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ON COMPLETELY SINGULAR VON NEUMANN SUBALGEBRAS

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Abstract Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. If $\mathcal{N}\bar{\otimes}\mathcal{B}(\mathcal{K})$ is singular in $\mathcal{M}\bar{\otimes}\mathcal{B}(\mathcal{K})$ for every Hilbert space $\mathcal{K}$, $\mathcal{N}$ is said to be *completely singular* in $\mathcal{M}$. We prove that if $\mathcal{N}$ is a singular abelian von Neumann subalgebra or if $\mathcal{N}$ is a singular subfactor of a type-II_1 factor $\mathcal{M}$, then $\mathcal{N}$ is completely singular in $\mathcal{M}$. If $\mathcal{H}$ is separable, we prove that $\mathcal{N}$ is completely singular in $\mathcal{M}$ if and only if, for every $\theta\in\mathrm{Aut}(\mathcal{N}')$ such that $\theta(X)=X$ for all $X\in\mathcal{M}'$, $\theta(Y)=Y$ for all $Y\in\mathcal{N}'$. As the first application, we prove that if $\mathcal{M}$ is separable (with separable predual) and $\mathcal{N}$ is completely singular in $\mathcal{M}$, then $\mathcal{N}\bar{\otimes}\mathcal{L}$ is completely singular in $\mathcal{M}\bar{\otimes}\mathcal{L}$ for every separable von Neumann algebra $\mathcal{L}$. As the second application, we prove that if $\mathcal{N}_1$ is a singular subfactor of a type-II_1 factor $\mathcal{M}_1$ and $\mathcal{N}_2$ is a completely singular von Neumann subalgebra of $\mathcal{M}_2$, then $\mathcal{N}_1\bar{\otimes}\mathcal{N}_2$ is completely singular in $\mathcal{M}_1\bar{\otimes}\mathcal{M}_2$.

Keywords: singular von Neumann subalgebras; completely singular von Neumann subalgebras; tensor products of von Neumann algebras

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1. Introduction

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ is *singular* if the only unitary operators in $\mathcal{M}$ satisfying the condition $U\mathcal{N}U^*=\mathcal{N}$ are those in $\mathcal{N}$. The study of singular von Neumann subalgebras has a long and rich history (see, for example, [**1**, **6**, **8**, **9**, **11**]). Recently, there has been remarkable progress on singular maximal abelian von Neumann subalgebras (masas) in type-II_1 factors (see [**12–14**]). In [**13**], Sinclair and Smith introduced the concept of the asymptotic homomorphism property. In [**12**], the concept of the weak asymptotic homomorphism property is introduced. Let $\mathcal{M}$ be a type-II_1 factor and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then $\mathcal{N}\subseteq\mathcal{M}$ has the weak asymptotic homomorphism property if, for all $X_1,\dots,X_n\in\mathcal{M}$ and $\varepsilon>0$, there exists a unitary operator $U\in\mathcal{N}$ such that

$$\|E_{\mathcal{N}}(X_iUX_j^*)-E_{\mathcal{N}}(E_{\mathcal{N}}(X_i)UE_{\mathcal{N}}(X_j)^*)\|_2<\varepsilon.$$

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Remarkably, in [**14**], it was shown that every singular masa in a type-II_1 factor satisfies the weak asymptotic homomorphism property. As a corollary, the tensor product of singular masas in type-II_1 factors is proved to be a singular masa in the tensor product of type-II_1 factors (see [**14**]), which was a long-standing problem.

It is very natural to ask the following question: if $\mathcal{N}_1$ and $\mathcal{N}_2$ are singular von Neumann subalgebras of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, is $\mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{N}_2$ singular in $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{M}_2$? It turns out that this is not always true. Let $\mathcal{M}_1 = M_3(\mathbb{C})$ and $\mathcal{N}_1 = M_2(\mathbb{C}) \oplus \mathbb{C}$. Then

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are central projections in $\mathcal{N}_1$ and $\mathcal{N}_1 = \{P, Q\}'$. Suppose that $U \in \mathcal{M}_1$ is a unitary matrix such that $U\mathcal{N}_1U^* = \mathcal{N}_1$. Then $UPU^* = P$ and $UQU^* = Q$ (because the automorphism $\theta(X) = UXU^*$ of $\mathcal{N}_1$ preserves the centre of $\mathcal{N}_1$ and $\tau(P) = \frac{2}{3}$, $\tau(Q) = \frac{1}{3}$, where τ is the normalized trace on $M_3(\mathbb{C})$). So $U \in \{P, Q\}' = \mathcal{N}_1$. This implies that $\mathcal{N}_1$ is singular in $\mathcal{M}_1$. Let $\mathcal{M}_2 = \mathcal{B}(l^2(\mathbb{N}))$ and $\mathcal{N}_2 = \mathcal{M}_2$. Then $\mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N})) = M_2(\mathbb{C}) \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N})) \oplus \mathbb{C} \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$ is not singular in $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N})) = M_3(\mathbb{C}) \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$. Indeed, let V be an isometry from $l^2(\mathbb{N})$ onto $\mathbb{C}^2 \otimes l^2(\mathbb{N})$. Then

$$U = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$$

is a unitary operator in $M_3(\mathbb{C}) \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$ such that

$$U(\mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N})))U^* = \mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N})).$$

Since U is not in $\mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$, $\mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$ is not singular in $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$. Indeed, $\mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$ is regular in $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$ (see Remark 2.14).

