Canad. J. Math. Vol. **55** (1), 2003 pp. 181–203

Homotopy Decompositions Involving the Loops of Coassociative Co-H Spaces

Stephen D. Theriault

Abstract. James gave an integral homotopy decomposition of $\Sigma\Omega\Sigma X$, Hilton-Milnor one for $\Omega(\Sigma X \vee \Sigma Y)$, and Cohen-Wu gave p-local decompositions of $\Omega\Sigma X$ if X is a suspension. All are natural. Using idempotents and telescopes we show that the James and Hilton-Milnor decompositions have analogues when the suspensions are replaced by coassociative co-H spaces, and the Cohen-Wu decomposition has an analogue when the (double) suspension is replaced by a coassociative, cocommutative co-H space.

1 Introduction

The thrust of this paper is to show that many common decompositions involving loop spaces which are valid for suspensions are also valid for coassociative co-H spaces. This is done through a straightforward use of idempotents and telescopes. In particular, we consider decomposition theorems of James, Hilton-Milnor, and Cohen-Wu. As the methods we use are general, they should be applicable in other contexts.

Suppose X is a simply connected space. One consequence of the James construction is a homotopy equivalence

$$\Sigma\Omega\Sigma X \simeq \bigvee_{k=1}^{\infty} \Sigma X^{(k)}$$

which is natural for maps $X \xrightarrow{f} Y$. We prove the following generalization.

Theorem 1.1 *Let A be a simply connected, homotopy coassociative co-H space. Then there is a homotopy equivalence*

$$\Sigma\Omega A \simeq \bigvee_{k=1}^{\infty} M_k,$$

where $\Sigma^{k-1}M_k \simeq A^{(k)}$. This is natural for co-H maps $A \xrightarrow{f} B$ between coassociative co-H spaces. Further, each M_k is a homotopy coassociative co-H space, and if A is homotopy cocommutative then so is each M_k.

Received by the editors October 26, 2001; revised March 21, 2002.
AMS subject classification: Primary: 55P35; secondary: 55P45.

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

The Hilton-Milnor theorem is a decomposition

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \prod_{\alpha \in \mathfrak{I}} \Omega\Sigma(X^{(\alpha_1)} \wedge Y^{(\alpha_2)}),$$

where each α corresponds to an iterated Whitehead product $\Sigma X^{(\alpha_1)} \wedge Y^{(\alpha_2)} \to \Sigma X \vee \Sigma Y$. This generalizes to:

Theorem 1.2 *If A and B are simply connected, homotopy coassociative co-H spaces then there is a homotopy equivalence*

$$\Omega(A \vee B) \simeq \prod_{\alpha \in \mathfrak{I}} \Omega N_\alpha,$$

where $\Sigma^{\alpha_1+\alpha_2-1} N_\alpha \simeq A^{(\alpha_1)} \wedge B^{(\alpha_2)}$, and each α corresponds to a "Whitehead product" $N_\alpha \to A \vee B$. This is natural for co-H maps $A \xrightarrow{f} C$ and $B \xrightarrow{g} D$ between coassociative co-H spaces. Further, each N_α is a homotopy coassociative co-H space and if at least one of A or B is homotopy cocommutative then so is N_α.

Note that for any simply connected spaces A and B there is a homotopy equivalence $\Omega(A \vee B) \simeq \Omega A \times \Omega B \times \Omega(\Sigma\Omega A \wedge \Omega B)$. So if A and B are homotopy coassociative co-H spaces then Theorem 1.1 allows one to iteratively obtain a product decomposition as in Theorem 1.2. The problem, as in the usual case when we deal with spaces which are suspensions, is to keep track of the terms in the decomposition. This is the purpose of the Hilton-Milnor theorem and Theorem 1.2.

Both Theorems 1.1 and 1.2 are valid integrally. Localizing at a prime p we can prove natural loop space decomposition theorems. Cohen and Wu [CW] prove that if X is a suspension and $(p, k) = 1$ then there is a map $\phi_k \colon U_k \to \Omega\Sigma X$ which in mod-p homology is an injection onto the module of primitives of tensor length k, and a natural decomposition $\Omega\Sigma X \simeq \Omega\Sigma U_k \times F_k$. This can be generalized by the following theorem. Recall that if A is a co-H space then $H_*(\Omega A; \mathbf{Z}/p\mathbf{Z})$ is isomorphic to a tensor algebra generated by $\Sigma^{-1}\tilde{H}_*(A; \mathbf{Z}/p\mathbf{Z})$.

Theorem 1.3 *Localize spaces and maps at a prime p. Let $(p, k) = 1$. Suppose A is a homotopy coassociative, cocommutative co-H space. Then there is a map $\phi_k \colon U_k \to A$ with $(\Omega\phi_k)_*$ an isomorphism onto the subalgebra of $H_*(\Omega A; \mathbf{Z}/p\mathbf{Z})$ generated by the primitives of tensor length k, and a natural homotopy decomposition*

$$\Omega A \simeq \Omega U_k \times F_k.$$

The tool for proving Theorems 1.1, 1.2, and 1.3 is the following proposition. For spaces $X_1, \dots, X_k$ let $\bigwedge_{i=1}^k X_i = X_1 \wedge \cdots \wedge X_k$.

Proposition 1.4 *Suppose $A_1, \dots, A_k$ are coassociative co-H spaces. Then there is a coassociative co-H space L_k and a co-H map $l_k \colon L_k \to \Sigma \bigwedge_{i=1}^k (\Omega A_i)$ satisfying the following properties:*

(a) *l_k has a left homotopy inverse,*
(b) *l_k is natural for co-H maps $A_i \xrightarrow{\varepsilon_i} B_i$ between coassociative co-H spaces,*
(c) $\Sigma^{k-1} L_k \simeq \bigwedge_{i=1}^{k} A_i$,
(d) *If $A_i = \Sigma Y_i$ for each i then $L_k \simeq \Sigma \bigwedge_{i=1}^{k} Y_i$,*
(e) *If one of A_1 through A_k is homotopy cocommutative then so is L_k.*

A notable corollary is the $k = 2$ case of Proposition 1.4(c).

Corollary 1.5 *The smash product of two coassociative co-H spaces is a suspension. In fact, it is the suspension of another coassociative co-H space.* ■

The space L_k is constructed as the telescope of an idempotent on $\Sigma \bigwedge_{i=1}^{k} (\Omega A_i)$ which is constructed by sequentially evaluating $\Sigma\Omega A_i \to A_i$ and including $A_i \to \Sigma\Omega A_i$ for each $1 \leq i \leq k$. The space M_k which appears in Theorem 1.1 is a renaming of L_k in the case when $A_1, \dots, A_k$ all equal the same coassociative co-H space A.

