resolution of $\mathrm{Spec}\, R$, in that any further resolution must involve crepant curves. This scheme was much studied in earlier works (e.g. [RRW90]), and is related to the deformation theory of $\mathrm{Spec}\, R$. We remark that $X^\mathcal{C}$ is often referred to as the rational double point resolution.

As a further consequence of Theorem 4.6, we have the following result.

THEOREM 4.10. *If $\mathcal{S} \subseteq \mathcal{C}$, then we have triangle equivalences*

$$\underline{\mathrm{SCM}}_{N^\mathcal{S}}(R) \simeq \underline{\mathrm{GP}}(\mathrm{End}_R(N^\mathcal{S})) \simeq \mathrm{D}_{\mathrm{sg}}(\mathrm{End}_R(N^\mathcal{S})) \simeq \mathrm{D}_{\mathrm{sg}}(X^\mathcal{S}) \simeq \bigoplus_{x\in \mathrm{Sing}\, X^\mathcal{S}} \underline{\mathrm{CM}}(\widehat{\mathcal{O}}_{X^\mathcal{S},x}),$$

where $\mathrm{Sing}\, X^\mathcal{S}$ denotes the set of singular points of $X^\mathcal{S}$. In particular, $\underline{\mathrm{SCM}}_{N^\mathcal{S}}(R)$ is 1-Calabi–Yau, and its shift functor satisfies $[2] = \mathrm{id}$.

Proof. Since R is complete local we already know that $\underline{\mathrm{SCM}}_{N^\mathcal{S}}(R)$ is idempotent complete, so the first equivalence is just Theorem 3.8(2). Since $\mathrm{End}_R(N^\mathcal{S})$ is Iwanaga–Gorenstein by Theorem 3.8(1), the second equivalence is a well-known theorem of Buchweitz [Buch86, Theorem 4.4.1(2)]. The third equivalence follows immediately from Theorem 4.6 (see, for example, [IW14, Lemma 4.1]). The fourth equivalence follows from [Orl09], [BK12] or [IW14, Theorem 3.2] since the singularities of $X^\mathcal{S}$ are isolated and the completeness of R implies that $\mathrm{D}_{\mathrm{sg}}(X^\mathcal{S}) \simeq \underline{\mathrm{SCM}}_{N^\mathcal{S}}(R)$ is idempotent complete. The final two statements hold since each $\widehat{\mathcal{O}}_{X^\mathcal{S},x}$ is Gorenstein ADE, and for these it is well known that $\underline{\mathrm{CM}}(\widehat{\mathcal{O}}_{X^\mathcal{S},x})$ are 1-Calabi–Yau [Aus78], satisfying $[2] = \mathrm{id}$ [Eis80]. □

Example 4.11. In Example 4.5 choose $\mathcal{S} = \{E_3, E_5\}$. Then by Theorem 4.10,

$$\underline{\mathrm{SCM}}_{N^\mathcal{S}}(R) \simeq \underline{\mathrm{CM}}\,\mathbb{C}[[x,y]]^{\frac{1}{2}(1,1)} \oplus \underline{\mathrm{CM}}\,\mathbb{C}[[x,y]]^{\frac{1}{2}(1,1)}.$$

Remark 4.12. It was remarked in [IW11, Remark 4.14] that often the category $\underline{\mathrm{SCM}}_D(R)$ is equivalent to that of a Gorenstein ADE singularity, but this equivalence was only known to be an

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additive equivalence, as the triangle structure on $\underline{\mathrm{SCM}}_D(R)$ was difficult to control algebraically. Theorem 4.10 improves this by lifting the additive equivalence to a triangle equivalence. It furthermore generalizes the equivalence to other Frobenius quotients of $\mathrm{SCM}(R)$ that were not considered in [IW11].

We now use Theorem 4.10 to extend Auslander's algebraic McKay correspondence. This requires the notion of the dual graph relative to a morphism.

DEFINITION 4.13. Consider $f^{\mathcal{S}} : Y \to X^{\mathcal{S}}$. The *dual graph* with respect to $f^{\mathcal{S}}$ is defined as follows: for each irreducible curve contracted by $f^{\mathcal{S}}$ draw a vertex, and join two vertices if and only if the corresponding curves in Y intersect. Furthermore, label every vertex with the self-intersection number of the corresponding curve.

The following corollary, which is immediate from Theorem 4.10, extends [IW11, Corollary 4.11].

COROLLARY 4.14. *If $\mathcal{S} \subseteq \mathcal{C}$, then the AR quiver of the category $\underline{\mathrm{SCM}}_{N^{\mathcal{S}}}(R)$ is the double of the dual graph with respect to the morphism $Y \to X^{\mathcal{S}}$.*

4.3 Iwanaga–Gorenstein rings from surfaces

The following corollary of Theorem 4.6 gives a geometric proof of Theorem 3.8(1).

COROLLARY 4.15. *Let $N \in \mathrm{SCM}(R)$ such that $\operatorname{add} D \subseteq \operatorname{add} N$. Then $e_N \Lambda e_N = \operatorname{End}_R(N)$ is an Iwanaga–Gorenstein ring.*

Proof. Since $\operatorname{add} D \subseteq \operatorname{add} N$, Theorem 4.6 shows that the algebra $\operatorname{End}_R(N)$ is derived equivalent, via a tilting bundle, to the Gorenstein scheme $X^{\mathcal{S}}$. Thus the result follows by Corollary 4.4. □

The point is that, using the geometry, we can sharpen Theorem 3.8(1) and Corollary 4.15, since we are explicitly able to determine the value of the injective dimension. The proof requires the following two lemmas, which we state and prove in greater generality.

LEMMA 4.16. *Suppose that $(S, \mathfrak{m})$ is local, Γ is a module-finite S-algebra, and $X, Y \in \operatorname{mod} \Gamma$. Then $\operatorname{Ext}^i_\Gamma(X, Y) = 0$ if $i > \operatorname{inj.dim}_\Gamma Y - \operatorname{depth}_S X$.*

Proof. Use induction on $t = \operatorname{depth}_S X$. The case $t = 0$ is clear. Take an X-regular element r and consider the sequence

$$0 \to X \xrightarrow{r} X \to X/rX \to 0.$$

By induction we have $\operatorname{Ext}^{i+1}_\Gamma(X/rX, Y) = 0$ for $i > \operatorname{inj.dim}_\Gamma Y - t$. By the exact sequence

$$\operatorname{Ext}^i_\Gamma(X, Y) \xrightarrow{r} \operatorname{Ext}^i_\Gamma(X, Y) \to \operatorname{Ext}^{i+1}_\Gamma(X/rX, Y) = 0$$

and Nakayama's lemma, we have $\operatorname{Ext}^i_\Gamma(X, Y) = 0$. □

Recall that if Γ is an S-order, then we denote by $\mathrm{CM}(\Gamma)$ the category consisting of those $X \in \operatorname{mod} \Gamma$ for which $X \in \mathrm{CM}(S)$.

LEMMA 4.17 [GN01, Proposition 1.1(3)]. *Suppose that S is an equicodimensional (i.e. $\dim S = \dim S_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max} S$) d-dimensional CM ring with canonical module ω_S, and let Γ be an S-order. Then:*

(1) $\operatorname{inj.dim}_\Gamma \operatorname{Hom}_S(\Gamma, \omega_S) = d = \operatorname{inj.dim}_{\Gamma^{\mathrm{op}}} \operatorname{Hom}_S(\Gamma, \omega_S)$;
(2) $\operatorname{inj.dim}_\Gamma X = \operatorname{proj.dim}_{\Gamma^{\mathrm{op}}} \operatorname{Hom}_S(X, \omega_S) + d$ *for all* $X \in \mathrm{CM}(\Gamma)$.

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Proof. We include a proof for the convenience of the reader. To simplify notation denote $\mathrm{Hom}_S(-, \omega_S) := (-)^\dagger$. This gives an exact duality $\mathrm{CM}(\Gamma) \leftrightarrow \mathrm{CM}(\Gamma^{\mathrm{op}})$. The statements are local, so we can assume that S is a local ring.