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. If $\mathcal{N} \mathbin{\bar{\otimes}} \mathcal{B}(\mathcal{K})$ is singular in $\mathcal{M} \mathbin{\bar{\otimes}} \mathcal{B}(\mathcal{K})$ for every Hilbert space $\mathcal{K}$, then $\mathcal{N}$ is said to be *completely singular* in $\mathcal{M}$. In § 2, we prove that if $\mathcal{N}$ is a singular masa or if $\mathcal{N}$ is a singular subfactor of a type-II_1 factor $\mathcal{M}$, then $\mathcal{N}$ is completely singular in $\mathcal{M}$. For every type-II_1 factor $\mathcal{M}$, we construct a singular von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ ($\mathcal{N} \neq \mathcal{M}$) such that $\mathcal{N} \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$ is regular in $\mathcal{M} \mathbin{\bar{\otimes}} \mathcal{B}(l^2(\mathbb{N}))$. Motivated by [**3**, Lemma 1.2], we obtain a nice characterization of complete singularity in § 3. As the first application, in § 4.1, we prove that if $\mathcal{M}$ is separable and $\mathcal{N}$ is completely singular in $\mathcal{M}$, then $\mathcal{N} \mathbin{\bar{\otimes}} \mathcal{L}$ is completely singular in $\mathcal{M} \mathbin{\bar{\otimes}} \mathcal{L}$ for every separable von Neumann algebra $\mathcal{L}$. As the second application, we prove that if $\mathcal{N}_1$ is a singular subfactor of a type-II_1 factor $\mathcal{M}_1$ and $\mathcal{N}_2$ is a completely singular von Neumann subalgebra of $\mathcal{M}_2$, then $\mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{N}_2$ is singular in $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{M}_2$. The following question seems to be interesting: if $\mathcal{N}_1$, $\mathcal{N}_2$ are completely singular von Neumann subalgebras of $\mathcal{M}_1$ and $\mathcal{M}_2$, is $\mathcal{N}_1 \mathbin{\bar{\otimes}} \mathcal{N}_2$ completely singular in $\mathcal{M}_1 \mathbin{\bar{\otimes}} \mathcal{M}_2$?

2. On singularity and complete singularity

2.1. Normalizer and normalizing groupoid of $\mathcal{N}$ in $\mathcal{M}$

Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ denotes the normalizer of $\mathcal{N}$ in $\mathcal{M}$:

$$\mathfrak{N}_{\mathcal{M}}(\mathcal{N}) = \{U \in \mathcal{M} : U \text{ is a unitary operator, } U\mathcal{N}U^* = \mathcal{N}\},$$

and $\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$ denotes the (two-sided) normalizing groupoid of $\mathcal{N}$ in $\mathcal{M}$:

$$\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N}) = \{V \in \mathcal{M} : V \text{ is a partial isometry with initial space } E \text{ and final space } F \text{ such that } E, F \in \mathcal{N} \text{ and } V\mathcal{N}_E V^* = \mathcal{N}_F\},$$

where $\mathcal{N}_E = E\mathcal{N}E$ and $\mathcal{N}_F = F\mathcal{N}F$. $\mathcal{N}$ is singular in $\mathcal{M}$ if and only if $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$, the von Neumann algebra generated by $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})$, is $\mathcal{N}$. Recall that $\mathcal{N}$ is *regular* in $\mathcal{M}$ if $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{M}$.

If $\mathcal{M}$ is a finite von Neumann algebra and $\mathcal{N}$ is a maximal abelian von Neumann subalgebra of $\mathcal{M}$, then $V \in \mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$ if and only if there is a unitary operator $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ and a projection $E \in \mathcal{N}$ such that $V = UE$ [**6**, Theorem 2.1]. In other words, any partial isometry that normalizes $\mathcal{N}$ extends to a unitary operator that normalizes $\mathcal{N}$. As a corollary, we have $\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$, i.e. the von Neumann algebra generated by the normalizing groupoid of $\mathcal{N}$ in $\mathcal{M}$ is the von Neumann algebra generated by the normalizer of $\mathcal{N}$ in $\mathcal{M}$. If $\mathcal{M}$ is an infinite factor, e.g. type-III, and $\mathcal{N} = \mathcal{M}$, then there is an isometry in $\mathcal{M}$ which cannot be extended to a unitary operator in $\mathcal{M}$. The following example tells us that even the weak form $\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$ can fail. Let $\mathcal{M} = M_3(\mathbb{C})$ and $\mathcal{N} = M_2(\mathbb{C}) \oplus \mathbb{C}$. As we pointed out in § 1, $\mathcal{N}$ is singular in $\mathcal{M}$, i.e. $\mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{N}$. Simple computations show that

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is in $\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$. Note that V is not in $\mathcal{N}$.

Let $V_1, V_2 \in \mathcal{M}$ be two partial isometries in $\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$ and $E_i = V_i^* V_i \in \mathcal{N}$, $i = 1, 2$. We say $V_1 \preceq V_2$ if $E_1 \leqslant E_2$ and $V_1 = V_2 E_1$. It is obvious that $\preceq$ is a partial order on the set of partial isometries in $\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$. Let $\{V_\alpha\}$ be a totally ordered subset of $\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$. Then $V = \lim_\alpha V_\alpha$ (in the strong operator topology) exists and $V \in \mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$.

Lemma 2.1. *If $\mathcal{M}$ is a finite von Neumann algebra and $\mathcal{N}$ is a subfactor of $\mathcal{M}$, then for every $V \in \mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$ there is a unitary operator $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ such that $V \preceq U$. In particular, $\mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$.*

Proof. By Zorn's lemma, there is a maximal element $W \in \mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$ such that $V \preceq W$. Let $E = W^*W$ and $F = WW^*$. Then $E, F \neq 0$ and $E, F \in \mathcal{N}$. We need to prove $E = I$. If $E \neq I$, then $F \neq I$ since $\mathcal{M}$ is finite. So $I - E, I - F \in \mathcal{N}$ are not 0.

Since $\mathcal{N}$ is a factor, there is a partial isometry $V_1 \in \mathcal{N}$ with initial space E_1, a non-zero subprojection of $I - E$, and final space E_2, a non-zero subprojection of E. Let F' be the range space of WE_2. Then $F' = WE_2W^* \in \mathcal{N}$. Since $\mathcal{N}$ is a factor, there is a partial isometry $V_2 \in \mathcal{N}$ with initial space F_2, a non-zero subprojection of F', and final space F_1, a non-zero subprojection of $I - F$. Now $W' = V_2WV_1$ is a partial isometry with initial space $E_1 \leqslant I - E$ and final space $F_1 \leqslant I - F$. Simple computation shows that $W + W' \in \mathfrak{GN}^{(2)}_{\mathcal{M}}(\mathcal{N})$. Note that $V \preceq W \preceq W + W'$ and $W \neq W + W'$. This contradicts the maximality of W. □

Lemma 2.2. *Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{N}$ be an abelian von Neumann subalgebra of $\mathcal{M}$. Then $\mathfrak{GN}^{(2)}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$.*

Proof. Let $\mathcal{M}_1 = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$. We only need to prove that $\mathfrak{GN}^{(2)}_{\mathcal{M}}(\mathcal{N})'' \subseteq \mathcal{M}_1$. For $V \in \mathcal{M}$ a partial isometry, define $\mathcal{S}(V) = \{W \in \mathcal{M}_1 : W$ is a partial isometry and $W \preceq V\}$. Suppose $V \notin \mathcal{M}_1$. By Zorn's lemma, we can choose a maximal element $W \in \mathcal{S}(V)$ such that $V - W \neq 0$ and $\mathcal{S}(V - W) = \{0\}$. Since $W \in \mathcal{M}_1$, $V \in \mathcal{M}_1$ if and only if $V - W \in \mathcal{M}_1$. Therefore, we can assume that $V \neq 0$ and $\mathcal{S}(V) = \{0\}$. Let $E = V^*V$ and $F = VV^*$. Then $E \neq 0$ and $F \neq 0$.