It should be emphasized that Theorems 1.1, 1.2, and 1.3 extend not just the list of spaces for which the respective decompositions hold, but also the list of maps. In particular, the theorems apply to maps $\Sigma X \xrightarrow{f} \Sigma Y$ which are co-H but not necessarily suspensions.

The combination of coassociativity and cocommutativity in Theorem 1.3 comes from needing to perform the arithmetic necessary to show that certain self-maps on the M_k's are idempotents (see Section 7 for details). In Theorems 1.1 and 1.2 the coassociative hypothesis is necessary. As an example, consider the (p-localized) cofibration

$$S^{2p} \xrightarrow{\alpha_1} S^3 \longrightarrow C,$$

where α_1 is the first nontrivial homotopy class. The standard argument for the co-H deviation of a map easily shows that α_1 is co-H. Thus C is a co-H space. To show that C is not homotopy coassociative, Berstein [B1] argues as follows (using mod-p coefficients in homology). If A is a co-H space then $H_*(\Omega A)$ is isomorphic as an algebra to the tensor algebra primitively generated by $\Sigma^{-1}\tilde{H}_*(A)$. If A has a homotopy coassociative comultiplication $\Delta\colon A \to A \vee A$ then by [B2] algebra generators $\{a_1, \dots, a_n, \dots\} \in H_*(\Omega A)$ can be chosen to satisfy $(\Omega\Delta)_*(a_i) = a_i \otimes 1 + 1 \otimes a_i + \Sigma_{\alpha_{j,k}} a_j \otimes a_k$ for some coefficients $\alpha_{j,k} \in \mathbf{Z}/p\mathbf{Z}$. In particular, $H_*(\Omega A)$ is primitively generated if $\alpha_{j,k} = 0$ for all j and k. Applying this to C, we see that $H_*(\Omega C)$ has two algebra generators, one in dimension 2 and one in dimension $2p$. For dimensional reasons, the coefficients $\alpha_{j,k}$ are all zero and so $H_*(\Omega C)$ is primitively generated. The condition of being primitively generated implies that p-th-powers in $H^*(\Omega A)$ are zero. Now consider the generator $x \in H^2(\Omega C)$. For dimensional reasons $\mathcal{P}^1(x) = x^p$ and so $\mathcal{P}^1(x) = 0$. On the other hand, the top and bottom cell in C are connected by $\mathcal{P}^1$, and since C is a retract of $\Sigma\Omega C$ we must have $\mathcal{P}^1(x) \neq 0$ in $H^*(\Omega C)$. The contradiction shows that C cannot be homotopy coassociative. It also shows that we do have $\mathcal{P}(x) = x^p \neq 0$. This nontrivial p-th-power then shows that Theorem 1.1 cannot hold for $\Sigma\Omega C$, as $\sigma(x)$ and $\sigma(x^p)$ would appear in different wedge summands but be connected by a Steenrod operation. As

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

for Theorem 1.2, including the wedge into the product gives a homotopy fibration $\Sigma\Omega C \wedge \Omega S^2 \to C \vee S^2 \to C \times S^2$. Here, $\Sigma\Omega C \wedge \Omega S^2$ is homotopy equivalent to a wedge of suspensions of $\Sigma\Omega C$. Theorem 1.2 would decompose these suspensions of $\Sigma\Omega C$ as in Theorem 1.1, which as we already have seen, cannot happen, even stably.

One interpretation of Theorem 1.1 is as follows. If A is a co-H space then [R] shows there is a homotopy equivalence $A \vee (A \wedge \Omega A) \xrightarrow{s \perp t} \Sigma\Omega A$ where t factors through the Hopf construction. Iterating this on $A \wedge \Omega A$ gives a homotopy decomposition

$$\Sigma^k \Omega A \simeq \Big(\bigvee_{i=1}^{k} \Sigma^{k-i} A^{(i)} \Big) \vee (A^{(k)} \wedge \Omega A).$$

In this context, Theorem 1.1 is a desuspension theorem. When A is coassociative, ΩA has a complete wedge decomposition after only a single suspension. Similar statements can be made about Theorems 1.2 and 1.3.

The results in this paper may be useful elsewhere. In [T] several of the constructions appearing here are used to help prove critical Lie algebra properties of the lifts of some mod-p homotopy classes to a certain coassociative, cocommutative space G_k. In another direction, Berstein and Harper [BH] are concerned with proving the existence of coassociative co-H spaces which are not suspensions. If A is such a space then Proposition 1.4 generates a supply of coassociative co-H spaces. It is not clear, however, when the L_k's are not suspensions.

This paper is organized as follows. Section 2 records some facts about co-H spaces while Section 3 does the same for idempotents and telescopes. In Section 4 we construct the idempotent which has the space L_k as its telescope and prove the properties of L_k in Proposition 1.4. Section 5 proves the generalized James decomposition of Theorem 1.1 and Section 6 proves the generalized Hilton-Milnor of Theorem 1.2. Section 7 constructs idempotents on the spaces M_k of Theorem 1.1 and these are used in Section 8 to prove the Cohen-Wu decomposition of Theorem 1.3.

Finally, note that unless otherwise indicated all statements in the paper are valid integrally. The only assumption we make from here on is that our co-H spaces are simply connected.

2 Preliminaries on Co-H Spaces

This section records some facts about co-H spaces. The first two describe the homology of looped co-H spaces, the remainder describe homotopical properties.

Lemma 2.1 *Let A be a co-H space. Then $H_*(\Omega A)$ is isomorphic as an algebra to the tensor algebra $T\big(\Sigma^{-1}\tilde{H}_*(A)\big)$ primitively generated by $\Sigma^{-1}\tilde{H}_*(A)$.*

Proof See [B2]. ∎

Lemma 2.2 *Let A be a coassociative, cocommutative co-H space. Give $T\big(\Sigma^{-1}\tilde{H}_*(A)\big)$ a coalgebra structure by requiring that the set of generators is primitive and then multiplicatively extending to the whole tensor algebra. Then $H_*(A)$ is isomorphic as a Hopf algebra to $T\big(\Sigma^{-1}\tilde{H}_*(A)\big)$.*

Proof Berstein [B2] does not explicitly state this, but it is a consequence of combining his Corollary 2.6 and Corollary 3.3. ■

Lemma 2.3 *Let A be a co-H space.*

(a) *There is a one-to-one correspondence between co-H structures on A and maps $A \to \Sigma\Omega A$ which are right homotopy inverses to the evaluation map $\Sigma\Omega A \to A$.*
(b) *The co-H structure on A is homotopy coassociative if and only if the corresponding map $A \to \Sigma\Omega A$ in part (a) is a co-H map.*

Proof See [Ga]. ■

Lemma 2.4 *Suppose A and B are co-H spaces and $f\colon A \to B$ is a co-H map. Then there is a homotopy commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"] & B \arrow[d, "t"] \\ \Sigma\Omega A \arrow[r, "\Sigma\Omega f"] & \Sigma\Omega B \end{tikzcd}$$

where s and t correspond as in Theorem 2.3 to the co-H structures on A and B respectively.