(1) Consider the minimal injective resolution of ω_S in $\mathrm{mod}\, S$, namely

$$0 \to \omega_S \to I_0 \to I_1 \to \cdots \to I_d \to 0.$$

Applying $\mathrm{Hom}_S(\Gamma, -)$, using the fact that $\Gamma \in \mathrm{CM}(S)$ we obtain an exact sequence

$$0 \to \Gamma^\dagger \to \mathrm{Hom}_S(\Gamma, I_0) \to \cdots \to \mathrm{Hom}_S(\Gamma, I_d) \to 0.$$

As in the proof of Lemma 4.2, each $\mathrm{Hom}_S(\Gamma, I_i)$ is an injective Γ-module. This shows that $\mathrm{inj.dim}_\Gamma \Gamma^\dagger \leqslant \dim S$. If $\mathrm{inj.dim}_\Gamma \Gamma^\dagger < \dim S$ then

$$0 \to \mathrm{Hom}_S(\Gamma, \Omega^{-d+1}\omega_S) \to \mathrm{Hom}_S(\Gamma, I_{d-1}) \to \mathrm{Hom}_S(\Gamma, I_d) \to 0 \qquad (4.\mathrm{B})$$

must split. Let T be some non-zero Γ-module which has finite length as an S-module (e.g. $T = \Gamma/\mathfrak{m}\Gamma$ for some $\mathfrak{m} \in \mathrm{Max}\, S$). Since (4.B) splits, applying $\mathrm{Hom}_\Gamma(T, -)$ shows that the top row in the following commutative diagram is exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_\Gamma(T, {}_S(\Gamma, \Omega^{-d+1}\omega_S)) & \longrightarrow & \mathrm{Hom}_\Gamma(T, {}_S(\Gamma, I_{d-1})) & \longrightarrow & \mathrm{Hom}_\Gamma(T, {}_S(\Gamma, I_{d-1})) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Hom}_S(T, \Omega^{-d+1}\omega_S) & \longrightarrow & \mathrm{Hom}_S(T, I_{d-1}) & \longrightarrow & \mathrm{Hom}_S(T, I_d) & \longrightarrow & 0
\end{array}$$

Hence the bottom row is exact. But T has finite length, so $\mathrm{Hom}_S(T, I_{d-1}) = 0$ since none of the associated primes of I_{d-1} is maximal by equicodimensionality of S. But by the above diagram this implies that $\mathrm{Hom}_S(T, I_d) = 0$, which is a contradiction since $\mathrm{Hom}_S(-, I_d)$ is a duality on finite length modules.

(2) Set $l := \mathrm{proj.dim}_{\Gamma^{\mathrm{op}}} X^\dagger$ and $m := \mathrm{inj.dim}_\Gamma X$. Consider a projective resolution of $X^\dagger$ over Γ^{op}:

$$\cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to X^\dagger \to 0. \qquad (4.\mathrm{C})$$

Applying $(-)^\dagger$ gives rise to an exact sequence

$$0 \to X \to P_0^\dagger \xrightarrow{f_1^\dagger} P_1^\dagger \xrightarrow{f_2^\dagger} \cdots . \qquad (4.\mathrm{D})$$

Since by (1) each $P_i^\dagger$ has injective dimension d, it follows that $m = \mathrm{inj.dim}_\Gamma X \leqslant l + d$. So m is infinity implies that l is infinity, and in this case the equality holds. Hence we can assume that $m < \infty$.

We first claim that $m \geqslant d$. This is true if $X \in \mathrm{add}\, \Gamma^\dagger$ by (1). Now we assume that $X \notin \mathrm{add}\, \Gamma^\dagger$, so $X^\dagger \notin \mathrm{add}\, \Gamma$. Thus

$$0 \neq \mathrm{Ext}^1_{\Gamma^{\mathrm{op}}}(X^\dagger, \Omega_{\Gamma^{\mathrm{op}}} X^\dagger) = \mathrm{Ext}^1_\Gamma((\Omega_{\Gamma^{\mathrm{op}}} X^\dagger)^\dagger, X).$$

Since $\mathrm{depth}_S(\Omega_{\Gamma^{\mathrm{op}}} X^\dagger)^\dagger = d$, by Lemma 4.16 we conclude that $m \geqslant d + 1$. Thus we have $m \geqslant d$ in both cases.

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Consider $\mathrm{Im}(f^\dagger_{m-d+1})$. Since $\mathrm{depth}_S(\mathrm{Im}(f^\dagger_{m-d+1})) = d$, by Lemma 4.16 it follows that $\mathrm{Ext}^{m-d+1}_\Gamma(\mathrm{Im}(f^\dagger_{m-d+1}), X) = 0$. But since $X \in \mathrm{CM}(\Gamma)$ and the $P^\dagger_i$ are injective in $\mathrm{CM}(\Gamma)$, $\mathrm{Ext}^j_\Gamma(X, P^\dagger_i) = 0$ for all $j > 0$ and so (4.D) shows that

$$\mathrm{Ext}^1_\Gamma(\mathrm{Im}(f^\dagger_{m-d+1}), \mathrm{Im}(f^\dagger_{m-d})) = \cdots = \mathrm{Ext}^{m-d+1}_\Gamma(\mathrm{Im}(f^\dagger_{m-d+1}), X) = 0.$$

This implies that the short exact sequence

$$0 \to \mathrm{Im}(f^\dagger_{m-d}) \to P^\dagger_{m-d} \to \mathrm{Im}(f^\dagger_{m-d+1}) \to 0$$

splits, which in turn implies that the sequence

$$0 \to \mathrm{Im}(f_{m-d+1}) \to P_{m-d} \to \mathrm{Im}(f_{m-d}) \to 0$$

splits, so $l \leqslant m - d$. In particular, $l < \infty$, so we may assume that $P_i = 0$ for $i > l$ in (4.C). So (4.D) shows that $m \leqslant l + d$. Combining inequalities, we have $m = l + d$, as required. □

The following result is the main result in this subsection. We remark that this gives a generalization of Proposition 3.2.

Theorem 4.18. *Let $N \in \mathrm{SCM}(R)$ such that $\mathrm{add}\, D \subseteq \mathrm{add}\, N$ and put $\Gamma := \mathrm{End}_R(N)$. Then*

$$\mathrm{inj.dim}_\Gamma \Gamma = \begin{cases} 2 & \text{if } R \text{ is Gorenstein,} \\ 3 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.17 we know that $\mathrm{inj.dim}_\Gamma \Gamma \geqslant 2$.

(1) Suppose that R is Gorenstein. In this case $\Gamma \in \mathrm{CM}(R)$ is a symmetric R-order, meaning $\Gamma \cong \mathrm{Hom}_R(\Gamma, R)$ as Γ-Γ bimodules [IR08, Proposition 2.4(3)]. Thus $\mathrm{inj.dim}_\Gamma \Gamma = \dim R = 2$ by Lemma 4.17.

(2) Suppose that R is not Gorenstein, so there exists an indecomposable summand N_i of N such that N_i corresponds to a non-(-2)-curve. Necessarily N_i is not free, and, furthermore, by Proposition 3.5(1), $\mathrm{Ext}^1_R(N_i, X) = 0$ for all $X \in \mathrm{SCM}(R)$.

If $\mathrm{inj.dim}_\Gamma \Gamma = \dim R = 2$ then, by Lemma 4.17, $\mathrm{Hom}_R(\Gamma, \omega_R)$ is a projective Γ-module. But

$$\mathrm{Hom}_R(\Gamma, \omega_R) = \mathrm{Hom}_R(\mathrm{End}_R(N), \omega_R) \cong \mathrm{Hom}_R(N, (N \otimes_R \omega_R)^{**}) \cong \mathrm{Hom}_R(N, \tau N)$$

where τ is the AR translation in the category $\mathrm{CM}(R)$, and the middle isomorphism holds, for example by [AG60, Proposition 4.1]. Hence by reflexive equivalence $\mathrm{Hom}_R(N, -) : \mathrm{CM}(R) \to \mathrm{CM}(\Gamma)$, we have $\tau N \in \mathrm{add}\, N$, so, in particular, $\tau N_i \in \mathrm{SCM}(R)$. But this implies that $\mathrm{Ext}^1_R(N_i, \tau N_i) = 0$ by the above, which by the existence of AR sequences is impossible. Hence $\mathrm{inj.dim}_\Gamma \Gamma \neq 2$. Now Theorem 3.8(1) implies that $\mathrm{inj.dim}_\Gamma \Gamma \leqslant 3$ and so consequently $\mathrm{inj.dim}_\Gamma \Gamma = 3$. □

4.4 Construction of Iwanaga–Gorenstein rings

In this subsection, we work over $\mathbb{C}$. If R is not Gorenstein and $N \in \mathrm{SCM}(R)$ such that $\mathrm{add}\, D \subseteq \mathrm{add}\, N$, then, by Theorems 4.10 and 4.18, $\Gamma := \mathrm{End}_R(N)$ is an Iwanaga–Gorenstein ring with $\mathrm{inj.dim}\, \Gamma = 3$, such that $\underline{\mathrm{GP}}(\Gamma)$ is a direct sum of stable CM categories of ADE singularities. In particular, each Γ has finite Gorenstein projective type. The simplest case is when Γ has only one non-free indecomposable GP module, i.e. the case $\underline{\mathrm{GP}}(\Gamma) \simeq \underline{\mathrm{CM}}(\mathbb{C}[[x, y]]^{\frac{1}{2}(1,1)})$.

