A CLASS OF ALGEBRAS WITHOUT UNITY ELEMENT

R. M. THRALL

1. Introduction. In a study of the commuting algebra of tensor space representations of the orthogonal group W. P. Brown encountered a class of algebras for which the existence of a unity element was equivalent to semi-simplicity, but which were of interest whether or not semisimple. He gave these algebras the name *generalized-total matrix algebras* and proved **(2)** that each such algebra was characterized by three integers l, r, m and was isomorphic to the algebra of all square matrices of degree $r + l + m$ which have zeros in the first l rows and in the last r columns.

Let F be any field, let K be an extension sfield of finite degree k over F, let m be a positive integer and let l, r be non-negative integers. We denote by $C = C(K, m, l, r)$ the F-algebra of order $k(m + l)(m + r)$ consisting of all K-matrices having zeros in the first l rows and in the last l-columns. We call C a *submatrix algebra*.

In the present paper we introduce a new family of algebras called *algebras of class* Q. These algebras are defined in terms of certain simple properties possessed by submatrix algebras. Our main result is a proof that each algebra of class Q is a factor algebra of a direct sum of submatrix algebras. We also touch on the topics of automorphisms, isomorphisms, and representations, of algebras of class Q.

2. Algebras of class Q and class Q'. An F-algebra A (of finite dimension) is said to be of class Q if there exists an idempotent ϵ in A such that the following three conditions hold:

(Q_1) $B = \epsilon A \epsilon$ is semisimple,

(Q_2) $A\epsilon A = A$,

(Q_3) $A = B + N$, where N is the radical of A.

If instead of (Q_1) we have the stronger condition

(Q_1') $B = \epsilon A \epsilon$ is simple,

then we say that A is of *class* Q'.

It is easy to see that every submatrix algebra $C(K, m, l, r)$ is of class Q'; for we may take as the idempotent the matrix

$$\epsilon' = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{1}$$

where the partitioning of rows and columns is given by l, m, r.

Received October 12, 1954. Part of this research was carried out under a contract of the Office of Ordnance Research with the University of Michigan.

THEOREM 1. *Let A be an algebra of class Q. Then $ANA = 0$.*

Since no non-zero element in a semisimple algebra can be in the radical we note that $B \cap N = \{0\}$; hence the sum $B + N$ is direct (in the vector space sense). Now multiply A on left and right by ϵ and we get from (Q_3) that

$$\epsilon N \epsilon = 0. \tag{2}$$

Finally, we have

$$ANA = A\epsilon ANA\epsilon A \subseteq A\epsilon N\epsilon A = 0.$$

COROLLARY 1. *If A is of class Q then $N^3 = 0$ and $AN^2 = N^2A = 0$.*

THEOREM 2. *If A is of class Q then*

$$A = B + \epsilon N + N\epsilon + N^2 \quad \text{(direct sum)}. \tag{3}$$

We establish the theorem by identifying (3) with the Pierce decomposition of A relative to ϵ. By (Q_1), $B = \epsilon A \epsilon$ consists of all elements of A having ϵ as two-sided unity. Next, suppose that α is an element of A for which $\alpha = \epsilon\alpha$ and $\alpha\epsilon = 0$. According to (Q_3) we can write $\alpha = \beta + \eta$ where $\beta \in B$ and $\eta \in N$. Now since $\alpha = \epsilon\alpha$ we must have $\epsilon\eta = \eta$, and then $\alpha\epsilon = \beta + \epsilon\eta\epsilon = \beta$ requires $\beta = 0$. This shows that ϵN contains all elements of A having ϵ as left unity and right annihilator; moreover, it follows from (2) that each element of ϵN has this property. We show similarly that $N\epsilon$ consists of all elements of A having ϵ as right unity and left annihilator.

This shows that the Pierce decomposition is

$$A = B + \epsilon N + N\epsilon + N_0 \qquad \text{(direct sum)} \tag{4}$$

where N_0 consists of all elements of A having ϵ as two-sided annihilator. All that remains is to show that $N_0 = N^2$. It is a consequence of Corollary 1 that $N^2 \subseteq N_0$. Next, it follows from (Q_2), (4), and the fact that N is an ideal that

$$\begin{aligned} A = A \cdot A &= (B + N\epsilon)(B + \epsilon N) \\ &= B + N\epsilon + \epsilon N + N\epsilon N, \end{aligned} \tag{5}$$

where $N\epsilon N \subseteq N^2 \subseteq N_0$. Comparison of (4) and (5) then shows that $N\epsilon N = N_0$ and hence

$$N^2 = N\epsilon N = N_0.$$

3. Structure theory. In this section we show how a general algebra of class Q can be built up from algebras of class Q' and in the following section we study the structure of algebras of class Q'.