Proof See, for example, [Gr1, 3.6]. ■

Lemma 2.5 *Suppose A retracts off a homotopy cocommutative co-H space X. Give A the co-H structure determined by this retraction. Then A is also homotopy cocommutative.*

Proof Let $\Delta\colon X \to X \vee X$ be the comultiplication on X. Let $\tau\colon X \vee X \to X \vee X$ be the twist map. Consider the diagram

$$\begin{tikzcd} A \arrow[r] & X \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & X \vee X \arrow[r] \arrow[d, "\tau"] & A \vee A \arrow[d, "\tau"] \\ & & X \vee X \arrow[r] & A \vee A. \end{tikzcd}$$

The middle triangle homotopy commutes because X is homotopy cocommutative. The right square homotopy commutes because τ is natural. The top row defines the co-H structure Δ_A on A. The diagram now shows that $\tau \circ \Delta_A \simeq \Delta_A$. ■

Two co-H spaces A and B are *co-H equivalent* if there is a map $f\colon A \to B$ which is both a co-H map and a homotopy equivalence. Note that, as a formal consequence, the inverse $f^{-1}\colon B \to A$ is also a co-H map.

Lemma 2.6 *Let A, B, and C be co-H spaces. Suppose there are co-H maps $A \to B$ and $A \to C$ such that their sum $A \to B \vee C$ is a co-H equivalence, where the co-H structure on $B \vee C$ is from the wedge. If A is homotopy coassociative (cocommutative) then the given co-H structures on B and C are also homotopy coassociative (cocommutative).*

Proof We prove the statements for B, those for C following symmetrically. Let $\Delta_A\colon A \to A \vee A$ be the co-H structure on A and similarly define Δ_B for B. Let $f\colon A \to B$ be the given co-H map. The homotopy equivalence $A \to B \vee C$ lets us define a left homotopy inverse $g\colon B \to A$ of f.

The given co-H structure Δ_B on B is homotopic to the composite

$$B \xrightarrow{g} A \xrightarrow{\Delta_A} A \vee A \xrightarrow{f \vee f} B \vee B$$

because $(f \vee f) \circ \Delta_A \circ g \simeq \Delta_B \circ f \circ g \simeq \Delta_B$.

For any space X, let $\tau\colon X \vee X \to X \vee X$ be the twist map. If A is homotopy cocommutative then by definition there is a homotopy $\Delta_A \simeq \tau \circ \Delta_A$. This implies $\tau \circ \Delta_B \simeq \tau \circ (f \vee f) \circ \Delta_A \circ g \simeq (f \vee f) \circ \tau \circ \Delta_A \circ g \simeq (f \vee f) \circ \Delta_A \circ g \simeq \Delta_B$. Thus B is also homotopy cocommutative.

If A is homotopy coassociative then by definition there is a homotopy $(1 \vee \Delta_A) \circ \Delta_A \simeq (\Delta_A \vee 1) \circ \Delta_A$. Arguing as in the cocommutative case, it follows that B is also homotopy coassociative. ■

3 Preliminaries on Telescopes of Idempotents

This section reviews some basic facts about the telescopes of idempotent maps. We will usually denote the telescope of a self-map $e\colon X \to X$ by $T(e)$.

Lemma 3.1 *Suppose Y is a space and $e\colon Y \to Y$ is an idempotent. Let $T(e)$ be the telescope of e and $j_e\colon Y \to T(e)$ be the canonical map. Then up to homotopy there is a unique map $j_e\colon T(e) \to Y$ such that $i_e \circ j_e \simeq e$ and $j_e \circ i_e$ is homotopic to the identity on $T(e)$. Further, if e is a co-H map then so are i_e and j_e.*

Proof See [Gr2]. ■

The telescopes of idempotents also satisfy a naturality property.

Lemma 3.2 *Let $f\colon Y \to Z$ be a map. Suppose $e_Y\colon Y \to Y$ and $e_Z\colon Z \to Z$ are idempotents such that $f \circ e_Y \simeq e_Z \circ f$. Then there is a homotopy commutative diagram*

$$\begin{array}{ccccc}
Y & \longrightarrow & T_{(e_Y)} & \longrightarrow & Y \\
\big\downarrow f & & \big\downarrow T(f) & & \big\downarrow f \\
Z & \longrightarrow & T_{(e_Z)} & \longrightarrow & Z,
\end{array}$$

where the top row is homotopic to e_Y, the bottom row is homotopic to e_Z, and $T(f)$ is an induced map of telescopes. Further, $T(f)$ is the unique map between the telescopes which makes the above diagram commute.

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

Proof The outer rectangle commutes by the hypothesis $f \circ e_Y \simeq e_Z \circ f$. The left square commutes by naturality in taking telescopes. Now precomposing the outer rectangle with $T(e_Y) \to Y$ shows that the right square commutes.

If $g\colon T(e_Y) \to T(e_Z)$ were another choice of an induced map between telescopes, then the same argument gives a homotopy commutative diagram as asserted by the lemma with $T(f)$ replaced by g. But then we have a homotopy commutative diagram

$$\begin{tikzcd} T(e_Y) \arrow[r] \arrow[d, "g"] & Y \arrow[r] \arrow[d, "f"] & T(e_Y) \arrow[d, "T(f)"] \\ T(e_Z) \arrow[r] & Z \arrow[r] & T(e_Z), \end{tikzcd}$$

where the top and bottom rows are both homotopic to the respective identity maps. Hence $g \simeq T(f)$. ∎

We will also need some additional information that comes out of Lemma 3.2 in two special cases.

Lemma 3.3 *Given the same setup as in Lemma 3.2.*

(a) *If $Y \xrightarrow{f} Z$ has a left homotopy inverse then so does $T(f)$.*

(b) *If $Y \xrightarrow{f} Z$ is a co-H map between co-H spaces then $T(e_Y) \xrightarrow{T(f)} T(e_Z)$ is also a co-H map between co-H spaces.*

Proof For part (a), assume f has a left homotopy inverse. Precompose the outer rectangle in the statement of Lemma 3.2 with the map $T(e_Y) \to Y$ and postcompose with $Z \to Y \to T(e_Y)$; a diagram chase shows that $T(f)$ also has a left homotopy inverse.