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The purpose of this section is to prove the following theorem.

THEOREM 4.19. *Let* $G \leqslant \mathrm{SL}(2,\mathbb{C})$ *be a finite subgroup, with* $G \ncong E_8$*. Then there are uncountably many non-isomorphic Iwanaga–Gorenstein rings* Γ *with* $\operatorname{inj.dim}\Gamma = 3$*, such that* $\underline{\mathrm{GP}}(\Gamma) \simeq \underline{\mathrm{CM}}(\mathbb{C}[[x,y]]^G)$.

The theorem is unusual, since commutative algebra constructions such as Knörrer periodicity only give countably many non-isomorphic Gorenstein rings S with $\underline{\mathrm{CM}}(S) \simeq \underline{\mathrm{CM}}(\mathbb{C}[[x,y]]^G)$, and, furthermore, no two of the S have the same injective dimension.

Remark 4.20. We remark that the omission of type $G \cong E_8$ from our theorem is also unusual; it may still be possible that there are uncountably many non-isomorphic Iwanaga–Gorenstein rings Γ with $\operatorname{inj.dim}\Gamma = 3$ such that $\underline{\mathrm{GP}}(\Gamma) \simeq \underline{\mathrm{CM}}([[x,y]]^{E_8})$, but our methods do not produce any. It is unclear to us whether this illustrates simply the limits of our techniques, or whether the finite type E_8 is much rarer.

To prove Theorem 4.19 requires some knowledge of complete local rational surface singularities over $\mathbb{C}$, which we now review. If R is a complete local rational surface singularity, then if we consider the minimal resolution $Y \to \operatorname{Spec} R$, then (as before) the fibre above the origin is well known to be a tree (i.e. a finite connected graph with no cycles) of $\mathbb{P}^1$s denoted $\{E_i\}_{i\in I}$. Their self-intersection numbers satisfy $E_i \cdot E_i \leqslant -2$, and the intersection matrix $(E_i \cdot E_j)_{i,j\in I}$ is negative definite. We encode the intersection matrix in the form of the labelled dual graph.

DEFINITION 4.21. We refer to the dual graph with respect to the morphism $Y \to \operatorname{Spec} R$ (in the sense of Definition 4.13) as the *dual graph* of R.

Thus, given a complete local rational surface singularity, we obtain a labelled tree. Before we state as a theorem the solution to the converse problem, we require some notation.

Suppose that T is a tree, with vertices denoted $E_1,\dots,E_n$, labelled with integers $w_1,\dots,w_n$. To this data we associate the symmetric matrix $M_T = (b_{ij})_{1\leqslant i,j\leqslant n}$ with b_{ii} defined by $b_{ii} := w_i$, and b_{ij} (with $i \neq j$) defined to be the number of edges linking the vertices E_i and E_j. We denote the free abelian group generated by the vertices E_i by $\mathcal{Z}$, and call its elements *cycles*. The matrix M_T defines a symmetric bilinear form $(-,-)$ on $\mathcal{Z}$ and by analogy with the geometry, we will often write $Y \cdot Z$ instead of (Y,Z). We define

$$\mathcal{Z}_{\mathrm{top}} := \left\{ Z = \sum_{i=1}^{n} a_i E_i \in \mathcal{Z} \mid Z \neq 0,\ \text{all } a_i \geqslant 0,\ \text{and } Z\cdot E_i \leqslant 0 \text{ for all } i \right\}.$$

If there exists $Z \in \mathcal{Z}_{\mathrm{top}}$ such that $Z \cdot Z < 0$, then automatically M_T is negative definite [Art66, Proposition 2(ii)]. In this case, $\mathcal{Z}_{\mathrm{top}}$ admits a unique smallest element Z_f, called the *fundamental cycle*.

THEOREM 4.22 [Art66, Gra62]. *Let T denote a labelled tree, with vertex set $\{E_i \mid i \in I\}$ and labels w_i. Suppose that T satisfies the following combinatorial properties:*

(1) $w_i \leqslant -2$ *for all* $i \in I$;
(2) *there exists* $Z \in \mathcal{Z}_{\mathrm{top}}$ *such that* $Z \cdot Z < 0$;
(3) *writing* Z_f *(which exists by (2)) as* $Z_f = \sum_{i\in I} a_i E_i$*, then*

$$Z_f \cdot Z_f + \sum_{i\in I} a_i(-w_i - 2) = -2.$$

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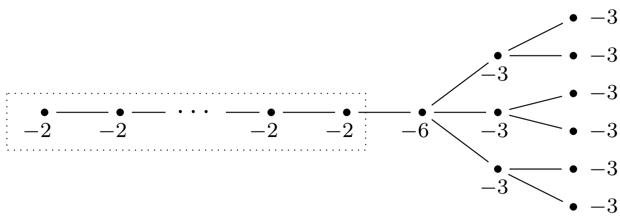
Then there exists some complete local rational surface singularity R, whose minimal resolution has labelled dual graph precisely T.

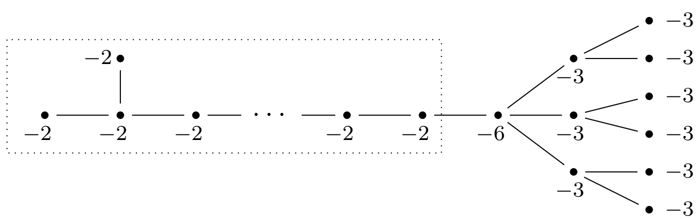
A labelled tree satisfying the combinatorial properties in Theorem 4.22 is called a *rational tree*. The above theorem says that every rational tree arises as the labelled dual graph of some complete local rational surface singularity; however, this singularity need not be unique.

We are now ready to prove Theorem 4.19.

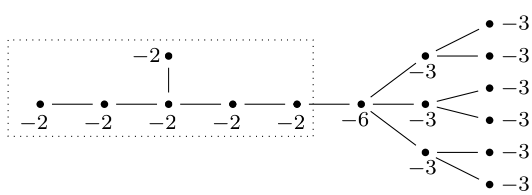
Proof. Consider the following labelled trees:



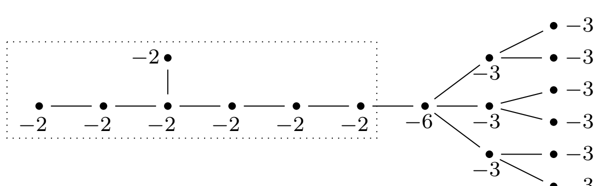
(4.E)



(4.F)

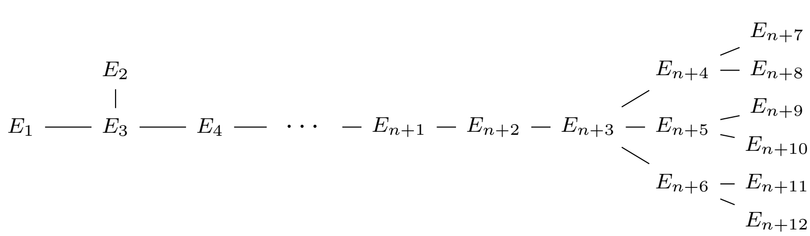


(4.G)



(4.H)

It is an easy combinatorial check to show that each labelled graph above satisfies the criteria in Theorem 4.22, so consequently there is a (not necessarily unique) complete rational surface singularity corresponding to each. We do this for (4.F), the rest being similar. Labelling the vertices in (4.F) by



