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ON PRODUCTS OF MODULES IN A TOPOS

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Abstract

In an elementary topos if R is a ring and X is a decidable object then there exists a canonical homomorphism from the coproduct of an X-family of R-modules to the product of the same family. In this paper it is shown that this homomorphisms is a monomorphism.

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In the category **Set** of sets if X is a set and R is a ring then for an X-family $\{M_x\}_{x \in X}$ of R-modules there is always a canonical monomorphism $\phi\colon \oplus_{x \in X} M_x \to \prod_{x \in X} M_x$, with $\pi_x \phi i_x = 1_{M_x}$, where π_x and i_x are the xth projection and injection, respectively. In [5] it is shown, by an example, that in an elementary topos such a homomorphism does not always exist. However, if we choose X, the index object, to be decidable, it is proved that such a canonical homomorphism exists.

In this paper we show that the canonical homomorphism given in [5] is a monomorphism. A closely related work can be found for the case of abelian groups in [1].

Throughout the paper, E denotes an elementary topos with natural numbers object, and R is a ring in E. All other notation, not explained here, can be found in [2] or [3].

Let A be an E-indexed category with $\varinjlim$ and small homs, let C be an internal category of E, let $\Gamma\colon C \to E$ be an internal functor with $\varinjlim \Gamma = I$ and for each J in E let $\Gamma^J(c) \xrightarrow{\lambda^J_c} J^*I$ be the canonical injection, where $c \in [J, C]$.

1. Lemma. *If* $A \in A^I$ *then*

$$\sum_I A \simeq \varinjlim_{C} \sum_{\Gamma^J(c)} \lambda_c^{J*} \pi_2^* A,$$

where $\pi_2 \colon J \times I \to I$ *is the projection.*

Proof. Let $B \in A^1$. Then we have the following natural isomorphisms:

$$\varinjlim_{C} \sum_{\Gamma^J(c)} \lambda_c^{J*} \pi_2^* A \to B;$$

$$\text{indexed cocone } \left\langle \Sigma_{\Gamma^J(c)} \lambda_c^{J*} \pi_2^* A \to J^* B \right\rangle_{c \in [J, C]};$$

$$\text{compatible families } \left\langle \lambda_c^{J*} \pi_2^* A \to \Gamma^j(c)^* J^* B \right\rangle_{c \in [J, C]};$$

$$\text{compatible families } \left\langle \lambda_c^{J*} \pi_2^* A \to \lambda_c^{J*} \pi_2^* I^* B \right\rangle_{c \in [J, C]};$$

$$\text{indexed cocone } \left\langle \pi_2 \lambda_c^J \to \mathrm{Hom}^I(A, I^* B) \right\rangle_{c \in [J, C]};$$

$$1 \simeq \varinjlim_{C} \pi_2 \lambda_c^J \to \mathrm{Hom}^I(A, I^* B);$$

$$A \to I^* B;$$

$$\sum_I A \to B.$$

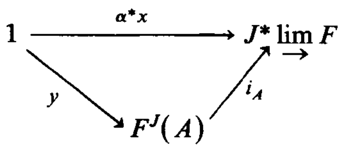
Hence by the Yoneda Lemma,

$$\varinjlim_{C} \sum_{\Gamma^J(c)} \lambda_c^{J*} \pi_2^* A \simeq \sum_I A.$$

We will use the following theorem, due to D. Schumacher, which is proved in [4].

2. Theorem. *Let* A *be a small filtered indexed category and let* $F \colon A \to E$ *be an indexed functor.*

(a) *For every* $1 \xrightarrow{x} I^* \varinjlim F$ *there exist* $J \xrightarrow{\alpha} I$, $A \in A^J$ *and* $1 \xrightarrow{y} F^J(A)$ *such that*



commutes, where i_A *is indexed.*

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(b) *For* $I \xrightarrow{x_1} F^I A_1$ *and* $I \xrightarrow{x_2} F^I A_2$, $i_{A_1}(x_1) = i_{A_2}(x_2)$ *if and only if there exist* $J \overset{\alpha}{\twoheadrightarrow} I$ *and*

$$\begin{array}{ccc} \alpha^* A_1 & & \\ & \searrow^{a_1} & \\ & & A \\ & \nearrow_{a_2} & \\ \alpha^* A_2 & & \end{array}$$

in A^J, *such that* $F^J(a_1)(\alpha^* x_1) = F^J(a_2)(\alpha^* x_2)$.