For part (b), the co-H property of f and Lemma 3.2 imply there is a homotopy commutative diagram,

$$\begin{tikzcd} T(e_Y) \arrow[r] \arrow[d, "T(f)"] & Y \arrow[r] \arrow[d, "f"] & Y \vee Y \arrow[r] \arrow[d, "f\vee f"] & T(e_Y) \vee T(e_Y) \arrow[d, "T(f)\vee T(f)"] \\ T(e_Z) \arrow[r] & Z \arrow[r] & Z \vee Z \arrow[r] & T(e_Z) \vee T(e_Z). \end{tikzcd}$$

The top row defines the co-H structure on $T(e_Y)$ from the retraction off Y, and similarly for $T(e_Z)$ along the bottom row. The diagram as a whole says that $T(f)$ is a co-H map. ∎

4 Construction and Properties of the Space L_k

The strategy of the proof is to construct the space L_k as the telescope of a certain idempotent on $\Sigma \bigwedge_{i=1}^{k}(\Omega A_i)$. It first takes a bit of work to identify the self-map on $\Sigma \bigwedge_{i=1}^{k}(\Omega A_i)$ as an idempotent. If A is a suspension then the identification is easy. If

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

A is not a suspension but is a coassociative co-H space then we use a co-H map $A \to \Sigma\Omega A$ to extrapolate from the easy case of the suspension $\Sigma\Omega A$ to the non-immediate case of A. The assorted properties of L_k are then extracted from the construction.

To begin, let X be a space. Let $\mathrm{ev}\colon \Sigma\Omega X \to X$ be the evaluation map. Let $j\colon X \to \Omega\Sigma X$ be the inclusion. Let $\bar{\lambda}$ be the composite

$$\bar{\lambda}\colon \Sigma\Omega\Sigma X \xrightarrow{\mathrm{ev}} \Sigma X \xrightarrow{\Sigma j} \Sigma\Omega\Sigma X.$$

Suppose $X_1, \ldots, X_k$ are spaces. Let $\bigwedge_{i=1}^k X_i = X_1 \wedge \cdots \wedge X_k$. For $1 \le i \le k$, define

$$\lambda_{k,i}\colon \Sigma \bigwedge_{i=1}^{k} (\Omega\Sigma X_i) \to \Sigma \bigwedge_{i=1}^{k} (\Omega\Sigma X_i)$$

as follows. Move the suspension coordinate so it is paired with the i-th smash factor. Do the identity map on the first $i-1$ and last $k-i$ smash factors, and do $\bar{\lambda}$ on the i-th smash factor. Define

$$\lambda_k\colon \Sigma \bigwedge_{i=1}^{k} (\Omega\Sigma X_i) \to \Sigma \bigwedge_{i=1}^{k} (\Omega\Sigma X_i)$$

by $\lambda_k = \lambda_{k,k} \circ \lambda_{k,k-1} \circ \cdots \circ \lambda_{k,1}$.

Lemma 4.1 *λ_k is an idempotent (in fact, a strict idempotent: $(\lambda_k)^2$ equals λ_k on the nose, no homotopy is involved). The telescope of λ_k is homotopy equivalent to $\Sigma \bigwedge_{i=1}^k X_i$. λ_k is natural for co-H maps $\varepsilon_i\colon \Sigma X_i \to \Sigma Y_i$.*

Proof First consider $\bar{\lambda}$. It's image is ΣX included—via Σj—into $\Sigma\Omega\Sigma X$. But $\bar{\lambda}$ is the identity when restricted to ΣX, and so $\bar{\lambda}$ is a strict idempotent whose telescope is homotopy equivalent to ΣX. In the same way λ_k has image $\Sigma \bigwedge_{i=1}^k X_i$ included—via $\Sigma \bigwedge_{i=1}^k j_i$—into $\Sigma \bigwedge_{i=1}^k (\Omega\Sigma X_i)$ and λ_k is the identity when restricted to $\Sigma \bigwedge_{i=1}^k X_i$. So λ_k is a strict idempotent whose telescope is homotopy equivalent to $\Sigma \bigwedge_{i=1}^k X_i$.

The naturality of the evaluation map together with Lemma 2.4 applied to a co-H map $\Sigma X \to \Sigma Y$ implies that $\bar{\lambda}$ is natural. Thus each $\lambda_{k,i}$ is natural and so λ_k is as well. ∎

We now construct the analogues of $\bar{\lambda}$, $\lambda_{k,i}$, and λ_k in which the suspended spaces have been replaced by coassociative co-H spaces. Let A be a coassociative co-H space. To simplify notation, let $X = \Omega A$. By Theorem 2.3 the coassociative co-H structure on A corresponds to a co-H map $s\colon A \to \Sigma X$ which is a right homotopy inverse of the evaluation map $\Sigma X = \Sigma\Omega A \xrightarrow{\mathrm{ev}} A$. Let $\bar{\gamma}$ be the composite

$$\bar{\gamma}\colon \Sigma\Omega A \xrightarrow{\mathrm{ev}} A \xrightarrow{s} \Sigma\Omega A.$$

Consider the following diagram,

$$\begin{tikzcd}
\Sigma\Omega A \arrow[r, "\mathrm{ev}"] \arrow[d, "\Sigma\Omega s"] & A \arrow[r, "s"] \arrow[d, "s"] & \Sigma\Omega A \arrow[d, "\Sigma\Omega s"] \\
\Sigma\Omega\Sigma X \arrow[r, "\mathrm{ev}"] & \Sigma X \arrow[r, "\Sigma j"] & \Sigma\Omega\Sigma X.
\end{tikzcd}$$

The left square homotopy commutes by the naturality of the evaluation map. The right square homotopy commutes by Lemma 2.4 since s is a co-H map. Note that the top row is the definition of $\bar{\gamma}$ and the bottom row is the definition of $\bar{\lambda}$.

Suppose $A_1, \ldots, A_k$ are coassociative co-H spaces. Let $X_i = \Omega A_i$. By Theorem 2.3 the coassociative co-H structure on A_i corresponds to a co-H map $s_i \colon A_i \to \Sigma X_i$ which is a right homotopy inverse of the evaluation map $\mathrm{ev}_i \colon \Sigma X_i = \Sigma\Omega A_i \xrightarrow{\mathrm{ev}} A_i$. The diagram above linking $\bar{\gamma}$ and $\bar{\lambda}$ then implies there is a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma \bigwedge_{i=1}^{k}(\Omega A_i) \arrow[r, "\gamma_k"] \arrow[d, "\Sigma f"] & \Sigma \bigwedge_{i=1}^{k}(\Omega A_i) \arrow[d, "\Sigma f"] \\
\Sigma \bigwedge_{i=1}^{k}(\Omega\Sigma X_i) \arrow[r, "\lambda_k"] & \Sigma \bigwedge_{i=1}^{k}(\Omega\Sigma X_i),
\end{tikzcd}$$

where $f = \bigwedge_{i=1}^{k}(\Omega s_i)$.