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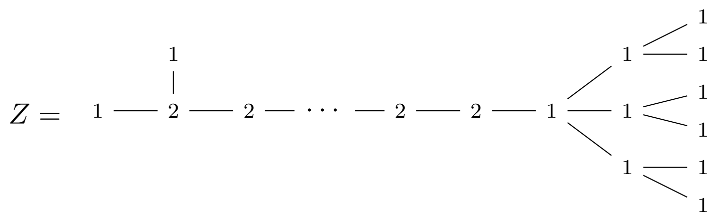
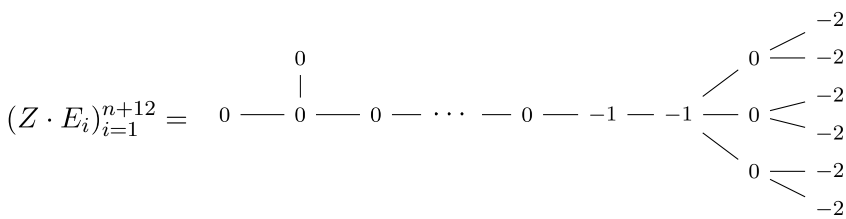
then it is easy to see that $Z := \sum_{i=1}^{2} E_i + \sum_{i=3}^{n+2} 2E_i + \sum_{i=n+3}^{n+12} E_i$ satisfies $Z \cdot E_i \leqslant 0$ for all $1 \leqslant i \leqslant n+12$, hence $Z \in \mathcal{Z}_{\mathrm{top}}$. We denote Z as



From this we see that



so $Z \in \mathcal{Z}_{\mathrm{top}}$ and $Z \cdot Z = Z \cdot (\sum_{i=1}^{2} E_i + \sum_{i=3}^{n+2} 2E_i + \sum_{i=n+3}^{n+12} E_i) = 0 + 2(-1) + (-1-2-2-2-2-2-2) = -15$. Hence condition (2) in Theorem 4.22 is satisfied. For condition (3), by the standard Laufer algorithm, $Z_f = Z$, so $Z_f \cdot Z_f = -15$. On the other hand, $\sum_{i \in I} a_i(-E_i^2 - 2) = 4+1+1+1+1+1+1+1+1+1 = 13$, so $Z_f \cdot Z_f + \sum_{i \in I} a_i(-E_i^2 - 2) = -15 + 13 = -2$, as required.

Now in the above diagrams, for clarity we have drawn a box around the curves that get contracted to form $X^{\mathcal{C}}$. Hence a $\Gamma = \mathrm{End}_R(N^{\mathcal{C}})$ corresponding to (4.E) has the GP finite type corresponding to cyclic groups, by Theorem 4.10 applied to $\mathrm{End}_R(N^{\mathcal{C}})$. Similarly, a Γ corresponding to (4.F) has the GP finite type corresponding to binary dihedral groups, (4.G) corresponds to binary tetrahedral groups, and (4.H) corresponds to binary octahedral groups.

Now each of the above trees has more than one vertex that meets precisely three edges, so by the classification [Lau73, § 1 p. 2] they are not pseudo-taut, and further in each of the above trees there exists a vertex that meets precisely four edges, so by the classification [Lau73, § 2 p. 2] they are not taut. This means that in Theorem 4.22 there are uncountably many (non-isomorphic) R corresponding to each of the above labelled trees. For each such R we thus obtain an Iwanaga–Gorenstein ring $\mathrm{End}_R(N^{\mathcal{C}})$ with the desired properties; furthermore, if R and R' both correspond to the same labelled graph, but $R \ncong R'$, then $\mathrm{End}_R(N^{\mathcal{C}}) \ncong \mathrm{End}_{R'}(N^{\mathcal{C}})$ since the centres of $\mathrm{End}_R(N^{\mathcal{C}})$ and $\mathrm{End}_{R'}(N^{\mathcal{C}})$ are R and R', respectively. Hence, since there are uncountably many such R, there are uncountably many such Iwanaga–Gorenstein rings. □

We give, in § 6.1, some explicit examples illustrating Theorem 4.19 in the case $G = \mathbb{Z}_2$.

Remark 4.23. We remark that the method in the above proof cannot be applied to E_8, since it is well known that the rational tree E_8 with all vertices labelled with -2 cannot be a (strict) subtree of any rational tree [TT04, Corollary 3.11].

5. Relationship to relative singularity categories

In the notation of § 4, let $Y \xrightarrow{f^{\mathcal{S}}} X^{\mathcal{S}} \xrightarrow{g^{\mathcal{S}}} \operatorname{Spec} R$ be a factorization of the minimal resolution of a rational surface singularity, with $\mathcal{S} \subseteq I$. Let Λ be the reconstruction algebra of R and $e \in \Lambda$

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be the idempotent corresponding to the identity endomorphism of the special Cohen–Macaulay R-module $N^{\mathcal{S}} = R \oplus (\bigoplus_{i\in I\setminus S} M_i)$.

Definition 5.1. (1) A triangle functor $Q : \mathcal{C} \to \mathcal{D}$ is called a *quotient functor* if the induced functor $\mathcal{C}/\ker Q \to \mathcal{D}$ is a triangle equivalence. Here $\ker Q \subseteq \mathcal{C}$ denotes the full subcategory of objects X such that $Q(X) = 0$.

(2) A sequence of triangulated categories and triangle functors $\mathcal{U} \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{Q}$ is called *exact* if G is a quotient functor with kernel $\mathcal{U}$, and F is the natural inclusion.

In this section, we extend triangle equivalences from Theorem 4.10 to exact sequences of triangulated categories. In particular, this yields triangle equivalences between the relative singularity categories studied in [BK12, KY12, Kal13].

Proposition 5.2. *There exists a commutative diagram of triangulated categories and functors such that the horizontal arrows are equivalences and the columns are exact.*

$$\begin{array}{ccc}
\operatorname{thick}\left(\bigoplus_{i\in\mathcal{S}} \mathcal{O}_{E_i}(-1)\right) & \xrightarrow{\sim} & \operatorname{thick}(\operatorname{mod}\Lambda/\Lambda e\Lambda) \\
\big\downarrow & & \big\downarrow \\
\dfrac{\mathrm{D^b}(\operatorname{coh} Y)}{\operatorname{thick}\left(\mathcal{O}_Y \oplus \left(\bigoplus_{i\in I\setminus\mathcal{S}} \mathcal{M}_i\right)\right)} & \xrightarrow[\sim]{\mathbf{R}\mathrm{Hom}_Y(\mathcal{V}_\emptyset,-)} & \dfrac{\mathrm{D^b}(\operatorname{mod}\Lambda)}{\operatorname{thick}(\Lambda e)} \\
\big\downarrow{\scriptstyle \mathbf{R}f^{\mathcal{S}}_*} & & \big\downarrow{\scriptstyle e(-)} \\
\mathrm{D_{sg}}(X^{\mathcal{S}}) & \xrightarrow[\sim]{\mathbf{R}\mathrm{Hom}_{X^{\mathcal{S}}}(\mathcal{V}_{\mathcal{S}},-)} & \mathrm{D_{sg}}(e\Lambda e)
\end{array} \qquad (5.A)$$

By an abuse of notation, the induced triangle functors in the lower square are labelled by the inducing triangle functors from the diagram in Theorem 4.6.

Proof. We start with the lower square. Since the corresponding diagram in Theorem 4.6 commutes, it suffices to show that the induced functors above are well-defined. Clearly, the equivalence $\mathbf{R}\mathrm{Hom}_Y(\mathcal{V}_\emptyset, -)$ from Theorem 4.6 maps $\mathcal{O}_Y \oplus (\bigoplus_{i\in I\setminus S} \mathcal{M}_i)$ to Λe. Hence, it induces an equivalence on the triangulated quotient categories. Since $\mathbf{R}\mathrm{Hom}_{X^{\mathcal{S}}}(\mathcal{V}_{\mathcal{S}}, -)$ is an equivalence by Theorem 4.6 and the subcategories $\mathrm{Perf}(X^{\mathcal{S}})$ and $\mathrm{Perf}(e\Lambda e)$ can be defined intrinsically, we get a well-defined equivalence on the bottom of diagram (5.A). The functor on the right is a well-defined quotient functor by Proposition 2.14. Now, the functor on the left is a well-defined quotient functor by the commutativity of the diagram in Theorem 4.6 and the considerations above.