If $E = F$, let $U = V + (I - E)$. Then $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ and $V = UE \in \mathcal{M}_1$. This is a contradiction. If $E \neq F$, we can assume that $E_1 = E(I - F) \neq 0$ (otherwise consider V^*). Let $V_1 = VE_1$ and F_1 be the final space of V_1. Then $V_1 \in \mathfrak{GN}^{(2)}_{\mathcal{M}}(\mathcal{N})$ with initial space $E_1 \leqslant I - F$ and final space $F_1 \leqslant F$ such that $0 \neq V_1 \preceq V$. Let $U = V_1 + V_1^* + (I - E_1 - F_1)$. Then $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ and $V_1 = UE_1 \in \mathcal{M}_1$. Note that $V_1 \neq 0$ and $V_1 \preceq V$. $\mathcal{S}(V) \neq \{0\}$. This is a contradiction. □

If $\mathcal{N}$ is singular in $\mathcal{M}$ and $E \in \mathcal{N}$ is a projection, $\mathcal{N}_E$ ($\mathcal{N}_E = E\mathcal{N}E$) may be not singular in $\mathcal{M}_E$. For example, let $\mathcal{M} = M_3(\mathbb{C})$, $\mathcal{N} = M_2(\mathbb{C}) \oplus \mathbb{C}$ and

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{N}.$$

Then $\mathcal{N}_E$ is not singular in $\mathcal{M}_E$. But we have the following result.

Lemma 2.3. *Let $\mathcal{N}$ be a singular von Neumann subalgebra of $\mathcal{M}$ and $E \in \mathcal{N}$ be a projection. If $\mathcal{N}$ is a countably decomposable, properly infinite von Neumann algebra, then $\mathcal{N}_E$ is singular in $\mathcal{M}_E$.*

Proof. Let P be the central support of E relative to $\mathcal{N}$. Then there are central projections P_1, P_2 of $\mathcal{N}$ such that $P_1 + P_2 = P$ and P_1E is finite and P_2E is properly infinite. Let $E_1 = P_1E$ and $E_2 = P_2E$. Then the central supports of E_1 and E_2 are P_1 and P_2, respectively. Since P_1 is a properly infinite countably decomposable projection and E_1 is a finite projection in $\mathcal{N}_{P_1}$ and the central support of E_1 is P_1, it follows that P_1 is a countably infinite sum of projections $\{E_{1n}\}$ in $\mathcal{N}$, each E_{1n} is equivalent to E_1 in $\mathcal{N}_{P_1}$(see, for example, [**7**, Corollary 6.3.12]). For $n \in \mathbb{N}$, let W_{1n} be a partial isometry in $\mathcal{N}_{P_1}$ such that $W_{1n}^*W_{1n} = E_{1n}$ and $W_{1n}W_{1n}^* = E_1$. Since P_2 and E_2 are properly infinite projections in $\mathcal{N}_{P_2}$ with the same central support P_2 and since $\mathcal{N}_{P_2}$ is countably decomposable, P_2 and E_2 are equivalent in $\mathcal{N}_{P_2}$ (see, for example, [**7**, Corollary 6.3.5]). Since P_2 is properly infinite in $\mathcal{N}$, it can be decomposed into a countably infinite sum of projections $\{E_{2n}\}$, where each E_{2n} is equivalent to P_2 and hence to E_2. For $n \in \mathbb{N}$, let W_{2n} be a partial isometry in $\mathcal{N}_{P_2}$ such that $W_{2n}^* W_{2n} = E_{2n}$ and $W_{2n} W_{2n}^* = E_2$. Let $W_n = W_{1n} + W_{2n} \in \mathcal{N}$. Then $W_n^* W_n = E_{1n} + E_{2n}$ and $W_n W_n^* = E_1 + E_2 = E$.

Suppose V is a unitary operator in $\mathcal{M}_E$ such that $V \mathcal{N}_E V^* = \mathcal{N}_E$. Define

$$U = \sum_{n=1}^{\infty} W_n^* V W_n + (I - P_1 - P_2).$$

Then U is a unitary operator and $U^* = \sum_{n=1}^{\infty} W_n^* V^* W_n + (I - P_1 - P_2)$. For any $T \in \mathcal{N}$, $UTU^* = \sum_{m,n=1}^{\infty} W_m^* V W_m T W_n^* V^* W_n + (I - P_1 - P_2)T$. Note that $W_m T W_n^* \in \mathcal{N}_E$, $V W_m T W_n^* V^* \in \mathcal{N}_E$. So $UTU^* \in \mathcal{N}$. Similarly, $U^* T U \in \mathcal{N}$. Thus, $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$. Since $\mathcal{N}$ is singular in $\mathcal{M}$, $U \in \mathcal{N}$. Therefore, $W_1^* V W_1 = U(E_{1n} + E_{2n}) \in \mathcal{N}$. So $V = EVE = W_1 W_1^* V W_1 W_1^* \in \mathcal{N}_E$. This implies that $\mathcal{N}_E$ is singular in $\mathcal{M}_E$. □

2.2. Singular masas and singular subfactors (of type-II$_1$ factor) are completely singular

There are close relations between $\mathfrak{N}_{\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))''$ and $\mathfrak{GN}^{(2)}_{\mathcal{M}}(\mathcal{N})'' \bar{\otimes} \mathcal{B}(\mathcal{K})$. In this subsection, we prove the following theorem.

Theorem 2.4. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ and $\mathcal{K}$ be a Hilbert space. If $\mathfrak{GN}^{(2)}_{\mathcal{M}}(\mathcal{N})'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$, then $\mathfrak{N}_{\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))'' = \mathfrak{N}_{\mathcal{M}}(\mathcal{N})'' \bar{\otimes} \mathcal{B}(\mathcal{K})$. In particular, if $\mathfrak{GN}^{(2)}_{\mathcal{M}}(\mathcal{N})'' = \mathcal{N}$, then $\mathcal{N}$ is completely singular.*

Combining Theorem 2.4 and Lemmas 2.1 and 2.2, we have the following corollaries.