Lemma 4.2 *γ_k is an idempotent. It is natural for co-H maps $\varepsilon_i \colon A_i \to B_i$ between coassociative co-H spaces.*

Proof The previous diagram implies $\Sigma f \circ \gamma_k \simeq \lambda_k \circ \Sigma f$ and also $\Sigma f \circ (\gamma_k)^2 \simeq (\lambda_k)^2 \circ \Sigma f$. But $(\lambda_k)^2 \simeq \lambda_k$ and so $\Sigma f \circ \gamma_k \simeq \Sigma f \circ (\gamma_k)^2$. Since $f = \bigwedge_{i=1}^{k}(\Omega s_i)$ has left homotopy inverse $\bigwedge_{i=1}^{k}(\Omega\,\mathrm{ev}_i)$, we obtain $(\gamma_k)^2 \simeq \gamma_k$.

The naturality argument is exactly as it is in Lemma 4.1 with λ's replaced by γ's. ■

Let $L_k = T(\gamma_k)$ be the telescope of γ_k. Since γ_k is an idempotent, Lemma 3.1 gives a map $l_k \colon L_k \to \Sigma \bigwedge_{i=1}^{k}(\Omega A_i)$ which has a left homotopy inverse. This fact and more are recorded in the following Lemma.

Lemma 4.3 *For $k \geq 1$, the map $L_k \xrightarrow{l_k} \Sigma \bigwedge_{i=1}^{k}(\Omega A_i)$ has a left homotopy inverse and is co-H. Further, if there are co-H maps $A_i \to B_i$ between coassociative co-H spaces then l_k is natural and the map of telescopes $L_k(A) \to L_k(B)$ is co-H.*

Proof The naturality of γ_k in Lemma 4.2 implies that l_k is natural, and Lemma 3.3(b) implies the induced map between telescopes is co-H.

It remains to prove that l_k is co-H. Essentially this comes from applying naturality to $\{A_i\}_{i=1}^{k}$ and $\{B_i = \Sigma\Omega A_i\}_{i=1}^{k}$, noting that by Lemma 4.1 $L_k(B) = \Sigma \bigwedge_{i=1}^{k} \Omega A_i$.

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

However, we need to check that the co-H telescope map is homotopic to the original map l_k.

If A is a co-H space then Lemma 2.3 implies that $\Omega A \xrightarrow{\Omega s} \Omega\Sigma\Omega A \xrightarrow{\Omega\,\mathrm{ev}} \Omega A$ is homotopic to the identity map. On the other hand, the composite $\Omega A \xrightarrow{j} \Omega\Sigma\Omega A \xrightarrow{\Omega\,\mathrm{ev}} \Omega A$ is also homotopic to the identity map.

Consider the diagram

$$\begin{tikzcd}
L_k \arrow[r, "l_k"] \arrow[d, "T(\Sigma f)"] & \Sigma \bigwedge_{i=1}^k (\Omega A_i) \arrow[d, "\Sigma f"] \arrow[dr, equal] & \\
\Sigma \bigwedge_{i=1}^k X_i \arrow[r, "\Sigma J"] & \Sigma \bigwedge_{i=1}^k (\Omega\Sigma X_i) \arrow[r, "\Sigma p"] & \Sigma \bigwedge_{i=1}^k (\Omega A_i).
\end{tikzcd}$$

The square commutes from an application of Lemma 3.2 to the idempotents γ_k and λ_k. Here, $T(\Sigma f)$ is the induced map of telescopes and J is the inclusion $\bigwedge_{i=1}^k j_i$. The triangle commutes because we define p as $\bigwedge_{i=1}^k (\Omega\,\mathrm{ev}_i)$, which is a left homotopy inverse of $f = \bigwedge_{i=1}^k (\Omega s_i)$. But p is also a left homotopy inverse of J. Thus $l_k \simeq T(\Sigma f)$. On the other hand, by Lemma 3.3(b), $T(\Sigma f)$ is a co-H map. ■

The next few lemmas describe some properties of L_k.

Lemma 4.4 *For $k \geq 1$, $\Sigma^{k-1} L_k \simeq \bigwedge_{i=1}^k A_i$. As well, there is a homotopy commutative diagram*

$$\begin{tikzcd}
\Sigma^{k-1} L_k \arrow[r, "\Sigma^{k-1} l_k"] \arrow[d, "\simeq"] & \Sigma^{k-1}\left(\Sigma \bigwedge_{i=1}^k (\Omega A_i)\right) \arrow[d, "\simeq"] \\
\bigwedge_{i=1}^k A_i \arrow[r, "\bigwedge_{i=1}^k s_i"] & \bigwedge_{i=1}^k (\Sigma\Omega A_i).
\end{tikzcd}$$

Proof To make the exposition as clear as possible, we do the $k = 2$ case, the others being similar.

Recall that $\lambda_2 = \lambda_{2,2} \circ \lambda_{2,1}$ where $\lambda_{2,1}$ is the composite $\Sigma\Omega\Sigma X_1 \wedge \Omega\Sigma X_2 \xrightarrow{\mathrm{ev} \wedge 1} \Sigma X_1 \wedge \Omega\Sigma X_2 \xrightarrow{\Sigma j \wedge 1} \Sigma\Omega\Sigma X_1 \wedge \Omega\Sigma X_2$ and $\lambda_{2,2}$ is the composite $\Sigma\Omega\Sigma X_1 \wedge \Omega\Sigma X_2 \xrightarrow{1 \wedge \mathrm{ev}} \Omega\Sigma X_1 \wedge \Sigma X_2 \xrightarrow{1 \wedge \Sigma j} \Sigma\Omega\Sigma X_1 \wedge \Omega\Sigma X_2$. The suspension coordinate in the middle term of both composites means we can rearrange the order of evaluations and inclusions so that λ_2 is homotopic to the composite

$$\Sigma\Omega\Sigma X_1 \wedge \Omega\Sigma X_2 \xrightarrow{\xi} \Sigma X_1 \wedge X_2 \xrightarrow{\Sigma j \wedge j} \Sigma\Omega\Sigma X_1 \wedge \Omega\Sigma X_2,$$

where ξ first evaluates on the left and then evaluates on the right.