THEOREM 3. *Let A, A_1, and A_2 be algebras of class Q and let M be an ideal (two-sided) in A. Then the direct sum $A_1 \oplus A_2$ and the residue class algebra $A - M$ are both of class Q.*

If ϵ, ϵ_1, ϵ_2 are the postulated idempotents in A, A_1, A_2, respectively, then (ϵ_1, ϵ_2) and $\epsilon + M$, respectively, are idempotents which satisfy (Q_1) and (Q_2) in $A_1 + A_2$ and $A - M$. It is also easy to check (Q_3) in both cases.

THEOREM 4. *Let A be an algebra of class Q. Then $A = A^* - M^*$ where A^* is a direct sum of algebras of class Q' and M^* is contained in the square of the radical of A^*.*

It follows from Theorem 3 that $A^* - M^*$ is of class Q if A^* and M^* satisfy the conditions of Theorem 4.

Let $A = B + N$ where B and N are related to A as in (Q_1), (Q_2), (Q_3), and suppose that

(6) $$B = B_1 + \ldots + B_p$$

is the (unique) expression of B as a direct sum of simple subalgebras. Let ϵ_i be the unity element of B_i $(i = 1, \ldots, p)$; then

(7) $$\epsilon = \epsilon_1 + \ldots + \epsilon_p$$

is an expression of ϵ as a sum of orthogonal idempotents in the centre of B.

Now set

(8) $$A_i = A\epsilon_i A \qquad (i = 1, \ldots, p).$$

and

(9) $$N_i = A_i \cap N \qquad (i = 1, \ldots, p).$$

LEMMA 1. *A_i is of class Q' with idempotent ϵ_i, simple summand B_i, and radical N_i.*

The equation

$$\epsilon_i A_i \epsilon_i = \epsilon_i(A\epsilon_i A)\epsilon_i = \epsilon_i \epsilon A \epsilon\epsilon_i \epsilon A \epsilon\epsilon_i = \epsilon_i B \epsilon_i B \epsilon_i = B_i B_i = B_i$$

verifies (Q_1').

Next, we have

$$A_i \supseteq A_i \epsilon_i A_i = A\epsilon_i A\epsilon_i A\epsilon_i A \supseteq A\epsilon_i A = A_i,$$

and hence $A_i = A_i \epsilon_i A_i$. This verifies (Q_2).

Finally,

$$A_i = (B + N)\epsilon_i(B + N) = B\epsilon_i B + (N\epsilon_i B + N\epsilon_i N + B\epsilon_i N) \subseteq B_i + N_i$$

since A_i and N_i are ideals in A. But $B_i \subseteq A_i$, $N_i \subseteq A_i$, hence

(10) $$A_i = B_i + N_i.$$

This sum is direct (in the vector space sense) since ϵ_i is unity element for B_i and $\epsilon_i N_i \epsilon_i \subseteq \epsilon_i N \epsilon_i = \epsilon_i \epsilon N \epsilon \epsilon_i = 0$; it follows that $A_i - N_i \cong B_i$, and hence the N_i is the radical of A_i. This establishes (Q_3).

LEMMA 2. *$A_i A_j = 0$ if $i \neq j$.*

If $i \neq j$ we have

$$\begin{aligned} \epsilon_i A \epsilon_j &= \epsilon_i B \epsilon_j + \epsilon_i N \epsilon_j \\ &= B \epsilon_i \epsilon_j + \epsilon_i \epsilon N \epsilon \epsilon_j \\ &= 0 + 0; \end{aligned}$$

hence $A_i A_j = A\epsilon_i AA\epsilon_j A = 0$.

https://doi.org/10.4153/CJM-1955-041-0 Published online by Cambridge University Press

We are now ready to prove Theorem 4. Let A^* be the (ring) direct sum of $A_1, \ldots, A_p$, i.e., A^* consists of all p-tuples $\alpha^* = (\alpha_1, \ldots, \alpha_p)$ with α_i in A_i and with addition and multiplication done componentwise. Consider the mapping

(11) $$T : \alpha^* = (\alpha_1, \ldots, \alpha_p) \rightarrow \alpha = \alpha_1 + \ldots + \alpha_p.$$

of A^* into A.