Corollary 2.5. *If $\mathcal{M}$ is a type-*II$_1$ *factor and $\mathcal{N}$ is a singular subfactor of $\mathcal{M}$, then $\mathcal{N}$ is completely singular in $\mathcal{M}$.*

Corollary 2.6. *If $\mathcal{N}$ is a singular masa of a von Neumann algebra $\mathcal{M}$, then $\mathcal{N}$ is completely singular in $\mathcal{M}$.*

To prove Theorem 2.4, we need the following lemmas. We consider $\dim \mathcal{K} = 2$ first, which motivates the general case.

Lemma 2.7. *Let*

$$U = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

be a unitary operator in $\mathcal{M} \bar{\otimes} M_2(\mathbb{C})$. Then the following conditions are equivalent:

(i) *$U(\mathcal{N} \bar{\otimes} M_2(\mathbb{C}))U^* = \mathcal{N} \bar{\otimes} M_2(\mathbb{C})$;*

(ii) *$A_i X A_j^* \in \mathcal{N}$ and $A_i^* X A_j \in \mathcal{N}$ for all $X \in \mathcal{N}$, $1 \leqslant i, j \leqslant 4$.*

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Proof. $U(\mathcal{N}\bar{\otimes}M_2(\mathbb{C}))U^* = \mathcal{N}\bar{\otimes}M_2(\mathbb{C})$ if and only if

$$U(\mathcal{N}\bar{\otimes}M_2(\mathbb{C}))U^* \subseteq \mathcal{N}\bar{\otimes}M_2(\mathbb{C}) \quad \text{and} \quad U^*(\mathcal{N}\bar{\otimes}M_2(\mathbb{C}))U \subseteq \mathcal{N}\bar{\otimes}M_2(\mathbb{C}).$$

$U(\mathcal{N}\bar{\otimes}M_2(\mathbb{C}))U^* \subseteq \mathcal{N}\bar{\otimes}M_2(\mathbb{C})$ if and only if

$$U\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}U^*, U\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}U^*, U\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}U^*, U\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}U^* \in \mathcal{N} \quad \text{for all } X \in \mathcal{N}.$$

Simple computations show that $U(\mathcal{N}\bar{\otimes}M_2(\mathbb{C}))U^* \subseteq \mathcal{N}\bar{\otimes}M_2(\mathbb{C})$ if and only if $A_iXA_j^* \in \mathcal{N}$ for all $X \in \mathcal{N}$, $1 \leqslant i, j \leqslant 4$. □

Since the proof of the following lemma is similar to the proof of Lemma 2.7, we omit it here.

Lemma 2.8. *Let $U = (A_{ij})$ be a unitary operator in $\mathcal{M}\bar{\otimes}\mathcal{B}(\mathcal{K})$. Then the following conditions are equivalent:*

(i) $U(\mathcal{N}\bar{\otimes}\mathcal{B}(\mathcal{K}))U^* = \mathcal{N}\bar{\otimes}\mathcal{B}(\mathcal{K})$;

(ii) $A_iXA_j^* \in \mathcal{N}$ *and* $A_i^*XA_j \in \mathcal{N}$ *for all* $X \in \mathcal{N}$, $1 \leqslant i, j \leqslant \dim\mathcal{K}$.

Let $X = I$ and $i = j$ in Lemma 2.8 (ii). We have the following corollary.

Corollary 2.9. *Let $U = (A_{ij})$ be a unitary operator in $\mathcal{M}\bar{\otimes}\mathcal{B}(\mathcal{K})$ such that*

$$U(\mathcal{N}\bar{\otimes}\mathcal{B}(\mathcal{K}))U^* = \mathcal{N}\bar{\otimes}\mathcal{B}(\mathcal{K}).$$

If $A_{ij} = V_{ij}H_{ij}$ is the polar decomposition of A_{ij}, then $H_{ij} \in \mathcal{N}$, $1 \leqslant i, j \leqslant \dim\mathcal{K}$.

Lemma 2.10. *Let $\mathcal{N}$ be a von Neumann algebra, let H be a positive operator in $\mathcal{N}$ and let E be the closure of the range space of H. Then the strong-operator closure of $\mathcal{T} = \{HXH : X \in \mathcal{N}\}$ is $\mathcal{N}_E$ ($\mathcal{N}_E = E\mathcal{N}E$).*

Proof. It is easy to see $\mathcal{T} \subseteq \mathcal{N}_E$. Let

$$H = \int_{\mathbb{R}} \lambda \, \mathrm{d}E(\lambda) \quad \text{and} \quad E_n = E([1/n, \infty)).$$

Then $\lim_{n\to\infty} E_n = E$ in strong-operator topology. Set $H_n = E_nH + (I - E_n)$. Then H_n is invertible in $\mathcal{N}$. For $X \in \mathcal{N}_E$, let $X_n = H_n^{-1}(E_nXE_n)H_n^{-1} \in \mathcal{N}$. Then $HX_nH = HH_n^{-1}E_nXE_nH_n^{-1}H = E_nXE_n \to EXE = X$ in strong-operator topology. Hence, the strong-operator closure of $\mathcal{T}$ contains $\mathcal{N}_E$. □

Lemma 2.11. *Suppose that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and that $A \in \mathcal{M}$ satisfies $A\mathcal{N}A^* \subseteq \mathcal{N}$ and $A^*\mathcal{N}A \subseteq \mathcal{N}$. Let $A = VH$ be the polar decomposition and $E = V^*V$, $F = VV^*$. Then $H, E, F \in \mathcal{N}$ and $V \in \mathfrak{GN}_{\mathcal{M}}^{(2)}(\mathcal{N})$.*

Proof. By the assumption, $A^*IA = H^2 \in \mathcal{N}$. So $H \in \mathcal{N}$ and $E = R(H) \in \mathcal{N}$, where $R(H)$ is the closure of range space of H. By symmetry, $F \in \mathcal{N}$. Note that $AXA^* = VHXHV^* \subseteq F\mathcal{N}F = \mathcal{N}_F$ for all $X \in \mathcal{N}$. By Lemma 2.10, $V\mathcal{N}_E V^* \subseteq \mathcal{N}_F$. By $A^*XA \subseteq \mathcal{N}$ for all $X \in \mathcal{N}$ and similar arguments, $V^*\mathcal{N}_F V \subseteq \mathcal{N}_E$. So $\mathcal{N}_F \subseteq V\mathcal{N}_E V^*$. Thus, $V\mathcal{N}_E V^* = \mathcal{N}_F$, i.e. $V \in \mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$. □