Let X be an object in E and let E_{fin} be the internalization of E_{fc}, the category of finite cardinals, in the sense that its external category of I-elements is equivalent to $(E/I)_{\text{fc}}$. For more details and the proof of the following lemma see [5] and [4].

3. LEMMA. *The internal category* E_{fin}/X *is filtered.*

Let M be an object in $\text{Mod}_R(E)^X$. Define a functor P: $E_{\text{fin}}/X \to \text{Mod}_R(E)$ as follows: for any I-object of E_{fin}/X, i.e. $(I \xrightarrow{p} N, [p] \xrightarrow{x} I^*X)$, $P^I([p] \xrightarrow{x} I^*X) = \bigoplus_{[p]} x^*\pi_2^*M$, where π_2: $I \times X \to X$, and for any I-morphism

$$\begin{array}{ccc} [p] & \xrightarrow{f} & [q] \\ & \searrow \quad \swarrow & \\ & I^*X & \end{array}$$

$P^I(f)$ is defined by

$$\begin{array}{ccc} \left[\bigoplus_{[p]} x^*\pi_2^*M, L\right] & \simeq & \left[\pi_2^*M, \prod_x [p]^*L\right] \\ {\scriptstyle [P^I(f), L]}\uparrow & & \uparrow{\scriptstyle [\pi_2^*M, f^*]} \\ \left[\bigoplus_{[q]} y^*\pi_2^*M, L\right] & \simeq & \left[\pi_2^*M, \prod_y [q]^*L\right], \end{array}$$

where L is in $\text{Mod}_R(E)^I$. It is easy to see that P is an indexed functor.

Let I be a decidable object in E, i.e. δ: $I \rightarrowtail I \times I$, the diagonal morphism, has a complement $J \xrightarrow{c} I \times I$ such that $\binom{\delta}{c}$: $I + J \to I \times I$ is an isomorphism. Then it is well known that $E/I \times I \xrightarrow{(\delta^*, c^*)} E/I \times E/J$ is an equivalence of categories. This extends to an equivalence α: $\text{Mod}_R(E)^{I \times I} \to \text{Mod}_R(E)^I \times \text{Mod}_R(E)^J$. For M in $\text{Mod}_R(E)^I$, let $\alpha(\pi_1^*M) = (M_1, M_1')$ and $\alpha(\pi_2^*M) = (M_2, M_2')$, where $I \xrightarrow{\pi_2} I \times I \xrightarrow{\pi_1} I$ are the projections and M_1', M_2' are in $\text{Mod}_R(E)^J$. Since $\pi_1\delta = \pi_2\delta = 1_I$, then $M_1 \simeq M \simeq M_2$ in $\text{Mod}_R(E)^I$, i.e. there is an isomorphism $M_1 \xrightarrow{\theta} M_2$. Thus we have a morphism

$$\alpha(\pi_1^*M) = (M_1, M_1') \overset{(\theta,0)}{\to} (M_2, M_2') = \alpha(\pi_2^*M)$$

and so there is a homomorphism $\psi\colon \pi_1^*M \to \pi_2^*M$, because α is an equivalence. By the Beck condition the canonical homomorphism $\gamma\colon \oplus_{\pi_2} \pi_1^*M \to I^* \oplus_I M$ is an isomorphism, and hence we have the following natural isomorphisms:

$$\begin{array}{cccl} \pi_1^*M & \overset{\psi}{\to} & \pi_2^*M & \text{in } \mathrm{Mod}_R(E)^{I\times I}; \\ \bigoplus\limits_{\pi_2} \pi_1^*M & \to & M & \text{in } \mathrm{Mod}_R(E)^I; \\ I^* \bigoplus\limits_I M & \overset{\bar\phi}{\to} & M & \text{in } \mathrm{Mod}_R(E)^I; \\ \bigoplus\limits_I M & \to & \prod\limits_I M & \text{in } \mathrm{Mod}_R(E). \end{array}$$

