LINEARIZATION OF A CONTRACTIVE HOMEOMORPHISM

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1. Introduction. Let X be a topological space and $\phi: X \to X$ a continuous self-mapping of X. We say that ϕ is *linearized in* L *by* Φ if there exists a topological embedding $\mu: X \to L$ of the space X into the linear topological vector space L such that for all $x \in X$, $\mu(\phi(x)) = \Phi(\mu(x))$, where Φ is a continuous linear operator on L.

Let X be metrizable and let $\alpha \in [0, 1)$. We say that $\phi: X \to X$ is a *topological* α*-contraction on* X if there exists a metric $\rho(x, y)$ on X inducing the given topology such that

$$\forall x, y \in X: \rho(\phi(x), \phi(y)) \leqq \alpha\rho(x, y).$$

If ϕ is a homeomorphism and at the same time a topological α-contraction, we shall say that ϕ is a *topologically* α*-contractive homeomorphism.*

Let $\alpha > 0$. We shall say that ϕ is a *topological* α*-homothety on* X if there is a metric $\rho(x, y)$ on X, inducing the given topology, such that

$$\forall x, y \in X: \rho(\phi(x), \phi(y)) = \alpha\rho(x, y).$$

Our main objective in this paper will be to show that if X is a compact metrizable space and $\phi: X \to X$ is a topologically α-contractive homeomorphism (for some $\alpha \in (0, 1)$), then ϕ can be linearized in a separable Hilbert space as a homothety.

2. Proof of the theorem.

THEOREM. *Let X be a compact metrizable space, $\alpha \in (0, 1)$, and $\phi: X \to X$ a topologically α-contractive homeomorphism. Then, for every $\beta \in (0, 1)$ there exists a topological embedding $\mu: X \to H$ of X into a separable Hilbert space H such that $\forall x \in X: \mu(\phi(x)) = \beta\mu(x)$.*

Proof of the theorem. According to the theorem, proved in (**1**), the mapping ϕ is a topological α-homothety for every $\alpha \in (0, 1)$, i.e., there exists a metric ρ on X such that

$$\forall x, y \in X: \rho(\phi(x), \phi(y)) = \alpha\rho(x, y).$$

We shall show, first of all, that (X, ρ) can be embedded isometrically in the larger metric space (X^*, ρ^*), over which ϕ can be extended as an α-homothety *onto*.

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Let $A_0 = X - \phi(X)$ and $A_{n+1} = \phi(A_n)$ for $n = 0, 1, 2 \ldots$. We observe that the sets A_n are all mutually homeomorphic and disjoint, and that X can be represented in the form:

$$X = \left[\bigcup_{n=0}^{\infty} A_n\right] \cup \{a\},$$

where $a \in X$ is the fixed point of ϕ. The mapping ϕ has an inverse on $\phi(X)$ and we have that $\phi^{-1}(A_n) = A_{n-1}$ for $n = 1, 2 \ldots$.

Let us now introduce the family of sets $A_{-1}, A_{-2}, \ldots$ as mutually disjoint abstract copies of A_0 and disjoint with X. Let us introduce mappings $\phi_n: A_n \to A_{n+1}$ for $n = -1, -2, \ldots$ to be one-to-one and onto. Now we can introduce the set X^* as

$$\left[\bigcup_{-\infty}^{+\infty} A_n\right] \cup \{a\}$$

and the mapping $\phi^*: X^* \to X^*$ in the following way:

if $x \in A_n$ for $n \geqq 0$, we put $\phi^*(x) = \phi(x)$,

if $x \in A_n$ for $n < 0$, we put $\phi^*(x) = \phi_n(x)$,

and finally we put $\phi^*(a) = \phi(a)$.

It is obvious that ϕ^* is one-to-one and maps X^* onto itself.

Define $n(x) = n$ for $x \in A_n$ and

$$n(a) = \infty, \qquad n(x, y) = \min\{n(x), n(y)\}.$$

With this notation we define a metric ρ^* on X^* by the formula

$$\rho^*(x, y) = \alpha^n \rho((\phi^*)^{-n}(x), (\phi^*)^{-n}(y)),$$

where $n = n(x, y)$.

Since $n(\phi^*(x), \phi^*(y)) = 1 + n(x, y)$, we see that $\rho^*(\phi^*(x), \phi^*(y)) = \alpha\rho^*(x, y)$ for all $x, y \in X^*$; henceforth, we shall denote the function ϕ^* by ϕ and ρ^* by ρ, on X^*.

Our next objective will be to show that for every $\beta \in (0, 1)$ there exists a countable family of functions $f_i(x) \in C(X^*)$ such that

(1) $f_i(\phi(x)) = \beta f_i(x)$ for all $i = 1, 2, \ldots$,

(2) the family is uniformly bounded on X, i.e., there exists $M \geqq 0$ such that $|f_i(x)| \leqq M$ for all $i = 1, 2, \ldots$ and all $x \in X$,

(3) the family is point-separating on X, i.e., for any $t_1, t_2 \in X$ there exists an index i such that $f_i(t_1) \neq f_i(t_2)$.