Proof of Theorem 2.4. Let $U_1 \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})$ and V be a unitary operator in $\mathcal{B}(\mathcal{K})$. Then $U_1 \otimes V \in \mathfrak{N}_{\mathcal{M}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})}(\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}))$. So

$$\mathfrak{N}_{\mathcal{M}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})}(\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}))'' \supseteq \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}).$$

Let $U = (A_{ij})$ be a unitary operator in $\mathcal{M}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$ such that $U(\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}))U^* = \mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$. Let $A_{ij} = V_{ij}H_{ij}$ be the polar decomposition of A_{ij}. By Lemmas 2.8 and 2.11 and Corollary 2.9, $H_{ij} \in \mathcal{N}$ and $V_{ij} \in \mathfrak{G}\mathfrak{N}^{(2)}_{\mathcal{M}}(\mathcal{N})$. By the assumption of Theorem 2.4, $V_{ij} \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''$. So $U \in \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$, i.e.

$$\mathfrak{N}_{\mathcal{M}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})}(\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}))'' \subseteq \mathfrak{N}_{\mathcal{M}}(\mathcal{N})''\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}).$$

□

2.3. On singular but not completely singular von Neumann subalgebras

Proposition 2.12. *If $\mathcal{N}$ is a singular but not a completely singular von Neumann subalgebra of $\mathcal{M}$, then there is a von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ and a Hilbert space $\mathcal{K}$ such that $\mathcal{N} \subsetneqq \mathcal{M}_1$, $\mathcal{N}$ is singular in $\mathcal{M}_1$ and $\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$ is regular in $\mathcal{M}_1\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$.*

Proof. Since $\mathcal{N}$ is not completely singular in $\mathcal{M}$, there is a Hilbert space $\mathcal{K}$ such that $\mathcal{L} = \mathfrak{N}_{\mathcal{M}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})}(\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}))'' \supsetneqq \mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$. Note that $\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}) \subseteq \mathcal{L} \subseteq \mathcal{M}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$. By Ge and Kadison's splitting theorem (see [**4**]), $\mathcal{L} = \mathcal{M}_1\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$ for some von Neumann algebra $\mathcal{M}_1$, $\mathcal{N} \subsetneqq \mathcal{M}_1 \subseteq \mathcal{M}$. Since $\mathcal{N}$ is singular in $\mathcal{M}$, $\mathcal{N}$ is singular in $\mathcal{M}_1$. Since $\mathcal{M}_1\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}) = \mathfrak{N}_{\mathcal{M}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})}(\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K}))$, $\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$ is regular in $\mathcal{M}_1\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$. □

Proposition 2.13. *If $\mathcal{M}$ is a type-II_1 factor, then there is a singular von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N} \neq \mathcal{M}$ and $\mathcal{N}\,\bar{\otimes}\,\mathcal{B}(l^2(\mathbb{N}))$ is regular in $\mathcal{M}\,\bar{\otimes}\,\mathcal{B}(l^2(\mathbb{N}))$. In particular, $\mathcal{N}$ is not completely singular.*

Proof. Let $\mathcal{M}_1$ be a type-I_3 subfactor of $\mathcal{M}$ and $\mathcal{M}_2 = \mathcal{M}_1' \cap \mathcal{M}$. Then $\mathcal{M}_2$ is a type-II_1 factor. We can identify $\mathcal{M}$ with $M_3(\mathbb{C})\,\bar{\otimes}\,\mathcal{M}_2$ and $\mathcal{M}_1$ with $M_3(\mathbb{C})\,\bar{\otimes}\,\mathbb{C}I$. With this identification, let $\mathcal{N} = (M_2(\mathbb{C}) \oplus \mathbb{C})\,\bar{\otimes}\,\mathcal{M}_2$. Then

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes I \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes I$$

are central projections in $\mathcal{N}$. $\mathcal{N} = \{P, Q\}' \cap \mathcal{M}$ and $\{P, Q\}''$ is the centre of $\mathcal{N}$. Let $U \in \mathcal{M}$ be a unitary operator such that $U\mathcal{N}U^* = \mathcal{N}$. Then $U\{P, Q\}''U^* = \{P, Q\}''$. Let

τ be the unique tracial state on $\mathcal{M}$. Then $\tau(P) = \frac{2}{3}$ and $\tau(Q) = \frac{1}{3}$. So $UPU^* = P$ and $UQU^* = Q$. This implies that $U \in \{P, Q\}' \cap \mathcal{M} = \mathcal{N}$ and $\mathcal{N}$ is singular in $\mathcal{M}$.

To see that $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ is not singular in $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$, we identify $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ with $M_3(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})) \bar{\otimes} \mathcal{M}_2$ and $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ with $(M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})) \bar{\otimes} \mathcal{M}_2$. Let V be an isometry from $l^2(\mathbb{N})$ onto $\mathbb{C}^2 \otimes l^2(\mathbb{N})$. Then

$$U = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}$$

is a unitary operator in $M_3(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ such that $U((M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N})))U^* = (M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$. Therefore, $U \otimes I$ is a unitary operator in the normalizer of $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ but $U \otimes I \notin \mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$.

By Proposition 2.12, there is a von Neumann subalgebra $\mathcal{L}$ of $\mathcal{M}$ such that $\mathcal{N} \subsetneqq \mathcal{L}$ and $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ is regular in $\mathcal{L} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$. Since $(M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{M}_2 \subsetneqq \mathcal{L} \subseteq M_3(\mathbb{C}) \bar{\otimes} \mathcal{M}_2$, by Ge and Kadison's splitting theorem [**4**], $\mathcal{L} = \mathcal{L}_1 \bar{\otimes} \mathcal{M}_2$ for some von Neumann algebra $\mathcal{L}_1$ such that $M_2(\mathbb{C}) \oplus \mathbb{C} \subsetneqq \mathcal{L}_1 \subseteq M_3(\mathbb{C})$. Since $M_3(\mathbb{C})$ is the unique von Neumann subalgebra of $M_3(\mathbb{C})$ satisfying the above condition, $\mathcal{L}_1 = M_3(\mathbb{C})$. This implies that $\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ is regular in $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$. □

Remark 2.14. By the proof of Proposition 2.13, $(M_2(\mathbb{C}) \oplus \mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ is regular in $M_3(\mathbb{C}) \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$.