This mapping is clearly a linear transformation. It follows from Lemma 2 that it is a ring homomorphism. It is "onto" since

$$A^*T = A_1 + \ldots + A_p = A\epsilon_1 A + \ldots + A\epsilon_p A = A\epsilon A = A.$$

Let M^* be the kernel of T, and let N^* be the radical of A^*. All that remains is to show that $M^* \subseteq (N^*)^2$. Suppose that $\alpha^* = (\alpha_1, \ldots, \alpha_p)$ lies in M^*;

(12) $$\alpha = \alpha_1 + \ldots + \alpha_h = 0.$$

It follows from (3) that we can write

(13) $$\alpha_i = \beta_i + \eta_i + \zeta_i + \tau_i \qquad (i = 1, \ldots, p)$$

where $\beta_i = \epsilon_i \alpha_i \epsilon_i$, $\eta_i = \epsilon_i \eta_i$, $\zeta_i = \zeta_i \epsilon_i$ and $\eta_i \epsilon_i = \epsilon_i \zeta_i = \epsilon_i \tau_i = \tau_i \epsilon_i = 0$.

Now $\epsilon_i \alpha \epsilon_i = \beta_i$, $\epsilon_i \alpha = \beta_i + \eta_i$, $\alpha\epsilon_i = \beta_i + \zeta_i$, and since $\alpha = 0$ this gives $\beta_i = \eta_i = \zeta_i = 0$ $(i = 1, \ldots, p)$ and hence

$$\alpha^* = (\tau_1, \ldots, \tau_p)$$

which is in $(N^*)^2$.

4. Structure theory (continued). Theorem 4 gives the structure of algebras of class Q in terms of algebras of class Q'. In this section we refine this result by an analysis of algebras of class Q'.

THEOREM 5. *Every algebra of class Q' is the homomorphic image of a submatrix algebra. The kernel of the homomorphism is contained in the square of the radical of the submatrix algebra.*

Let A be an algebra of class Q' with simple summand $B = \epsilon A \epsilon$. According to Wedderburn's Theorem **(1)**, B is isomorphic to a total matrix algebra over a finite extension sfield K of F. Let $\{\kappa_1, \ldots, \kappa_k\}$ be a basis for K over F, let m be the degree of B over K, and let e_{ij} $(i, j = 1, \ldots, m)$ be a matrix unit K-basis for B.

Let $e = e_{11}$; then K is isomorphic to $K' = eBe$ and every irreducible left-B-space W is isomorphic to eB. In particular, $eW \neq 0$ and there exists a vector w in W such that $\{w(= e_{11} w), e_{21} w, \ldots, e_{m1} w\}$ is a K-basis for W. Then the mk vectors $\kappa_h e_{i1} w$ $(h = 1, \ldots, k; i = 1, \ldots, m)$ form an F-basis for W.

ϵN is a left B-space and as such is a direct sum of irreducible left B-spaces. Hence, there exist vectors $\eta_1, \ldots, \eta_l$ in eN such that

(14) $$\{\kappa_h e_{i1} \eta_s \ (h = 1, \ldots, k; i = 1, \ldots, m; s = 1, \ldots, l)\}$$

is an F-basis for ϵN.

https://doi.org/10.4153/CJM-1955-041-0 Published online by Cambridge University Press

Similarly, there exist vectors $\zeta_1, \ldots, \zeta_r$ in Ne such that

(15) $$\{\zeta_t e_{ij} \kappa_h \quad (h = 1, \ldots, k; j = 1, \ldots, m; t = 1, \ldots, r)\}$$

is an F-basis for $N\epsilon$.