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

This suspension coordinate is not present in the middle term $A_1 \wedge \Omega A_2$ of $\gamma_{2,1}$ and $\gamma_{2,2}$ so there is no analogous homotopy. There is, however, after suspending. Now we have $\Sigma\gamma_2 = \Sigma(\gamma_{2,2} \circ \gamma_{2,1})$ homotopic to the composite

$$\delta\colon \Sigma\Omega A_1 \wedge \Sigma\Omega A_2 \xrightarrow{\mathrm{ev}\wedge\mathrm{ev}} A_1 \wedge A_2 \xrightarrow{s_1\wedge s_2} \Sigma\Omega A_1 \wedge \Sigma\Omega A_2.$$

Thus the telescope ΣL_2 of $\Sigma\gamma_2$ is homotopic to the telescope $A_1 \wedge A_2$ of δ. Applying Lemma 3.2 with $Y = Z = \Sigma\Omega A_1 \wedge \Omega A_2$, $e_Y = \Sigma\gamma_2$, and $e_Z = \delta$, we obtain the $k = 2$ case of the homotopy commutative diagram asserted by the lemma. ■

Since each A_i is co-H, Lemma 2.1 implies that $\tilde{H}_*(\bigwedge_{i=1}^k \Omega A_i) \cong \bigotimes_{i=1}^k T\big(\Sigma^{-1}\tilde{H}_*(A_i)\big)$. Lemma 4.4 therefore has the following corollary.

Corollary 4.5 *$\tilde{H}_*(L_k) \cong \Sigma\big(\bigotimes_{i=1}^k \Sigma^{-1}\tilde{H}_*(A_i)\big)$ and the inclusion $L_k \xrightarrow{l_k} \Sigma \bigwedge_{i=1}^k \Omega A_i$ in homology is the tensor algebra inclusion*

$$\Sigma\Big(\bigotimes_{i=1}^{k} \Sigma^{-1}\tilde{H}_*(A_i)\Big) \to \Sigma\Big(\bigotimes_{i=1}^{k} T(\Sigma^{-1}\tilde{H}_*(A_i))\Big). \qquad ■$$

The next lemma makes sure that if the coassociative co-H spaces A_i are already suspensions then the construction of L_k gives what would be expected from Lemma 4.1. Recall that two co-H spaces A and B are *co-H equivalent* if there is a homotopy equivalence $A \xrightarrow{\simeq} B$ which is also a co-H map.

Lemma 4.6 *Suppose each coassociative co-H space A_i is co-H equivalent to ΣY_i. Then L_k is co-H equivalent to $\Sigma \bigwedge_{i=1}^k Y_i$.*

Proof Let $e\colon A \to \Sigma Y$ be a co-H equivalence. Applying Lemma 2.4 implies the commutativity of the right square in the diagram

$$\begin{tikzcd} \Sigma\Omega A \arrow[r, "\mathrm{ev}"] \arrow[d, "\Sigma\Omega e"] & A \arrow[r, "s"] \arrow[d, "e"] & \Sigma\Omega A \arrow[d, "\Sigma\Omega e"] \\ \Sigma\Omega\Sigma Y \arrow[r, "\mathrm{ev}"] & \Sigma Y \arrow[r, "\Sigma j"] & \Sigma\Omega\Sigma Y. \end{tikzcd}$$

The left square homotopy commutes by the naturality of the evaluation map. The top row is the definition of $\bar{\gamma}$; the bottom row is the definition of $\bar{\lambda}$ for ΣY. These two maps were the essential ingredients that went into defining the idempotents γ_k on $\Sigma \bigwedge_{i=1}^k(\Omega A_i)$ and λ_k on $\Sigma \bigwedge_{i=1}^k(\Omega\Sigma Y_i)$ respectively. So if $e_i\colon A_i \to \Sigma Y_i$ is a co-H equivalence for $1 \le i \le k$, then the diagram above implies there is a homotopy commutative diagram

$$\begin{tikzcd} \Sigma \bigwedge_{i=1}^k(\Omega A_i) \arrow[r, "\gamma_k"] \arrow[d, "E"] & \Sigma \bigwedge_{i=1}^k(\Omega A_i) \arrow[d, "E"] \\ \Sigma \bigwedge_{i=1}^k(\Omega\Sigma Y_i) \arrow[r, "\lambda_k"] & \Sigma \bigwedge_{i=1}^k(\Omega\Sigma Y_i), \end{tikzcd}$$

where $E = \Sigma\bigwedge_{i=1}^{k}(\Omega e_i)$ is a co-H equivalence. Hence by Lemma 3.3(b) there is a co-H equivalence of telescopes $L_k = T(\gamma_k) \simeq T(\lambda_k) \simeq \Sigma\bigwedge_{i=1}^{k} Y_i$. ■

Remark The proof of Lemma 4.6 actually works if we only demand that $k-1$ of the A_i's are co-H equivalent to a suspension. Assuming without loss of generality that the exception is A_1, then $L_k \simeq A_1 \wedge (\bigwedge_{i=2}^{k} Y_i)$.

Lemma 4.7 *L_k is a coassociative co-H space.*

Proof By Lemma 4.3, there is a co-H map $L_k \xrightarrow{l_k} \Sigma\bigwedge_{i=1}^{k}(\Omega A_i)$ which has a left homotopy inverse $r\colon \Sigma\bigwedge_{i=1}^{k}(\Omega A_i) \to L_k$. Thus there is a homotopy commutative diagram

$$\begin{tikzcd}
L_k \arrow[r, "l_k"] & \Sigma\bigwedge_{i=1}^{k}(\Omega A_i) \arrow[r, "\Sigma J"] \arrow[dr, equal] & \Sigma\Omega\left(\Sigma\bigwedge_{i=1}^{k}(\Omega A_i)\right) \arrow[r, "\Sigma\Omega r"] \arrow[d, "ev"] & \Sigma\Omega L_k \arrow[d, "ev"] \\
 & & \Sigma\bigwedge_{i=1}^{k}(\Omega A_i) \arrow[r, "r"] & L_k,
\end{tikzcd}$$

where $J = \bigwedge_{i=1}^{k} j_i$. The top row is a composite of co-H maps and so is co-H. The lower direction is homotopic to the identity on L_k. So by Theorem 2.3, L_k is homotopy coassociative. ■

We next show that the inclusion $L_k \xrightarrow{l_k} \Sigma\bigwedge_{i=1}^{k}\Omega A_i$ factors through the telescopes of all the possible sub-idempotents on $\Sigma\bigwedge_{i=1}^{k}\Omega A_i$ formed by sequences of evaluations and inclusions.

