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A REMARK ON THE UNITS OF FINITE ORDER IN THE GROUP RING OF A FINITE GROUP

BY
GERALD LOSEY

Let G be a group, ZG its integral group ring and $U(ZG)$ the group of units of ZG. The elements $\pm g \in U(ZG)$, $g \in G$, are called the trivial units of ZG. In this note we will prove

THEOREM. *Let G be a finite group. If ZG contains a non-trivial unit of finite order then it contains infinitely many non-trivial units of finite order.*

In [1] S. D. Berman has shown that if G is finite then every unit of finite order in ZG is trivial if and only if G is abelian or G is the direct product of a quaternion group of order 8 and an elementary abelian 2-group. Thus we have

COROLLARY. *Let G be a finite group. If G is neither abelian nor the direct product of a quaternion group and an elementary abelian 2-group then ZG contains infinitely many non-trivial units of finite order.*

For the proof of the theorem we need the following results.

LEMMA 1. (Dietzmann, [3]). *If N is a finite normal subset of elements of finite order in a group G then $\langle N\rangle$, the subgroup generated by N, is a finite normal subgroup of G.*

LEMMA 2. (Cohn—Livingstone, [2]). *Let G be a finite group and $u=\sum_{g\in G} u(g)g$ a unit of finite order in ZG. If $u(1)\neq 0$ then $u=\pm 1$.*

Note that the group ring ZG admits an involution $\alpha\to\alpha^*$ defined as follows: If $\alpha=\sum_{g\in G}\alpha(g)g$ then $\alpha^*=\sum_{g\in G}\alpha(g^{-1})g$. If u is a unit of finite order in ZG then so is u^*.

Proof of the theorem. Suppose there are only finitely many units of finite order in ZG. Then they constitute a finite *normal* subset N of $U(ZG)$. By Dietzmann's lemma, N generates a finite normal subgroup of $U(ZG)$. But this subgroup then consists of units of finite order and, therefore, coincides with N. Hence N is a subgroup of $U(ZG)$. Let $u=\sum_{g\in G} u(g)g \in N$. Then u^* and $w=uu^* \in N$ also. Now

$$w(1) = \sum_{g\in G} u(g)^2 > 0$$

and so, by lemma 2, $w=\pm 1$. Clearly, in this case $w=1$ and, thus,

$$w(1) = \sum_{g\in G} u(g)^2 = 1.$$

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Since the $u(g)$ are integers it follows that there exists $g_0 \in G$ such that $u(g_0) = \pm 1$ and $u(g) = 0$ for all $g \neq g_0$. Thus we have shown that N is just the set of trivial units.

References

1. S. D. Berman, *On the equation* $x^m = 1$ *in an integral group ring*, Ukrain. Math. Z., **7** (1955), pp. 253–261.

2. J. A. Cohn and D. Livingstone, *On the structure of group algebras*, *I*, Canadian J. Math., **17** (1965), pp. 583–593.

3. A. P. Dietzmann, *Uber p-gruppen*, *Doklady Akad. Nauk SSSR*, **15** (1937), pp. 71–76.