The category $\operatorname{thick}(\operatorname{mod}\Lambda/\Lambda e\Lambda)$ is the kernel of the quotient functor $e(-)$, by Proposition 2.14. Since R has isolated singularities, the algebra $\Lambda/\Lambda e\Lambda$ is always finite-dimensional and so $\operatorname{thick}(\operatorname{mod}\Lambda/\Lambda e\Lambda) = \operatorname{thick}(\bigoplus_{i\in\mathcal{S}} S_i)$, where S_i denotes the simple Λ-module corresponding to the vertex i in the quiver of Λ. But under the derived equivalence $\mathbf{R}\mathrm{Hom}_Y(\mathcal{V}_\emptyset, -)$, S_i corresponds to $\mathcal{O}_{E_i}(-1)[1]$ [vdB04a, Proposition 3.5.7], so it follows that we can identify the subcategory $\operatorname{thick}(\operatorname{mod}\Lambda/\Lambda e\Lambda) = \operatorname{thick}(\bigoplus_{i\in\mathcal{S}} S_i)$ with $\operatorname{thick}(\bigoplus_{i\in\mathcal{S}} \mathcal{O}_{E_i}(-1))$, inducing the top half of the diagram. □

Remark 5.3. The functor $\mathbf{R}\mathrm{Hom}_{X^{\mathcal{S}}}(\mathcal{V}_{\mathcal{S}}, -)$ identifies $\mathrm{Perf}(X^{\mathcal{S}})$ with $\mathrm{Perf}(e\Lambda e) \cong \operatorname{thick}(\Lambda e) \subseteq \mathrm{D^b}(\operatorname{mod}\Lambda)$. Hence, applying the quasi-inverse of $\mathbf{R}\mathrm{Hom}_Y(\mathcal{V}_\emptyset, -)$ to $\operatorname{thick}(\Lambda e)$ yields a triangle

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equivalence $\operatorname{Perf}(X^{\mathcal{S}}) \cong \operatorname{thick}(\mathcal{O}_Y \oplus (\bigoplus_{i\in I\setminus\mathcal{S}} \mathcal{M}_i))$. In particular, there is an equivalence

$$\frac{\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)}{\operatorname{Perf}(X^{\mathcal{S}})} \xrightarrow{\sim} \frac{\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)}{\operatorname{thick}(\Lambda e)}. \tag{5.B}$$

Analysing the commutative diagram in Theorem 4.6 shows that $\operatorname{Perf}(X^{\mathcal{S}}) \cong \operatorname{thick}(\mathcal{O}_Y \oplus (\bigoplus_{i\in I\setminus\mathcal{S}} \mathcal{M}_i))$ is obtained as a restriction of $\mathbf{L}(f^{\mathcal{S}})^*$.

If we contract only (-2)-curves (i.e. if $\mathcal{S} \subseteq \mathcal{C}$ holds), then we know that $\mathrm{D}_{\mathrm{sg}}(X^{\mathcal{S}})$ splits into a direct sum of singularity categories of ADE surface singularities (Theorem 4.10). In this case, it turns out that the diagram above admits an extension to the right and that in fact all the triangulated categories in our (extended) diagram split into blocks indexed by the singularities of the Gorenstein scheme $X^{\mathcal{S}}$.

Let us fix some notation. For a singular point $x \in \operatorname{Sing} X^{\mathcal{S}}$ let $R_x = \widehat{\mathcal{O}}_{X^{\mathcal{S}},x}$, and let $f_x : Y_x \to \operatorname{Spec} R_x$ be the minimal resolution of singularities.

PROPOSITION 5.4. *Assume $\mathcal{S} \subseteq \mathcal{C}$. There exists a commutative diagram of triangulated categories and functors such that the horizontal arrows are equivalences and the columns are exact.*

$$\begin{array}{ccc}
\operatorname{thick}(\operatorname{mod}\Lambda/\Lambda e\Lambda) & \xrightarrow{\sim} & \bigoplus_{x\in\operatorname{Sing} X^{\mathcal{S}}} \ker(\mathbf{R}(f_x)_*) \\
\big\downarrow & & \big\downarrow \\
\dfrac{\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)}{\operatorname{thick}(\Lambda e)} & \xrightarrow{\sim} & \bigoplus_{x\in\operatorname{Sing} X^{\mathcal{S}}} \dfrac{\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y_x)}{\operatorname{Perf}(R_x)} \\
\big\downarrow{\scriptstyle e(-)} & & \big\downarrow{\scriptstyle \bigoplus_{x\in\operatorname{Sing} X^{\mathcal{S}}} \mathbf{R}(f_x)_*} \\
\mathrm{D}_{\mathrm{sg}}(e\Lambda e) & \xrightarrow{\sim} & \bigoplus_{x\in\operatorname{Sing} X^{\mathcal{S}}} \mathrm{D}_{\mathrm{sg}}(R_x)
\end{array} \tag{5.C}$$

Proof. We need some preparation. Note that by the derived McKay correspondence [KV00, BKR01], there are derived equivalences $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y_x) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Pi_x)$, where Π_x is the Auslander algebra of the Frobenius category of maximal Cohen–Macaulay R_x-modules $\operatorname{CM}(R_x)$. Now we have two Frobenius categories $\mathcal{E}_1 := \operatorname{SCM}_{\mathcal{N}^{\mathcal{S}}}(R)$ and $\mathcal{E}_2 := \bigoplus_{x\in\operatorname{Sing} X^{\mathcal{S}}} \operatorname{CM}(R_x)$, which clearly satisfy the conditions (FM1)–(FM4) in [KY12, Setup 5.1] and whose stable categories are Hom-finite and idempotent complete. Further, $\mathcal{E}_1$ and $\mathcal{E}_2$ are stably equivalent by Theorem 4.10.

Now, by [KY12, Theorem 5.5(a)], there are triangle equivalences

$$\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)/\operatorname{thick}(\Lambda e) \cong \operatorname{per}\big(\Lambda_{dg}(\underline{\mathcal{E}_1})\big), \tag{5.D}$$

$$\bigoplus_{x\in\operatorname{Sing} X^{\mathcal{S}}} \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Pi_x)/\operatorname{Perf}(R_x) \cong \operatorname{per}\big(\Lambda_{dg}(\underline{\mathcal{E}_2})\big), \tag{5.E}$$

where by definition $\Lambda_{dg}(\underline{\mathcal{E}_1})$ and $\Lambda_{dg}(\underline{\mathcal{E}_2})$ are differential graded algebras that depend only on (the triangulated structure of) the stable Frobenius categories $\underline{\mathcal{E}_1}$ and $\underline{\mathcal{E}_2}$ (the quotient category $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)/\operatorname{thick}(\Lambda e)$ is idempotent complete by [Kal13, Proposition 2.69] combined with Proposition 3.2 and the completeness of R). Hence, since $\mathcal{E}_1$ and $\mathcal{E}_2$ are stably equivalent, these two differential graded algebras are isomorphic. Thus the combination of the equivalences (5.D) and (5.E) yields a triangle equivalence

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$$\frac{\mathrm{D^b}(\operatorname{mod}\Lambda)}{\operatorname{thick}(\Lambda e)} \longrightarrow \bigoplus_{x\in \operatorname{Sing} X^{\mathcal{S}}} \frac{\mathrm{D^b}(\operatorname{mod}\Pi_x)}{\operatorname{Perf}(R_x)} \tag{5.F}$$

which, in conjunction with the derived McKay correspondence, yields the equivalence of triangulated categories in the middle of (5.C).

Furthermore, the functors $\operatorname{Hom}_\Lambda(\Lambda e, -)$ and $\bigoplus_{x\in \operatorname{Sing} X^{\mathcal{S}}} \mathbf{R}(f_x)_*$ are quotient functors with kernels $\operatorname{thick}(\operatorname{mod}\Lambda/\Lambda e\Lambda)$ and $\bigoplus_{x\in \operatorname{Sing} X^{\mathcal{S}}} \ker(\mathbf{R}(f_x)_*)$, respectively. These subcategories admit intrinsic descriptions (cf. [KY12, Corollary 6.17]). Hence, there is an induced equivalence, which renders the upper square commutative. This in turn induces an equivalence on the bottom of (5.C), such that the lower square commutes. □

Remark 5.5. Using (5.B) together with an appropriate adaption of the techniques developed in [BK12] may yield a more direct explanation for the block decomposition in (5.C).

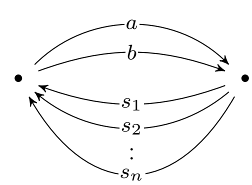
6. Examples

In this section we illustrate some of the previous results with some examples. Our construction in § 2 relies on finding some M such that $\operatorname{gl.dim}\operatorname{End}_\Lambda(\Lambda\oplus M) < \infty$, so we give explicit examples of when this occurs both in finite-dimensional algebras and in geometry.

6.1 Iwanaga–Gorenstein rings of finite GP type

As a special case of Theorem 4.19, there are uncountably many Iwanaga–Gorenstein rings Γ with the property that $\underline{\mathrm{GP}}(\Gamma) \simeq \underline{\mathrm{CM}}(\mathbb{C}[[x,y]]^{\frac{1}{2}(1,1)})$. This category has only one indecomposable object, and is the simplest possible triangulated category. Here we show that the abstract setting in Theorem 4.19 can be used to give explicit examples of such Γ, presented as a quiver with relations.