Suppose we are given a sequence $b\colon \Sigma\bigwedge_{i=1}^{k}\Omega A_i \to \Sigma\bigwedge_{i=1}^{k}\Omega A_i$ of evaluations $\Sigma\Omega A_i \xrightarrow{\mathrm{ev}} A_i$ and inclusions $A_i \to \Sigma\Omega A_i$. We may assume that any individual factor is evaluated and included only once, since an inclusion followed by an evaluation is homotopic to the identity. Let $a\colon \Sigma\bigwedge_{i=1}^{k}\Omega A_i \to \Sigma\bigwedge_{i=1}^{k}\Omega A_i$ be the sequence of evaluations and inclusions which do not appear in b. So in particular, $b \circ a = \gamma_k \circ \sigma$ where σ is some permutation of the smash factors of $\bigwedge_{i=1}^{k}\Omega A_i$. As with γ_k, the map b is an idempotent. Let $\mathcal{L}$ be its telescope. By Lemma 3.1 there is a factorization

$$\Sigma\bigwedge_{i=1}^{k}\Omega A_i \xrightarrow{g} \mathcal{L} \xrightarrow{h} \Sigma\bigwedge_{i=1}^{k}\Omega A_i,$$

for maps g and h where $b \simeq h \circ g$ and $g \circ h$ is homotopic to the identity on $\mathcal{L}$. Since $\mathcal{L}$ is a retract of a co-H space it too is a co-H space. Also, as in Lemma 4.3, the map h is co-H.

Lemma 4.8 *The co-H inclusion $L_k \xrightarrow{l_k} \Sigma\bigwedge_{i=1}^{k}\Omega A_i$ factors as a composition of co-H maps $L_k \to \mathcal{L} \xrightarrow{h} \Sigma\bigwedge_{i=1}^{k}\Omega A_i$.*

Proof The idea is to construct an idempotent on $\mathcal{L}$ which is compatible with the idempotent γ_k on $\Sigma \bigwedge_{i=1}^k \Omega A_i$, and show their telescopes are homotopy equivalent.

Let $m = g \circ a$ and $\theta = m \circ h$. Then $h \circ \theta = h \circ m \circ h = h \circ g \circ a \circ h \simeq b \circ a \circ h = \gamma_k \circ \sigma \circ h$. Also, $\theta \circ g = m \circ h \circ g \simeq m \circ b = g \circ a \circ b \simeq g \circ \gamma_k \circ \sigma$. These two homotopies say there is a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma \bigwedge_{i=1}^k \Omega A_i \arrow[r, "\gamma_k \circ \sigma"] \arrow[d, "g"] & \Sigma \bigwedge_{i=1}^k \Omega A_i \arrow[d, "g"] \\
\mathcal{L} \arrow[r, "\theta"] \arrow[d, "h"] & \mathcal{L} \arrow[d, "h"] \\
\Sigma \bigwedge_{i=1}^k \Omega A_i \arrow[r, "\gamma_k \circ \sigma"] & \Sigma \bigwedge_{i=1}^k \Omega A_i .
\end{tikzcd}$$

The diagram tells us several things. (1) Since σ is just a permutation of the smash factors of $\Sigma \bigwedge_{i=1}^k \Omega A_i$, the map $\gamma_k \circ \sigma$ is a sequence of evaluations and inclusions just like γ_k, but in a different order. In particular, like γ_k, $\gamma_k \circ \sigma$ is an idempotent with telescope L_k. Since g is a left homotopy inverse of h, the lower square implies $\theta \simeq g \circ \gamma_k \circ \sigma \circ h$. Juxtaposing the lower square with itself also shows that $\theta^2 \simeq g \circ (\gamma_k \circ \sigma)^2 \circ h$. Since $\gamma_k \circ \sigma$ is an idempotent, θ must also be an idempotent. (2) Let $T(\mathcal{L})$ be the telescope of θ. Note that the horizontal composite is $h \circ g \simeq b$. Recall that $\gamma_k \circ \sigma = b \circ a$. Thus when we take telescopes horizontally we obtain a sequence $L_k \to T(\mathcal{L}) \to L_k$. which is homotopic to the identity map on L_k. (3) On the other hand, $g \circ h$ is homotopic to the identity on $\mathcal{L}$, so we also obtain a sequence of telescopes $T(\mathcal{L}) \to L_k \to T(\mathcal{L})$ which is homotopic to the identity on $T(\mathcal{L})$. Hence $T(\mathcal{L}) \simeq L_k$.

Using Lemma 3.2 we therefore have a homotopy commutative diagram

$$\begin{tikzcd}
T(\mathcal{L}) \arrow[r] \arrow[d, "\simeq"] & \mathcal{L} \arrow[d, "h"] \\
L_k \arrow[r, "l_k"] & \Sigma \bigwedge_{i=1}^k \Omega A_i ,
\end{tikzcd}$$

When h was defined prior to the Lemma, we saw that it was co-H. The telescope map in the top row of the diagram is co-H just as l_k was shown to be co-H in Lemma 4.3. The factorization in the diagram now proves the Lemma. ■

One particular case of Lemma 4.8 is used to give a condition under which L_k is cocommutative as well as coassociative.

Lemma 4.9 *If one of A_1 through A_k is homotopy cocommutative as well as coassociative then L_k is also homotopy cocommutative as well as coassociative.*

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

Proof Assume without loss of generality that A_k is homotopy cocommutative. The evaluation and inclusion $b\colon \Sigma\bigwedge_{i=1}^{k}\Omega A_i \to \Sigma\bigwedge_{i=1}^{k-1}\Omega A_i \wedge A_k \to \Sigma\bigwedge_{i=1}^{k}\Omega A_i$ is an idempotent with telescope $\mathcal{L}$. By Lemma 4.8 there is a factorization of the inclusion $L_k \xrightarrow{l_k} \Sigma\bigwedge_{i=1}^{k}\Omega A_i$ as a composite of co-H maps $L_k \xrightarrow{c} (\bigwedge_{i=1}^{k-1}\Omega A_i)\wedge A_k \xrightarrow{d} \Sigma\bigwedge_{i=1}^{k}\Omega A_i$. The coassociative co-$H$ structure on L_k comes from a left homotopy inverse r of l_k. This co-H structure is equivalent to the one obtained by retracting L_k off $(\bigwedge_{i=1}^{k-1}\Omega A_i)\wedge A_k$ by c and $r\circ d$ because d is co-H. But the latter co-H structure is cocommutative because A_k is. ■

Proof of Proposition 1.4 The assertions of the Proposition follow from Lemmas 4.3, 4.4, 4.6, 4.7, and 4.9. ■

5 The Generalized James Decomposition

To make sure the decomposition in Theorem 1.1 can be chosen naturally, a momentary digression is necessary. Recall that the join of two spaces X and Y is defined by $X * Y = (X\times I\times Y)/\sim$ where $(x,1,*)\sim * \sim (*,0,y)$. The suspension of $X\times Y$ is defined by $\Sigma(X\times Y) = X\times Y\times I/\sim'$ where $(x,y,1)\sim' * \sim' (x,y,0)$. The definitions yield a quotient map $X*Y \to \Sigma(X\times Y)$ which is natural in both variables. The homotopy equivalence $X*Y \simeq \Sigma X\wedge Y$ is also natural in both variables, and so we have a map $\Sigma X\wedge Y \to \Sigma(X\times Y)$ which is natural in both variables. Further, this map is a right homotopy inverse for the suspension of the usual quotient map $X\times Y\to X\wedge Y$. The following lemma says that this sort of thing happens for more than two variables.