Finally, $N^2 = N\epsilon N$ is spanned by all products of basis vectors for $N\epsilon$ and ϵN; i.e. by all products of the form

$$\zeta_t e_{1j} \kappa_h \kappa_{h'} e_{i1} \eta_s.$$

Next, we observe that the matrix units e_{ij} commute with elements of K, hence $e_{1j} \kappa_h \kappa_{h'} e_{i1}$ is equal to zero unless $i = j$ and then it is equal to $e \kappa_h \kappa_{h'} e$. Now $\zeta_t e = \zeta_t$ and $e\eta_s = \eta_s$; hence N^2 is spanned by the klr products

(16) $$\{\zeta_t \kappa_h \eta_s \quad (h = 1, \ldots, k; s = 1, \ldots, l; t = 1, \ldots, r)\}.$$

In general these products will not be independent but will satisfy certain linear conditions

(17) $$\sum_{t,h,s} a_q(t, h, s) \zeta_t \kappa_h \eta_s = 0 \qquad (q = 1, \ldots, g).$$

We may suppose that these conditions are independent; then the F-dimension of N^2 is $klr - g$.

We now let $C = C(K, m, l, r)$. Let ϵ' be the matrix given in (1), and let $C = B' + N'$ be the decomposition (Q_3) for C. Then we can choose matrix units e_{ij}' for B' and elements $\eta_1', \ldots, \eta_l'$ in $e_{11}' N'$, $\zeta_1', \ldots, \zeta_r'$ in $N' e_{11}'$ such that the $k(m + l)(m + r)$ elements

(18) $$\kappa_h e'_{ij}, \; \kappa_h e'_{i1}\eta'_s, \; \zeta'_t e'_{1j}\kappa_h, \; \zeta'_t \kappa_h \eta'_s$$

$$(h = 1, \ldots, k; i, j = 1, \ldots, m; s = 1, \ldots, l; t = 1, \ldots, r)$$

form an F-basis for C. Then the unique linear transformation of C onto A which sends each of these basis vectors into the corresponding unprimed element of A is clearly a ring homomorphism whose kernel lies in $(N')^2$. This completes the proof of Theorem 5.

Next we combine the results of Theorems 4 and 5 and get our main structure theorem.

THEOREM 6. *Let A be an algebra of class Q. Then $A = A^* - M^*$ where A^* is a direct sum of submatrix algebras and M^* is contained in the square of the radical of A^*.*

5. Representation theory. Let A be an algebra of class Q and let V be a (left) representation space for A. Consider the chain $V \supseteq AV \supseteq NAV \supseteq ANAV = 0$. Clearly both the spaces V/AV and NAV are annihilated by every element of A, and AV/NAV is a completely reducible non-degenerate A-space. Hence, by suitable choice of basis vectors we get the following matrix form for the representation:

https://doi.org/10.4153/CJM-1955-041-0 Published online by Cambridge University Press

$$\alpha \to V_{(\alpha)} = \begin{Vmatrix} 0 & 0 & 0 \\ V_{21}(\alpha) & V_{22}(\alpha) & 0 \\ V_{31}(\alpha) & V_{32}(\alpha) & 0 \end{Vmatrix} \tag{19}$$

where V_{22} is in completely reduced form. We may suppose the basis elements so chosen that $V_{22}(\epsilon)$ is the identity matrix, and if $\alpha = \beta + \eta + \zeta + \tau$ is a splitting of α according to (3) then

$$V_{22}(\alpha) = V_{22}(\beta),\ V_{21}(\alpha) = V_{21}(\eta),\ V_{32}(\alpha) = V_{32}(\zeta),$$

and

$$V_{31}(\alpha) = V_{31}(\tau).$$

It is easy to show that if any of the integers l_i, r_i defined by the ideals A_i of A given by (8) exceeds unity, then A has unbounded representation type **(3)**. Consequently, it is not likely that there is any simple classification of the indecomposable representations of algebras of class Q.

6. Uniqueness and automorphisms. Since the structure theorems depend on the decomposition $A = B + N$ it seems desirable to study its uniqueness. Since N is the radical any lack of uniqueness must come from the semisimple summand B. But since $B = \epsilon A \epsilon$ is uniquely determined by ϵ any second decomposition must correspond to a second idempotent ϵ'.

It is easy to verify for any η_0, ζ_0 in ϵN, $N\epsilon$, respectively, that

$$\epsilon' = \epsilon + \eta_0 + \zeta_0 + \zeta_0 \eta_0 = (\epsilon + \zeta_0)(\epsilon + \eta_0) \tag{20}$$

is an idempotent for which Q_1, Q_2, and Q_3 hold. Moreover, if either η_0 or ζ_0 is different from zero, $B' = \epsilon' A \epsilon'$ is not the same as B. Hence all we can expect for B is uniqueness to within an automorphism and this is established in the following theorem.