Definition 6.1. For all $n \geqslant 3$, we define the algebra Λ_n to be the path algebra of the quiver



(where there are n arrows from right to left), subject to the relations

$$\begin{aligned} s_{n-1}bs_n &= s_nbs_{n-1},\\ as_n &= (bs_{n-1})^2,\\ s_na &= (s_{n-1}b)^2,\\ \left.\begin{aligned} as_{i+1} &= bs_i\\ s_{i+1}a &= s_ib \end{aligned}\right\}& \quad \text{for all } 1\leqslant i\leqslant n-2. \end{aligned}$$

Our main result (Theorem 6.2) shows that, for all $n \geqslant 3$, the completion $\widehat{\Lambda}_n$ is an Iwanaga–Gorenstein ring with $\operatorname{inj.dim}\widehat{\Lambda}_n = 3$, such that $\underline{\mathrm{GP}}(\widehat{\Lambda}_n) \simeq \underline{\mathrm{CM}}(\mathbb{C}[[x,y]]^{\frac{1}{2}(1,1)})$. Before we can prove this, we need some notation. Let $n \geqslant 3$, set $m := 2n-1$ and consider the group

$$\frac{1}{m}(1,2) := \left\langle \begin{pmatrix} \varepsilon_m & 0\\ 0 & \varepsilon_m^2 \end{pmatrix} \right\rangle$$

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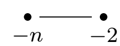
where ε_m is a primitive mth root of unity. The invariants $\mathbb{C}[x,y]^{\frac{1}{m}(1,2)}$ are known to be generated by

$$a := x^m,\ b_1 := x^{m-2}y,\ b_2 := x^{m-4}y^2, \dots,\ b_{n-1} := xy^{n-1},\ c := y^m$$

which abstractly as a commutative ring is $\mathbb{C}[a, b_1, \dots, b_{n-1}, c]$ factored by the relations given by the 2×2 minors of the matrix

$$\begin{pmatrix} a & b_1 & b_2 & \dots & b_{n-2} & b_{n-1}^2 \\ b_1 & b_2 & b_3 & \dots & b_{n-1} & c \end{pmatrix}.$$

We denote this (non-complete) commutative ring by R. This singularity is toric, and the minimal resolution of $\operatorname{Spec} R$ is well known to have dual graph



THEOREM 6.2. *Let $n \geqslant 3$, set $m := 2n-1$ and consider $G := \frac{1}{m}(1,2)$. Denote $R := \mathbb{C}[x,y]^G$, presented as $\mathbb{C}[a, b_1, \dots, b_{n-1}, c]/(2 \times 2$ minors$)$ as above. Then the following statements hold.*

(1) *The R-ideal (a, b_1) is the non-free special CM R-module corresponding to the $(-n)$-curve in the minimal resolution of* $\operatorname{Spec} R$.

(2) $\Lambda_n \cong \operatorname{End}_R(R \oplus (a, b_1))$.

In particular, by completing both sides of (2), $\widehat{\Lambda}_n$ is an Iwanaga–Gorenstein ring with $\operatorname{inj.dim} \widehat{\Lambda}_n = 3$, *such that $\underline{\mathrm{GP}}(\widehat{\Lambda}_n) \simeq \underline{\mathrm{CM}}(\mathbb{C}[[x,y]]^{\frac{1}{2}(1,1)})$. Furthermore, $\widehat{\Lambda}_{n'} \not\cong \widehat{\Lambda}_n$ whenever $n' \neq n$.*

Proof. (1) Let $\rho_0, \dots, \rho_{m-1}$ be the irreducible representations of $G \cong \mathbb{Z}_m$ over $\mathbb{C}$. Since $R = \mathbb{C}[x,y]^G$, we can consider the CM R modules $S_i := (\mathbb{C}[x,y] \otimes_{\mathbb{C}} \rho_i)^G$. It is a well-known result of Wunram [Wun87] that the special CM R-modules in this case are $R = S_0$, S_1 and S_2, with S_2 corresponding to the $(-n)$-curve. We remark that Wunram proved this result under the assumption that R is complete, but the result is still true in the non-complete case [Cra11, Wem11b]. Furthermore, S_2 is generated by x^2, y as an R-module [Wun87]. It is easy to check that under the new coordinates, S_2 is isomorphic to (a, b_1).

(2) We prove this using key varieties.

Step 1. Consider the commutative ring $\mathbb{C}[a, b_1^{(1)}, b_1^{(2)}, \dots, b_{n-1}^{(1)}, b_{n-1}^{(2)}, c]$ factored by the relations given by the 2×2 minors of the matrix

$$\begin{pmatrix} a & b_1^{(1)} & b_2^{(1)} & \cdots & b_{n-2}^{(1)} & b_{n-1}^{(1)} \\ b_1^{(2)} & b_2^{(2)} & b_3^{(2)} & \cdots & b_{n-1}^{(2)} & c \end{pmatrix}.$$

We denote this factor ring by S. We regard $\operatorname{Spec} S$ as a key variety which we then cut (in Step 4) to obtain our ring R.

Step 2. We blow up the ideal $(a, b_1^{(2)})$ of S to give a variety, denoted Y, covered by the two affine opens

$$\mathbb{C}\left[b_1^{(2)}, b_2^{(2)}, \dots, b_{n-1}^{(2)}, c, \frac{a}{b_1^{(2)}}\right], \quad \mathbb{C}\left[a, b_1^{(1)}, b_2^{(1)}, \dots, b_{n-1}^{(1)}, \frac{b_1^{(2)}}{a}\right].$$

The resulting map $f : Y \to \operatorname{Spec} S$ has fibres at most one-dimensional, so we know from [vdB04a] that Y has a tilting bundle. Using the above explicit open cover and morphism, there is an ample

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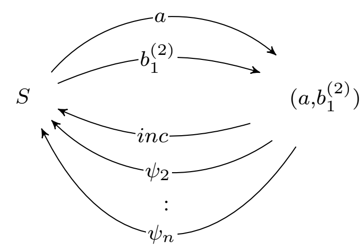
line bundle $\mathcal{L}$ on Y generated by global sections, satisfying $\mathcal{L} \cdot E = 1$ (where E is the $\mathbb{P}^1$ above the origin), with the property that $H^1(\mathcal{L}^\vee) = 0$. This means, by [vdB04a, Proposition 3.2.5], that $\mathcal{V} := \mathcal{O} \oplus \mathcal{L}$ is a tilting bundle. As is always true in the one-dimensional fibre tilting setting (where f is projective birational between integral normal schemes), $\mathrm{End}_Y(\mathcal{O} \oplus \mathcal{L}) \cong \mathrm{End}_S(S \oplus f_*\mathcal{L})$. In the explicit construction of Y above, it is clear that $f_*\mathcal{L} = (a, b_1^{(2)})$. This shows that $\mathrm{End}_S(S \oplus (a, b_1^{(2)}))$ is derived equivalent to Y.

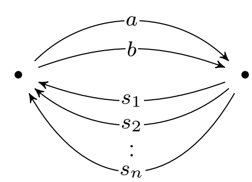
Step 3. We present $\mathrm{End}_S(S \oplus (a, b_1^{(2)}))$ as a quiver with relations. This is easy, since Y is smooth. We have

$$\mathrm{End}_S(S \oplus (a, b_1^{(2)})) \cong \begin{pmatrix} S & (a, b_1^{(2)}) \\ (a, b_1^{(2)})^* & S \end{pmatrix},$$

and we can check that all generators can be seen on the diagram



where $\psi_i := b_{i-1}^{(1)}/a = b_i^{(2)}/b_1^{(2)}$ for all $2 \leqslant i \leqslant n-1$, and $\psi_n := b_{n-1}^{(1)}/a = c/b_1^{(2)}$. Thus if we consider the quiver Q,



with relations $\mathcal{R}$,

$$\begin{aligned} as_ib &= bs_ia \quad \text{for all } 1 \leqslant i \leqslant n \\ s_ias_j &= s_jas_i \quad \text{for all } 1 \leqslant i < j \leqslant n, \end{aligned}$$

then there is a natural surjective ring homomorphism

$$\mathbb{C}Q/\mathcal{R} \to \mathrm{End}_S(S \oplus (a, b_1^{(2)})).$$

But everything above is graded (with arrows all having grade one), and so a Hilbert series calculation shows that the above ring homomorphism must also be bijective.