Remark 2.15. Let $\mathcal{M}$ be a type-II_1 factor and let $\mathcal{N}$ be the singular von Neumann subalgebra constructed as in the proof of Proposition 2.13. It is easy to see that $\mathcal{N} \bar{\otimes} \mathcal{N}$ is not singular in $\mathcal{M} \bar{\otimes} \mathcal{M}$.

3. A characterization of complete singularity

Theorem 3.1. *Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ and let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then the following conditions are equivalent.*

(i) *$\mathcal{N}$ is completely singular in $\mathcal{M}$.*

(ii) *$\mathcal{N} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ is singular in $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$.*

(iii) *If $\theta \in \mathrm{Aut}(\mathcal{N}')$ and $\theta(X) = X$ for all $X \in \mathcal{M}'$, then $\theta(Y) = Y$ for all $Y \in \mathcal{N}'$.*

Proof. (iii) $\Longrightarrow$ (i). Let $\mathcal{K}$ be a Hilbert space and let $U \in \mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$ be a unitary operator such that $U(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))U^* = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$. Note that $(\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K}))' = \mathcal{M}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$ and $(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))' = \mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$. $\theta(Z) = UZU^*$ is an automorphism of $\mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$. Since $U \in \mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$, $\theta(X \otimes I_{\mathcal{K}}) = X \otimes I_{\mathcal{K}}$ for all $X \in \mathcal{M}'$. By the assumption of (iii), $Y \otimes I_{\mathcal{K}} = \theta(Y \otimes I_{\mathcal{K}}) = U(Y \otimes I_{\mathcal{K}})U^*$ for all $Y \in \mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$. This implies that $U \in \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$. Therefore, $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ is singular in $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$.

(i) $\Longrightarrow$ (ii) is trivial.

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(ii) $\Longrightarrow$ (iii). By [**5**], there is a separable Hilbert space $\mathcal{H}_1$ and a faithful normal representation ϕ of $\mathcal{N}'$ such that $\phi(\mathcal{N}')$ acts on $\mathcal{H}_1$ in standard form. Let $\theta_1 = \phi \cdot \theta \cdot \phi^{-1}$. Then $\theta_1 \in \operatorname{Aut} \phi(\mathcal{N}')$ and $\theta_1(\phi(X)) = \phi(X)$ for all $X \in \mathcal{M}'$. Now there is a unitary operator $U_1 \in \mathcal{B}(\mathcal{H}_1)$ such that $\theta_1(\phi(Y)) = U_1\phi(Y)U_1^*$ for all $Y \in \mathcal{N}'$. Let $\mathcal{N}_1$ and $\mathcal{M}_1$ be the commutants of $\phi(\mathcal{N}')$ and $\phi(\mathcal{M}')$ relative to $\mathcal{H}_1$, respectively. Then $\mathcal{N}_1$ is a von Neumann subalgebra of $\mathcal{M}_1$. Since $\theta_1(\phi(X)) = U_1\phi(X)U_1^* = \phi(X)$ for all $X \in \mathcal{M}'$, $U_1 \in \mathcal{M}_1$. Since $\theta_1(Z) = U_1 Z U_1^*$ is an automorphism of $\phi(\mathcal{N}'$, $\theta_1(Z) = U_1 Z U_1^*$ is also an automorphism of $\mathcal{N}_1$. Now we only need to prove that $\mathcal{N}_1$ is a singular von Neumann subalgebra of $\mathcal{M}_1$. Then $U_1 \in \mathcal{N}_1$ and $\theta_1(\phi(Y)) = U_1\phi(Y)U_1^* = \phi(Y)$ for all $Y \in \mathcal{N}'$. This implies that $\theta(Y) = Y$ for all $Y \in \mathcal{N}'$.

By [**2**, Theorem 3, p. 61], $\phi = \phi_3 \cdot \phi_2 \cdot \phi_1$, where $\phi_1(\mathcal{N}') = \mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}$, $\mathcal{K} = l^2(\mathbb{N})$, $\phi_2(\mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}}) = (\mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}})E$, $E \in (\mathcal{N}' \bar{\otimes} \mathbb{C}I_{\mathcal{K}})' = \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ and ϕ_3 is a spacial isomorphism. We may assume that $\phi_3 = \mathrm{id}$. Then $\mathcal{N}_1 = E(\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K}))E$ and $\mathcal{M}_1 = E(\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K}))E$, where $E \in \mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$. By (ii), $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ is singular in $\mathcal{M} \bar{\otimes} \mathcal{B}(\mathcal{K})$. Note that $\mathcal{N} \bar{\otimes} \mathcal{B}(\mathcal{K})$ is a countably decomposable, properly infinite von Neumann algebra. By Lemma 2.3, $\mathcal{N}_1$ is singular in $\mathcal{M}_1$. □

Note that in the proof of (iii) $\Longrightarrow$ (i) of Theorem 3.1, we do not need the assumption that $\mathcal{H}$ is a separable Hilbert space.

Let $\mathcal{M}$ be a von Neumann algebra. A von Neumann subalgebra $\mathfrak{B}$ of $\mathcal{M}$ is called *maximal injective* if it is injective and if it is maximal with respect to inclusion in the set of all injective von Neumann subalgebras of $\mathcal{M}$ (see [**10**]).

Proposition 3.2. *If $\mathfrak{B}$ is a maximal injective von Neumann subalgebra of $\mathcal{M}$, then $\mathfrak{B}$ is completely singular in $\mathcal{M}$.*

Proof. We can assume that $\mathcal{M}$ acts on $\mathcal{H}$ in standard form. Then $\mathfrak{B}'$ is a minimal injective von Neumann algebra extension of $\mathcal{M}'$ [**3**, 1.3]. Let $\theta \in \operatorname{Aut}(\mathfrak{B}')$ such that $\theta(X) = X$ for all $X \in \mathcal{M}'$. Then $\theta(Y) = Y$ for all $Y \in \mathfrak{B}'$ by [**3**, Lemma 1.2]. By Theorem 3.1, $\mathfrak{B}$ is completely singular in $\mathcal{M}$. □

4. Completely singular von Neumann subalgebras in tensor products of von Neumann algebras

4.1. Complete singularity is stable under the tensor product

The proof of the following lemma is similar to the proof of [**16**, Lemma 6.6].