If η' is the unit for $\oplus_{\pi_2} \to \pi_2^*$ and ε is the counit for $I^* \vdash \prod_I$, then ψ and ϕ are related by the equation $(\pi_2^*\bar\phi)(\pi_2^*\gamma)\eta'_{\pi_1^*M} = \psi$, where $\bar\phi = \varepsilon_M(I^*\phi)$. On the other hand the canonical morphism γ satisfies $(\pi_2^*\gamma)\eta'_{\pi_1^*M} = \pi_1^*\eta_M$, where η is the unit for $I^* \vdash \prod_I$, so we have $(\pi_2^*\bar\phi)(\pi_1^*\eta_M) = \psi$. Apply δ^* to this equality to get $\bar\phi\eta_M = \delta^*\psi = 1_M$ (by definition of ψ), i.e. $\varepsilon_M(I^*\phi)\eta_M = 1_M$.

The main result of this paper is as follows.

4. Theorem. *Let X be a decidable object in E and M be an object in* $\mathrm{Mod}_R(E)^X$. *Then the homomorphism $\oplus_X M \overset{\phi}{\to} \prod_X M$, defined above, is a monomorphism.*

Proof. Let $T \overset{m}{\to} \oplus_X M$ be a T-element of $\oplus_X M$ such that $\phi(m) = 0$. We have to show that $m = 0$. By Lemma 1, $\oplus_X M \simeq \varinjlim_C \oplus_{[p]} x^*\pi_2^*M$, where $[p] \overset{x}{\to} I^*X$ is in E/I and $C = E_{\text{fin}}/X$. Apply the first part of Theorem 2 to the filtered indexed category E_{fin}/X and the indexed functor P, defined above (we can do that because filtered colimits in $\mathrm{Mod}_R(E)$ are the same as in E) to get $L \overset{\alpha}{\twoheadrightarrow} T$, $p\colon L \to N$, $x\colon [p] \to L^*X$, and $1 \to \oplus_{[p]} x^*\pi_2^*M$ such that $\alpha^*m = i_x y$, where $i_x\colon P^L(x) \to L^* \varinjlim_C P$ is the indexed injection. But by the properties of colimit the following diagram commutes:

$$\begin{array}{ccc} [p]^*L^* \varinjlim\limits_C \bigoplus\limits_{[p]} x^*\pi_2^*M \simeq x^*\pi_2^*X^* \bigoplus\limits_X M & \xrightarrow{x^*\pi_2^*X^*\phi} & x^*\pi_2^*X^* \prod\limits_X M \\ \uparrow {\scriptstyle i_{[p]^*x} = [p]^*i_x} & \searrow {\scriptstyle x^*\pi_2^*\bar\phi} & \downarrow {\scriptstyle x^*\pi_2^*\varepsilon_M} \\ [p]^* \bigoplus\limits_{[p]} x^*\pi_2^*M & \xrightarrow{\quad\bar\phi\quad} & x^*\pi_2^*M. \end{array}$$

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Since $x^*\pi_2^*X^* = [p]^*L^*$, then by transpose of the above diagram along $[p]^*$ we get

$$\begin{array}{ccc}
L^* \varinjlim_{C} \bigoplus_{[p]} x^*\pi_2^*M \simeq L^* \bigoplus_{X} M & \xrightarrow{L^*\phi} & L^*\prod_{X} M \\
\uparrow{\scriptstyle i_x} & & \downarrow{\scriptstyle j_x} \\
\bigoplus_{[p]} x^*\pi_2^*M & \longrightarrow & \prod_{[p]} x^*\pi_2^*M,
\end{array}$$

where j_x is the transpose of $x^*\pi_2^*\varepsilon_M$. Now, by Theorem 2.3 [5],

$$\bigoplus_{[p]} x^*\pi_2^*M \xrightarrow{\phi} \prod\nolimits_{[p]} x^*\pi_2^*M$$

is an isomorphism. Therefore $j_x L^*\phi(\alpha^*m) = j_x L^*\phi(i_x y) = \phi(y) = 0$ implies $y = 0$, so $\alpha^*m = 0$. But α is an epimorphism and hence $m = 0$, as required.

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