THEOREM 7. *Let A be an algebra of class Q and let ϵ, ϵ' be idempotents for which Q_1, Q_2, Q_3 hold. Then there is an automorphism T of A which sends ϵ into ϵ'. More precisely, if $\alpha = \beta + \eta + \zeta + \tau$ is a splitting of α according to* (3) *then the mapping*

$$T : \alpha \to \alpha T = \alpha' = \epsilon' \beta \epsilon' + \epsilon' \eta + \zeta \epsilon' + \tau$$

is an automorphism of A which sends ϵ into ϵ'.

Both ϵ and ϵ' are mapped into the identity element of $A - N$ under the natural mapping; hence $\epsilon' - \epsilon$ is in N and so from (3) we get

$$\epsilon' = \epsilon + \eta_0 + \zeta_0 + \tau_0.$$

Now $\epsilon' = (\epsilon')^2 = \epsilon + \eta_0 + \zeta_0 + \zeta_0 \eta_0$; i.e., ϵ' has the form (20). Note in particular that $\epsilon\epsilon'\epsilon = \epsilon$.

Next, we have

$$\begin{aligned} \epsilon T = \epsilon' \epsilon \epsilon' &= (\epsilon + \zeta_0)(\epsilon + \eta_0)\epsilon(\epsilon + \zeta_0)(\epsilon + \eta_0) \\ &= (\epsilon + \zeta_0)(\epsilon + \eta_0) = \epsilon'. \end{aligned}$$

Clearly T is a linear transformation and a direct computation shows that $(\alpha_1 \alpha_2)T = (\alpha_1 T)(\alpha_2 T)$; i.e., T is an endomorphism on A.

To complete the proof that T is an automorphism, i.e. that it is one-to-one and onto we construct its inverse. Let $\alpha = \beta' + \eta' + \zeta' + \tau$ be the splitting (3) for α relative to ϵ' and let $\alpha T' = \epsilon\beta'\epsilon + \epsilon\eta' + \zeta'\epsilon + \tau$. Then the equations $TT' = T'T = I$ follow from $\epsilon\epsilon'\epsilon = \epsilon$ and $\epsilon'\epsilon\epsilon' = \epsilon'$.

The automorphism T of the theorem is completely defined by ϵ' and hence by η_0 and ζ_0; we denote it by $T(\eta_0, \zeta_0)$. It is easy to verify that the set W of all $T(\eta, \zeta)$ is a commutative group with composition rule

$$T(\eta_1, \zeta_1) \cdot T(\eta_2, \zeta_2) = T(\eta_1 + \eta_2, \zeta_1 + \zeta_2). \tag{21}$$

Let U_ϵ denote the subgroup consisting of all automorphisms of A which leave ϵ fixed; then the group G of all automorphisms has the factorization $U_\epsilon W$.

Let γ, γ' be elements of B for which $\gamma\gamma' = \gamma'\gamma = \epsilon$. Then the mapping $S(\gamma)$ given by

$$\alpha = \beta + \eta + \zeta + \tau \rightarrow \alpha S(\gamma) = \gamma'\beta\gamma + \gamma'\eta + \zeta\gamma + \tau \tag{22}$$

is an automorphism of A. The set V of all $S(\gamma)$ is an subgroup of U. We observe that N^2 is elementwise fixed under the automorphisms in V and in W.

According to Theorem 6, the general question of conditions for isomorphism of algebras of class Q can be reduced to the study of conjugacy (under automorphisms) of ideals contained in the square of the radical of a direct sum of submatrix algebras.

In this paper we shall limit our study of isomorphism to the case of algebras of class Q' and in particular those for which $K = F$, i.e. for which B is a total matrix algebra. According to Theorem 5 we can reduce this to the following question. Let $C = C(F, m, l, r)$, and let M, M' be two ideals in the square of the radical of C. We ask for necessary and sufficient conditions for the conjugacy of M and M' under automorphisms of C. We first determine the group G of automorphisms of C which leave F elementwise fixed.