Lemma 4.1. *Let $\mathcal{M}$ be a separable von Neumann algebra and let $\mathcal{N}$ be a singular von Neumann subalgebra of $\mathcal{M}$. If $\mathcal{A}$ is an abelian von Neumann algebra, then $\mathcal{N} \bar{\otimes} \mathcal{A}$ is a singular von Neumann subalgebra of $\mathcal{M} \bar{\otimes} \mathcal{A}$.*

Proof. We can assume that $\mathcal{M}$ acts on a separable Hilbert space $\mathcal{H}$ in standard form and $\mathcal{A}$ is countably decomposable. Then there is a $*$-isomorphism from $\mathcal{A}$ onto $L^\infty(\Omega, \mu)$ with μ a probability Radon measure on some compact space Ω. To the $*$-isomorphism $\mathcal{A} \to L^\infty(\Omega, \mu)$ corresponds canonically a $*$-isomorphism Φ from $\mathcal{B}(\mathcal{H}) \bar{\otimes} \mathcal{A}$ onto

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$L^\infty(\Omega,\mu;\mathcal{B}(\mathcal{H}))$. Note that $\Phi(\mathcal{M}\,\bar{\otimes}\,\mathcal{A})(\omega) = \mathcal{M}$ and $\Phi(\mathcal{N}\,\bar{\otimes}\,\mathcal{A})(\omega) = \mathcal{N}$ for almost all $\omega \in \Omega$. Let $U \in \mathcal{M}\,\bar{\otimes}\,\mathcal{A}$ be a unitary operator such that $U(\mathcal{N}\,\bar{\otimes}\,\mathcal{A})U^* = \mathcal{N}\,\bar{\otimes}\,\mathcal{A}$. Then $\Phi(U) = U(\omega)$ such that $U(\omega)$ is a unitary operator in $\mathcal{M}$ for almost all $\omega \in \Omega$. Because $U(\mathcal{N}\,\bar{\otimes}\,\mathcal{A})U^* = \mathcal{N}\,\bar{\otimes}\,\mathcal{A}$, we have $U(\omega)\mathcal{N}U(\omega)^* = \mathcal{N}$ for almost all $\omega \in \Omega$. Since $\mathcal{N}$ is singular in $\mathcal{M}$, $U(\omega) \in \mathcal{N}$ for almost all $\omega \in \Omega$. Hence, $U \in \mathcal{N}\,\bar{\otimes}\,\mathcal{A}$. □

Note that for every Hilbert space $\mathcal{K}$, the von Neumann algebra $\mathcal{M}\,\bar{\otimes}\,\mathcal{A}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})$ is canonically isomorphic to the von Neumann algebra $\mathcal{M}\,\bar{\otimes}\,\mathcal{B}(\mathcal{K})\,\bar{\otimes}\,\mathcal{A}$. We have the following corollary.

Corollary 4.2. *Let $\mathcal{M}$ be a separable von Neumann algebra and $\mathcal{N}$ be a completely singular von Neumann subalgebra. If $\mathcal{A}$ is an abelian von Neumann algebra, then $\mathcal{N}\,\bar{\otimes}\,\mathcal{A}$ is a completely singular von Neumann subalgebra of $\mathcal{M}\,\bar{\otimes}\,\mathcal{A}$.*

Theorem 4.3. *Let $\mathcal{M}$ be a separable von Neumann algebra and let $\mathcal{N}$ be a completely singular von Neumann subalgebra. Then $\mathcal{N}\,\bar{\otimes}\,\mathcal{L}$ is completely singular in $\mathcal{M}\,\bar{\otimes}\,\mathcal{L}$ for every separable von Neumann algebra $\mathcal{L}$.*

Proof. We can assume that $\mathcal{M}$ and $\mathcal{L}$ act on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ in standard form, respectively. Let θ be in $\mathrm{Aut}(\mathcal{N}'\,\bar{\otimes}\,\mathcal{L}')$ such that $\theta(X \otimes Z) = X \otimes Z$ for all $X \in \mathcal{M}'$ and $Z \in \mathcal{L}'$. Let $\mathcal{A}$ be the centre of $\mathcal{L}'$. Then $(\mathbb{C}I_{\mathcal{H}}\,\bar{\otimes}\,\mathcal{L}')' \cap (\mathcal{N}'\,\bar{\otimes}\,\mathcal{L}') = (\mathcal{B}(\mathcal{H})\,\bar{\otimes}\,\mathcal{L}) \cap (\mathcal{N}'\,\bar{\otimes}\,\mathcal{L}') = (\mathcal{B}(\mathcal{H}) \cap \mathcal{N}')\,\bar{\otimes}\,(\mathcal{L} \cap \mathcal{L}') = \mathcal{N}'\,\bar{\otimes}\,\mathcal{A}$. So for $T \in \mathcal{N}'\,\bar{\otimes}\,\mathcal{A}$ and $Z \in \mathcal{L}'$, $T(I_{\mathcal{H}} \otimes Z) = (I_{\mathcal{H}} \otimes Z)T$ and $\theta(T)\theta(I_{\mathcal{H}} \otimes Z) = \theta(I_{\mathcal{H}} \otimes Z)\theta(T)$. Since $\theta(I_{\mathcal{H}} \otimes Z) = I_{\mathcal{H}} \otimes Z$, $\theta(T)(I_{\mathcal{H}} \otimes Z) = (I_{\mathcal{H}} \otimes Z)\theta(T)$. This implies that $\theta(T) \in \mathcal{N}'\,\bar{\otimes}\,\mathcal{A}$. So $\theta \in \mathrm{Aut}(\mathcal{N}'\,\bar{\otimes}\,\mathcal{A})$ when θ is restricted on $\mathcal{N}'\,\bar{\otimes}\,\mathcal{A}$ such that $\theta(X \otimes Z) = X \otimes Z$ for all $X \in \mathcal{M}'$ and $Z \in \mathcal{A}$.