Step 4. We base change, and show that we can add central relations to the presentation of $\mathrm{End}_S(S \oplus (a, b_1^{(2)}))$ in Step 3 to obtain a presentation for $\mathrm{End}_R(R \oplus (a, b_1))$.

Factoring S by the regular element $b_1^{(1)} - b_1^{(2)}$, we obtain a ring denoted R_1. Factoring R_1 by the regular element $b_2^{(1)} - b_2^{(2)}$, we obtain a ring denoted R_2. Continuing in this manner, factor R_{n-3} by $b_{n-2}^{(1)} - b_{n-2}^{(2)}$ to obtain R_{n-2}. Finally, factor R_{n-2} by $b_{n-1}^{(1)} - (b_{n-1}^{(2)})^2$ to obtain R_{n-1}, which by definition is the ring R in the statement of the theorem. At each step, we are factoring

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by a regular element. Taking the pullbacks, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
Y_{n-1} & \xrightarrow{i_{n-1}} & Y_{n-2} & \xrightarrow{i_{n-2}} & \cdots & \xrightarrow{i_2} & Y_1 & \xrightarrow{i_1} & Y \\
\downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_{n-2}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\
\operatorname{Spec} R & \xrightarrow{j_{n-1}} & \operatorname{Spec} R_{n-2} & \xrightarrow{j_{n-2}} & \cdots & \xrightarrow{j_2} & \operatorname{Spec} R_1 & \xrightarrow{j_1} & \operatorname{Spec} S
\end{array}$$

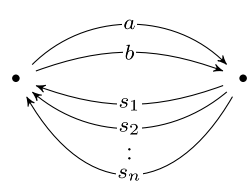
Under the set-up above, $\mathcal{V}_{n-1} := i_{n-1}^* \dots i_1^* \mathcal{V}$ is a tilting bundle on Y_{n-1} with $\operatorname{End}_{Y_{n-1}}(\mathcal{V}_{n-1}) \cong j_{n-1}^* \dots j_1^* \operatorname{End}_S(f_* \mathcal{V}) \cong j_{n-1}^* \dots j_1^* \operatorname{End}_S(S \oplus (a, b_1^{(2)}))$. But on the other hand, f_{n-1} is a projective birational morphism with fibres at most one-dimensional between integral normal schemes, and so

$$\operatorname{End}_{Y_{n-1}}(\mathcal{V}_{n-1}) \cong \operatorname{End}_R((f_{n-1})_* \mathcal{V}_{n-1}) \cong \operatorname{End}_R(j_{n-1}^* \dots j_1^* f_* \mathcal{V}) \cong \operatorname{End}_R(R \oplus (a, b_1)),$$

where the middle isomorphism follows by iterating [IU09, Lemma 8.1]. Thus $\operatorname{End}_R(R \oplus (a, b_1)) \cong j_{n-1}^* \dots j_1^* \operatorname{End}_S(S \oplus (a, b_1^{(2)}))$. Since by definition each j_t^* factors by a regular element, we obtain $\operatorname{End}_R(R \oplus (a, b_1))$ from the presentation of $\operatorname{End}_S(S \oplus (a, b_1^{(2)}))$ in Step 3 by factoring out by the central relations corresponding to the regular elements. Now, via the explicit form in Step 3, these are

$$\begin{array}{rcl}
b_1^{(1)} - b_1^{(2)} & \leftrightarrow & (as_2 + s_2 a) - (bs_1 + s_1 b) \\
& \vdots & \\
b_{n-2}^{(1)} - b_{n-2}^{(2)} & \leftrightarrow & (as_{n-1} + s_{n-1} a) - (bs_{n-2} + s_{n-2} b) \\
b_{n-1}^{(1)} - (b_{n-1}^{(2)})^2 & \leftrightarrow & (as_n + s_n a) - (bs_{n-1} + s_{n-1} b)^2.
\end{array}$$

Step 5. We justify that $\Lambda_n \cong \operatorname{End}_R(R \oplus (a, b_1))$. From Step 4 we know that $\operatorname{End}_R(R \oplus (a, b_1))$ can be presented as



subject to the relations

$$\begin{array}{rll}
as_i b = bs_i a & & \text{for all } 1 \leqslant i \leqslant n, \\
s_i a s_j = s_j a s_i & & \text{for all } 1 \leqslant i < j \leqslant n, \\
as_n = (bs_{n-1})^2, & & \\
s_n a = (s_{n-1} b)^2, & & \\
as_{i+1} = bs_i & & \text{for all } 1 \leqslant i \leqslant n-2, \\
s_{i+1} a = s_i b & & \text{for all } 1 \leqslant i \leqslant n-2.
\end{array}$$

This is a non-minimal presentation, since some relations can be deduced from others. It is not difficult to show that the non-minimal presentation above can be reduced to the relations defining Λ_n. This proves (2).

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For the final statement in the theorem, by completing both sides we see that $\widehat{\Lambda}_n \cong \mathrm{End}_{\widehat{R}}(N^{\mathcal{C}})$, which by Corollary 4.8 is derived equivalent to the rational double point resolution $X^{\mathcal{C}}$ of $\mathrm{Spec}\,\widehat{R}$. Since by construction $X^{\mathcal{C}}$ has only one singularity, of type $\frac{1}{2}(1,1)$, $\underline{\mathrm{GP}}(\widehat{\Lambda}_n) \simeq \underline{\mathrm{CM}}(\mathbb{C}[[x,y]]^{\frac{1}{2}(1,1)})$ follows from Theorem 4.10. Finally, since the centre of $\widehat{\Lambda}_n$ is $\mathbb{C}[[x,y]]^{\frac{1}{2n-1}(1,2)}$, it follows that $n' \neq n$ implies $\widehat{\Lambda}_{n'} \not\cong \widehat{\Lambda}_n$. □

6.2 Frobenius structures on module categories

Let K be a field and denote $\mathbb{D} := \mathrm{Hom}_K(-,K)$. Here we illustrate our main theorem, Theorem 2.7, in the setting of finite-dimensional algebras. Using both Theorem 2.7 and Proposition 2.16, we recover the following result due to Auslander and Solberg [AS93a], which is rediscovered and generalized by Kong [Kon12].

Proposition 6.3. *Let Λ be a finite-dimensional algebra and $\mathcal{N}$ a functorially finite subcategory of* $\mathrm{mod}\,\Lambda$ *satisfying $\Lambda \oplus \mathbb{D}\Lambda \in \mathcal{N}$ and $\tau\underline{\mathcal{N}} = \overline{\mathcal{N}}$, where τ is the AR translation. Then* $\mathrm{mod}\,\Lambda$ *has a structure of a Frobenius category such that the category of projective objects is* $\mathrm{add}\,\mathcal{N}$*, and we have an equivalence* $\mathrm{mod}\,\Lambda \to \mathrm{GP}(\mathcal{N})$, $X \mapsto \mathrm{Hom}_\Lambda(X,-)|_{\mathcal{N}}$.

Proof. By Proposition 2.16, we have a new structure of a Frobenius category on $\mathrm{mod}\,\Lambda$ whose projective-injective objects are $\mathrm{add}\,\mathcal{N}$. Applying Theorem 2.8 to $(\mathcal{E},\mathcal{M},\mathcal{P}) := (\mathrm{mod}\,\Lambda, \mathrm{mod}\,\Lambda, \mathrm{add}\,\mathcal{N})$, we have the assertion since $\mathrm{mod}(\mathrm{mod}\,\Lambda)$ has global dimension at most two and $\mathrm{mod}\,\Lambda$ is idempotent complete. □

The following result supplies a class of algebras satisfying the conditions in Proposition 6.3. It generalizes [Kon12, Theorem 3.4] in which Γ is the path algebra of a Dynkin quiver. Below $\otimes := \otimes_K$.