Let $x \in A_0$ and denote by $d(x)$ the distance between x and $\phi(X)$, namely, $d(x) = \rho(x, \phi(X))$. The function $d(x)$ is positive since $\phi(X)$ is compact. Denote by $N(x, r)$ a spherical neighbourhood of the radius $r > 0$ about $x \in A_0$ in X^* (we are working in X^*):

$t \in N(x, r) \Leftrightarrow \rho(x, t) < r$,

if $r < d(x)$, then $N(x, r) \cap \phi(X) = 0$,

and it is easily seen that if $r < \frac{1}{2}d(x)$, then all images $\phi^n(N(x, r))$ are mutually disjoint.

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Let us denote by $\mathscr{R}_x$ the set of all $r > 0$ such that

(i) $N(x, r) \cap \phi(X) = 0$,

(ii) the family $\phi^n(N(x, r))$ is disjoint.

It is easy to see that $\mathscr{R}_x$ is an interval $(0, R_x)$, where $R_x > 0$.

Let us now associate to every $x \in A_0$ and every $r \in (0, R_x)$ a continuous function $g(x, r; t)$ of t on X^* in such a way that:

(i) $g(x, r; t)$: $X^* \to [0, 1]$,

(ii) $g(x, r; x) = 1$,

(iii) $g(x, r; t) = 0$ for $t \in N^c(x, r)$ (complement of $N(x, r)$ in X^*).

The fact that all $\phi^n[N(x, r)]$ are disjoint enables us to define the function $f(x, r; t)$: $X^* \to [0, \infty)$ as follows:

$$\begin{aligned} f(x, r; t) &= \beta^n g(x, r; \phi^{-n}(t)) \quad \text{if } t \in \phi^n[N(x, r)], \\ &= 0 \quad \text{if } t \notin \bigcup_{-\infty}^{\infty} \phi^n[N(x, r)]. \end{aligned}$$

The number β is chosen arbitrarily from $(0, 1)$. Continuity of $f(x, r; t)$ can be easily seen since it can be represented in the form

$$\sum_{-\infty}^{+\infty} \beta^n g(x, r; \phi^{-n}(t)),$$

the sum being uniformly converging on each set of the form

$$\left[\bigcup_{i=n}^{\infty} A_i \right] \cup \{a\}.$$

The function $f(x, r; t)$ obviously satisfies the equation

$$f(x, r; \phi(t)) = \beta f(x, r; t)$$

and is bounded on X since $\phi^{-n}[N(x, r)] \cap X = 0$ for all $n = 1, 2, \ldots$ and therefore

$$\sup_{t \in X} f(x, r; t) = \sup_{t \in X} g(x, r; t) = 1.$$

Let Q be a dense and countable subset of A_0 and let $t_1, t_2 \in X$ be arbitrarily given different points of X: $t_1 \neq t_2$. Then at least one of them, say t_2, is not equal to a and, therefore, there exists $x \in A_0$ such that $\phi^n(x) = t_2$ for some $n \geqq 0$. Consider the neighbourhood $N(x, r)$ for some rational $r \in (0, R_x)$ and choose $q \in Q$, $q \in N(x, \frac{1}{2}r)$. Then, evidently, $\frac{1}{2}r \in (0, R_q)$, and the function $f(q, \frac{1}{2}r, t)$ separates points t_2 and a, since $f(q, \frac{1}{2}r, t_2) = \alpha^n f(q, \frac{1}{2}r, x) > 0$ for $x \in N(q, \frac{1}{2}r)$, and $f(q, \frac{1}{2}r, a) = 0$; thus, if $t_1 = a$, our proof is complete. If $t \neq a$, then $y = \phi^{-n}(t_1)$ for some $y \in A_0$ and $n \geqq 0$. If we choose the rational number r such that $r \in (0, R_x)$, $r < \rho(x, y)$, and choose $q \in N(x, \frac{1}{2}r)$, then we have, again, that $f(q, \frac{1}{2}r, t_2) \neq 0$, $f(q, \frac{1}{2}r, y) = 0$, and therefore

$$f(q, \tfrac{1}{2}r, t) = 0,$$

and we have shown that the family $f(q, r, t)$ where $q \in Q$ and r rational numbers from $(0, R_2)$, satisfies our conditions. If we index this function of our

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family by natural numbers $f_n(t)$, we may construct the desired embedding $\mu: X \to H$ by the formula

$$\mu(t) = \{f_1(t), \tfrac{1}{2}f_2(t), \tfrac{1}{3}f_3(t), \ldots\}.$$

References

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