Consider the standard representation ϕ of $\mathcal{A}$ on a separable Hilbert space $\mathcal{K}_1$. Then $\phi(\mathcal{A})' = \phi(\mathcal{A})$. By Corollary 4.2, $\mathcal{N}\,\bar{\otimes}\,\phi(\mathcal{A})$ is completely singular in $\mathcal{M}\,\bar{\otimes}\,\phi(\mathcal{A})$. On $\mathcal{H} \otimes \mathcal{K}_1$, $(\mathcal{N}\,\bar{\otimes}\,\phi(\mathcal{A}))' = \mathcal{N}'\,\bar{\otimes}\,\phi(\mathcal{A})$ and $(\mathcal{M}\,\bar{\otimes}\,\phi(\mathcal{A}))' = \mathcal{M}'\,\bar{\otimes}\,\phi(\mathcal{A})$. Note that $\theta_1 = (\mathrm{id}\,\bar{\otimes}\,\phi)\cdot\theta\cdot(\mathrm{id}\,\bar{\otimes}\,\phi^{-1}) \in \mathrm{Aut}(\mathcal{N}'\,\bar{\otimes}\,\phi(\mathcal{A}))$ and $\theta_1(X \otimes Z') = (\mathrm{id}\,\bar{\otimes}\,\phi)\cdot\theta(X \otimes \phi^{-1}(Z')) = (\mathrm{id}\,\bar{\otimes}\,\phi)(X \otimes \phi^{-1}(Z')) = X \otimes Z'$ for all $X \in \mathcal{M}'$ and $Z' \in \phi(\mathcal{A})$. By Theorem 3.1, $\theta_1(Y \otimes Z') = Y \otimes Z'$ for all $Y \in \mathcal{M}'$ and $Z' \in \phi(\mathcal{A})$. This implies that $\theta(Y \otimes \phi^{-1}(Z')) = Y \otimes \phi^{-1}(Z')$ for all $Y \in \mathcal{N}'$ and $Z' \in \phi(\mathcal{A})$. Let $Z' = I_{\mathcal{K}_1}$. Then $\theta(Y \otimes I_{\mathcal{K}}) = Y \otimes I_{\mathcal{K}}$ for all $Y \in \mathcal{N}'$. Hence, $\theta(Y \otimes Z) = Y \otimes Z$ for all $Y \in \mathcal{N}'$ and $Z \in \mathcal{L}'$. By Theorem 3.1, $\mathcal{N}\,\bar{\otimes}\,\mathcal{L}$ is completely singular in $\mathcal{M}\,\bar{\otimes}\,\mathcal{L}$. □

4.2. Tensor product with completely singular subfactors

Theorem 4.4. *Let $\mathcal{M}_i$ be a separable von Neumann algebra,and let $\mathcal{N}_i$ be a completely singular von Neumann subalgebra of $\mathcal{M}_i$, $i = 1, 2$. If $\mathcal{N}_1$ is a factor, then $\mathcal{N}_1\,\bar{\otimes}\,\mathcal{N}_2$ is completely singular in $\mathcal{M}_1\,\bar{\otimes}\,\mathcal{M}_2$.*

Proof. We can assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ act on separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in standard form, respectively. Let θ be in $\mathrm{Aut}(\mathcal{N}_1'\,\bar{\otimes}\,\mathcal{N}_2')$ such that $\theta(X_1 \otimes X_2) = X_1 \otimes X_2$ for all $X_1 \in \mathcal{M}_1'$ and $X_2 \in \mathcal{M}_2'$.

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Since $\mathcal{N}_1$ is a singular subfactor in $\mathcal{M}_1$, $\mathcal{N}_1' \cap \mathcal{M}_1 = \mathcal{N}_1' \cap \mathcal{N}_1 = \mathbb{C}I_{\mathcal{H}_1}$. Note that

$$\begin{aligned}(\mathcal{M}_1' \bar{\otimes} \mathbb{C}I_{\mathcal{H}_2})' \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2') &= (\mathcal{M}_1 \bar{\otimes} \mathcal{B}(\mathcal{H}_2)) \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2') \\ &= (\mathcal{M}_1 \cap \mathcal{N}_1') \bar{\otimes} (\mathcal{B}(\mathcal{H}_2) \cap \mathcal{N}_2') \\ &= \mathbb{C}I_{\mathcal{H}_1} \bar{\otimes} \mathcal{N}_2'.\end{aligned}$$

We have

$$\begin{aligned}\theta(\mathbb{C}I_{\mathcal{H}_1} \bar{\otimes} \mathcal{N}_2') &= \theta((\mathcal{M}_1 \cap \mathcal{N}_1') \bar{\otimes} (\mathcal{B}(\mathcal{H}_2) \cap \mathcal{N}_2')) \\ &= \theta((\mathcal{M}_1 \bar{\otimes} \mathcal{B}(\mathcal{H}_2)) \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2')) \\ &= \theta((\mathcal{M}_1' \bar{\otimes} \mathbb{C}I_{\mathcal{H}_2})' \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2')) \\ &= \theta(\mathcal{M}_1' \bar{\otimes} \mathbb{C}I_{\mathcal{H}_2})' \cap \theta(\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2') \\ &= (\mathcal{M}_1' \bar{\otimes} \mathbb{C}I_{\mathcal{H}_2})' \cap (\mathcal{N}_1' \bar{\otimes} \mathcal{N}_2') \\ &= \mathbb{C}I_{\mathcal{H}_1} \bar{\otimes} \mathcal{N}_2'.\end{aligned}$$

Since $\mathcal{N}_2$ is completely singular in $\mathcal{M}_2$ and $\theta(I_{\mathcal{H}_1} \otimes X_2) = I_{\mathcal{H}_1} \otimes X_2$ for all $X_2 \in \mathcal{M}_2'$, $\theta(I_{\mathcal{H}_1} \otimes Y_2) = I_{\mathcal{H}_1} \otimes Y_2$ for all $Y_2 \in \mathcal{M}_2'$ by Theorem 3.1. Therefore, $\theta(X_1 \otimes Y_2) = X_1 \otimes Y_2$ for all $X_1 \in \mathcal{M}_1'$ and $Y_2 \in \mathcal{N}_2'$. By Theorem 4.3, $\mathcal{N}_1 \bar{\otimes} \mathcal{M}_2$ is completely singular in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. Since $\theta(X_1 \otimes Y_2) = X_1 \otimes Y_2$ for all $X_1 \in \mathcal{M}_1'$ and $Y_2 \in \mathcal{N}_2'$, by Theorem 3.1, $\theta(Y_1 \otimes Y_2) = Y_1 \otimes Y_2$ for all $Y_1 \in \mathcal{N}_1'$ and $Y_2 \in \mathcal{N}_2'$. By Theorem 3.1 again, $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$ is completely singular in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. □

Combining Theorem 4.4 and Corollary 2.5, we obtain the following corollary, which generalizes [**15**, Corollary 4.4].

Corollary 4.5. *If $\mathcal{N}_1$ is a singular subfactor of a type-*II$_1$ *factor $\mathcal{M}_1$ and $\mathcal{N}_2$ is a completely singular von Neumann subalgebra of $\mathcal{M}_2$, then $\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2$ is completely singular in $\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$.*

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