Proposition 6.4. *Let Δ and Γ be finite-dimensional K-algebras. Assume that Δ is self-injective. Then $\Lambda = \Delta \otimes \Gamma$ and $\mathcal{N} = \Delta \otimes \mathrm{mod}\,\Gamma := \{\Delta \otimes M \mid M \in \mathrm{mod}\,\Gamma\}$ satisfy the conditions in Proposition 6.3. Consequently, we have an equivalence*

$$\mathrm{mod}\,\Lambda \cong \mathrm{GP}(\Delta \otimes \mathrm{mod}\,\Gamma).$$

Proof. Since Δ is self-injective, both $\Lambda = \Delta \otimes \Gamma$ and $\mathbb{D}\Lambda = \mathbb{D}(\Delta \otimes \Gamma) = \mathbb{D}\Delta \otimes \mathbb{D}\Gamma = \Delta \otimes \mathbb{D}\Gamma$ belong to $\mathcal{N} = \Delta \otimes \mathrm{mod}\,\Gamma$. For $M \in \mathrm{mod}\,\Gamma$, it follows from the next lemma that $\tau_\Lambda(\Delta \otimes M) = \nu_\Delta(\Delta) \otimes \tau_\Gamma(M)$. Since Δ is self-injective, we have $\nu_\Delta(\Delta) = \Delta$, and hence $\tau_\Lambda(\Delta \otimes M) = \Delta \otimes \tau_\Gamma(M) \in \Delta \otimes \mathrm{mod}\,\Gamma$. Thus the conditions in Proposition 6.3 are satisfied. □

Lemma 6.5. *Let Δ and Γ be finite-dimensional K-algebras and $\Lambda = \Delta \otimes \Gamma$. Then, for a finite-dimensional Γ-module M and a finitely generated projective Δ-module P, we have $\tau_\Lambda(P \otimes M) = \nu_\Delta(P) \otimes \tau_\Gamma(M)$, where $\nu_\Delta = \mathbb{D}\,\mathrm{Hom}_\Delta(-,\Delta)$ is the Nakayama functor.*

Proof. This is shown in the proof of [Kon12, Theorem 3.4] for the case when Δ is self-injective and Γ is the path algebra of a Dynkin quiver. The proof there works more generally in our setting. For the convenience of the reader we include it here.

Let $Q^{-1} \xrightarrow{f} Q^0$ be a minimal projective presentation of M over Γ. Then

$$P \otimes Q^{-1} \xrightarrow{\mathrm{id}_P \otimes f} P \otimes Q^0$$

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is a minimal projective presentation of $P \otimes M$ over $\Delta \otimes \Gamma$. We apply $\nu_\Lambda = \mathbb{D}\operatorname{Hom}_{\Delta\otimes\Gamma}(-, \Delta\otimes\Gamma)$, and by the definition of τ we obtain an exact sequence

$$0 \to \tau_\Lambda(P \otimes M) \to \nu_\Lambda(P \otimes Q^{-1}) \xrightarrow{\nu(\mathrm{id}_P \otimes f)} \nu_\Lambda(P \otimes Q^0). \tag{6.A}$$

Observe that for a finitely generated projective Γ-module Q we have

$$\begin{aligned}\nu_\Lambda(P \otimes Q) &= \mathbb{D}\operatorname{Hom}_{\Delta\otimes\Gamma}(P\otimes Q, \Delta\otimes\Gamma) = \mathbb{D}(\operatorname{Hom}_\Delta(P,\Delta)\otimes\operatorname{Hom}_\Gamma(Q,\Gamma))\\ &= \nu_\Delta(P) \otimes \nu_\Gamma(Q).\end{aligned}$$

Therefore the sequence (6.A) is equivalent to

$$0 \to \tau_\Lambda(P \otimes M) \to \nu_\Delta(P)\otimes\nu_\Gamma(Q^{-1}) \xrightarrow{\nu(\mathrm{id}_P)\otimes\nu(f)} \nu_\Delta(P)\otimes\nu_\Gamma(Q^0).$$

It follows that $\tau_\Lambda(P \otimes M) = \nu_\Delta(P) \otimes \tau_\Gamma(M)$, as desired. □

Remark 6.6. Let Δ, Γ and Λ be as in Proposition 6.4. Assume further that Γ has finite representation type and let $\operatorname{Aus}(\Gamma)$ denote the Auslander algebra of Γ, i.e. the endomorphism algebra of an additive generator of $\operatorname{mod}\Gamma$.

(1) The algebra $\Delta \otimes \operatorname{Aus}(\Gamma)$ is Iwanaga–Gorenstein and we have an equivalence

$$\operatorname{mod}\Lambda \cong \operatorname{GP}(\Delta \otimes \operatorname{Aus}(\Gamma)).$$

(2) If, in addition, $\operatorname{mod}\Gamma$ has no stable τ-orbits, then any subcategory of $\Delta \otimes \operatorname{mod}\Gamma$ satisfying the conditions in Proposition 6.3 already additively generates $\Delta \otimes \operatorname{mod}\Gamma$. In this sense, $\Delta \otimes \operatorname{Aus}(\Gamma)$ is smallest possible.

6.3 Frobenius categories arising from preprojective algebras

Let Q be a finite quiver without oriented cycles and let W be the Coxeter group associated to Q with generators s_i, $i \in Q_0$. Let K be a field, let Λ be the associated preprojective algebra over K and let e_i be the idempotent of Λ corresponding to the vertex i of Q. Denote $I_i = \Lambda(1-e_i)\Lambda$.

For an element $w \in W$ with reduced expression $w = s_{i_1}\cdots s_{i_k}$, let $I_w = I_{i_1}\cdots I_{i_k}$ and set $\Lambda_w = \Lambda/I_w$. As a concrete example, if Q is the quiver of type A_3 and $w = s_2s_1s_3s_2$, then Λ_w is given by the following quiver with relations:

$$\underset{1}{\bullet} \underset{a^*}{\overset{a}{\rightleftarrows}} \underset{2}{\bullet} \underset{b^*}{\overset{b}{\rightleftarrows}} \underset{3}{\bullet} \qquad \begin{aligned} aa^* = 0, \quad b^*b = 0, \quad a^*a = bb^*,\\ ab = 0, \quad b^*a^* = 0.\end{aligned}$$

Note that I_w and Λ_w do not depend on the choice of the reduced expression. By [BIRS09, Proposition III.2.2], Λ_w is finite-dimensional and is Iwanaga–Gorenstein of dimension at most 1. In this case, the category of Gorenstein projective Λ_w-modules coincides with the category $\operatorname{Sub}\Lambda_w$ of submodules of finitely generated projective Λ_w-modules. By [BIRS09, Proposition III.2.3 and Theorem III.2.6], $\operatorname{Sub}\Lambda_w$ is a Hom-finite stably 2-Calabi–Yau Frobenius category and admits a cluster-tilting object M_w. These results were stated in [BIRS09] only for non-Dynkin quivers, but they also hold for Dynkin quivers.

Another family of Hom-finite stably 2-Calabi–Yau Frobenius categories with cluster-tilting object are constructed by Geiß, Leclerc and Schröer in [GLS07]. Specifically, for a terminal module M over KQ (i.e. M is preinjective and $\operatorname{add} M$ is closed under taking the inverse AR translation), consider $\mathcal{C}_M = \pi^{-1}(\operatorname{add} M) \subseteq \operatorname{nil}\Lambda$, where $\operatorname{nil}\Lambda$ is the category of finite-dimensional

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nilpotent representations over Λ and $\pi : \mathrm{nil}\,\Lambda \to \mathrm{mod}\, kQ$ is the restriction along the canonical embedding $KQ \to \Lambda$. Geiß *et al.* show that $\mathcal{C}_M$ admits the structure of a Frobenius category which is stably 2-Calabi–Yau with a cluster tilting object $T_M^\vee$. To M is naturally associated an element w of W. By comparing $T_M^\vee$ with M_w, they show that there is an anti-equivalence $\mathcal{C}_M \to \mathrm{Sub}\,\Lambda_w$ [GLS07, § 22.7].

We now explain how the results in this paper can be used to give a different proof of the equivalence $\mathcal{C}_M \cong \mathrm{Sub}\,\Lambda_w^{\mathrm{op}}$.

In [GLS07, § 8.1], an explicit construction of a projective generator I_M of the Frobenius category $\mathcal{C}_M$ is given. One can check that $\mathrm{End}_{\mathcal{C}_M}(I_M) \cong \Lambda_w^{\mathrm{op}}$. By [GLS07, Theorem 13.6(2)], $\mathrm{End}_{\mathcal{C}_M}(T_M^\vee)$ has global dimension 3. Since $T_M^\vee$ has I_M as a direct summand, it follows from Theorem 2.7 that

$$\mathcal{C}_M \cong \mathrm{GP}(\Lambda_w^{\mathrm{op}}),$$

and since $\mathrm{inj.dim}\,\Lambda_w^{\mathrm{op}} = 1$, we have $\mathrm{GP}(\Lambda_w^{\mathrm{op}}) = \mathrm{Sub}\,\Lambda_w^{\mathrm{op}}$. Thus $\mathcal{C}_M \cong \mathrm{Sub}\,\Lambda_w^{\mathrm{op}}$ follows.

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