We have initially the factorization $G = U_\epsilon W$. Let U_B denote the subgroup of G consisting of automorphisms which leave B elementwise fixed and let T be any element of U_ϵ. Then since B is a total matrix algebra T must agree on B with an inner automorphism of B, i.e., there exists γ in B such that $T(\gamma)^{-1}$ leaves B elementwise fixed **(1)**. This shows that $U_\epsilon = U_B V$. Now since W and V both leave N^2 elementwise fixed, M and M' will be conjugate under G if and only if they are conjugate under U_B.

THEOREM 8. *Let G be the group of automorphisms of a submatrix F-algebra $C = C(F, m, l, r)$. Then G has the factorization*

$$G = U_B VW$$

where W is isomorphic to a vector space of dimension m^2lr over F, where V is isomorphic to the full linear group $GL(m)$, and where U_B is isomorphic to the direct product of $GL(l)$ and $GL(r)$.

The statement about W follows from (21) and the fact that for C we have $\dim \epsilon N = lm$ and $\dim N\epsilon = rm$. The statement about V follows from (22) since $GL(m)$ is the group of inner automorphisms of a total matrix algebra of degree m.

Next, let T be any element of U_B, and choose e as in the proof of Theorem 5. Then, since e is in B, $eT = e$ and hence $(eN)T = eN$ and $(Ne)T = Ne$; moreover, T induces non-singular linear transformations T_L on eN and T_R on Ne. Let τ be an element of N^2. Then we have (cf. (16))

$$\tau = \sum_{t,s} a_{ts}\zeta_t\eta_s. \tag{23}$$

(The factor κ_h appearing in (16) does not appear here since $K = F$.) Now,

$$\tau T = \sum_{t,s} a_{ts}(\zeta_t T_R)(\eta_s T_L); \tag{24}$$

hence T is completely determined by T_L and T_R. It follows that the mapping $T \to (T_L, T_R)$ is a homomorphism of U_B into $GL(l) \times GL(r)$. Moreover, it follows from (24) that if $T \to (I_l, I_r)$, then $T = I$, i.e. this mapping is an isomorphism. Finally we show that the mapping is onto. Let T_L, T_R be any elements of $GL(l)$ $GL(r)$ respectively. Then the mapping $T = T(T_L, T_R)$ defined by

$$(\beta + \eta + \zeta + \tau)T = \beta + \eta T_L + \zeta T_R + \tau', \tag{25}$$

where τ' is given by (24), is clearly an element of U_B which maps into (T_L, T_R).

Now let M be an ideal of C contained in N^2, and let M have dimension g over F. Then (cf. (17)) M has a basis of the form

$$\{\tau_q = \sum_{t,s} a_q(t,s)\,\zeta_t\eta_s\ (q = 1, \ldots, g)\} \tag{26}$$

where the $a_q(t, s)$ are elements of F. We can associate M and the given basis with the trilinear form

$$f_M : f_M(x, y, z) = \sum_{q,t,s} a_q(t,s)\, x_q y_s z_t.$$

A change of basis for M replaces f_M according to a non-singular linear transformation on the x_q. Under an automorphism $T(T_L, T_R)$, M is replaced by a new ideal M' whose corresponding trilinear form is obtained from f_M by applying the substitutions T_L to the y_s and T_R to the z_t. Thus we see that *two ideals M and M' are conjugate under U_B and therefore under G if and only if their corresponding trilinear forms f and f' are equivalent.* Thus the problem of isomorphism of two algebras of class Q' (having $K = F$) is reduced to the equivalence of trilinear forms.

To extend this result to the case where $K \neq F$ would involve equivalence of quadrilinear forms under the full linear group on three of the sets of variables and under a finite group corresponding to automorphisms of K over F on the fourth set of variables. If the centre of K is inseparable over F, then Theorem 8 still remains valid except that U_B must be enlarged to account for automorphisms of K. The factorization $G = U_B VW$ is no longer valid (unless the centre of K is inseparable over F). We leave the detailed analysis of this case as well as the general problem of isomorphism of algebras of class Q for future treatment.

References

1. E. Artin, C. J. Nesbitt, and R. M. Thrall, *Rings with minimum condition* (University of Michigan Press, 1944).

2. W. P. Brown, *Generalized matrix algebras*, Can. J. Math., 7 (1955), 188–190.

3. J. P. Jans, *On the indecomposable representations of algebras*. 2200–5–T Engineering Research Institute, University of Michigan.

University of Michigan