Lemma 5.1 *For $n\geq 2$, there is a map*

$$\Sigma X_1\wedge\cdots\wedge X_n \longrightarrow \Sigma(X_1\times\cdots\times X_n)$$

which is natural in each of the n variables and which is a right homotopy inverse to the suspension of the usual quotient map $X_1\times\cdots\times X_n\to X_1\wedge\cdots\wedge X_n$.

Proof The proof is by induction. Let $Y = X_1\times\cdots\times X_{n-1}$. The $n=2$ case gives a map $\theta\colon \Sigma Y\wedge X_n \to \Sigma(Y\times X_n)$ which is natural in both variables. The $n-1$ case, smashed with X_n, gives a map $\psi\colon (\Sigma X_1\wedge\cdots\wedge X_{n-1})\wedge X_n \to \bigl(\Sigma(X_1\times\cdots\times X_{n-1})\bigr)\wedge X_n$ which is natural in each of the variables $X_1,\dots X_{n-1}$. The composite $\theta\circ\psi$ then gives the asserted map. ■

Remark The naturality in Lemma 5.1 is not free of choice. The induction used an ordering of the n variables, in much the same way as an order is chosen when multiplying in an H-space which is not associative. One wonders whether there is a choice of the natural map in Lemma 5.1 which does not depend on a preset ordering.

We now set ourselves up for proving Theorem 1.1. Suppose A is a homotopy coassociative co-H space. In Proposition 1.4 let A_1 through A_k all equal A. Let M_k be the space L_k in this case. Then M_k is coassociative, it is cocommutative if A is, it is a retract of $\Sigma(\Omega A)^{(k)}$, and $\Sigma^{k-1}M_k \simeq A^{(k)}$.

Suspend the multiplication $\prod_{i=1}^{k} \Omega A \to \Omega A$ and use Lemma 5.1 to obtain a map

$$\mu^*\colon \Sigma(\Omega A)^{(k)} \longrightarrow \Sigma\Big(\prod_{i=1}^{k} \Omega A\Big) \longrightarrow \Sigma\Omega A.$$

Let ψ_k be the composite

$$\psi_k\colon M_k \longrightarrow \Sigma(\Omega A)^{(k)} \xrightarrow{\mu^*} \Sigma\Omega A.$$

By Lemma 2.1, $H_*(\Sigma\Omega A)$ is isomorphic as an algebra to the suspension of the tensor algebra generated by $\Sigma^{-1}\tilde{H}_*(A)$. Corollary 4.5 then implies the following.

Lemma 5.2 *The inclusion $M_k \to \Sigma\Omega A$ maps $H_*(M_k)$ isomorphically onto the suspension of the submodule of tensor algebra elements in $H_*(\Omega A) \cong T\big(\Sigma^{-1}\tilde{H}_*(A)\big)$ of length k.* ■

Proof of Theorem 1.1 Let

$$\psi\colon \bigvee_{k=1}^{\infty} M_k \to \Sigma\Omega A$$

be the universal map defined by piecing together the ψ_k's. By Lemma 5.2, ψ_* is an isomorphism and so ψ is a homotopy equivalence.

For naturality, if $A \to B$ is a co-H map between coassociative co-H spaces, then μ^* is natural by Lemma 5.1, the idempotent γ_k on $\Sigma(\Omega A)^{(k)}$ is natural by Lemma 4.2 and so the map $M_k \to \Sigma(\Omega A)^{(k)}$ is natural by Lemma 3.2. Thus ψ_k and hence ψ are natural.

Also note that if $A = \Sigma A'$ then Lemma 4.6 implies that the decomposition of $\Sigma\Omega A$ above is equivalent to the usual one of $\Sigma\Omega\Sigma A'$. ■

6 The Generalized Hilton-Milnor Theorem

Theorem 1.2 generalizes the Hilton-Milnor theorem. Recall the usual setup. Let $i_1\colon \Sigma X \to \Sigma X \vee \Sigma Y$ and $i_2\colon \Sigma Y \to \Sigma X \vee \Sigma Y$ be the inclusions of the left and right wedge summands respectively. Let $L\langle i_1, i_2\rangle$ be the free Lie algebra on the two indicated generators. Let $\mathfrak{I}$ be an indexing set running over a vector space basis of $L\langle i_1, i_2\rangle$. For $\alpha \in \mathfrak{I}$, let α_1, α_2 respectively be the number of occurrences of i_1, i_2 in α. The basis element in $L\langle i_1, i_2\rangle$ represented by α corresponds to an iterated Whitehead product $\Sigma X^{(\alpha_1)} \wedge Y^{(\alpha_2)} \to \Sigma X \vee \Sigma Y$ which will also be denoted by α. Looping so we can multiply gives the Hilton-Milnor decomposition

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \prod_{\alpha\in\mathfrak{I}} \Omega\Sigma(X^{(\alpha_1)} \wedge Y^{(\alpha_2)}).$$

To make the move from Whitehead product on suspensions to Whitehead products on coassociative co-H spaces, we need to factor α. The identity map on ΣX

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

factors as the composite $\Sigma X \xrightarrow{\Sigma j} \Sigma\Omega\Sigma X \xrightarrow{\mathrm{ev}} \Sigma X$, and similarly for ΣY. The iterated Whitehead product $\Sigma X^{(\alpha_1)} \wedge Y^{(\alpha_2)} \xrightarrow{\alpha} \Sigma X \vee \Sigma Y$ is built from the inclusions i_1 and i_2 of ΣX and ΣY respectively into $\Sigma X \vee \Sigma Y$. Let α' be the iterated Whitehead product $\alpha'\colon \Sigma(\Omega\Sigma X)^{(\alpha_1)} \wedge (\Omega\Sigma Y)^{(\alpha_2)} \to \Sigma X \vee \Sigma Y$, where each occurrence of i_1 and i_2 in α has been replaced by $i_1 \circ \mathrm{ev}$ and $i_2 \circ \mathrm{ev}$ respectively. Naturality of the Whitehead product then implies that α factors as the composite

$$f_\alpha\colon \Sigma X^{(\alpha_1)} \wedge Y^{(\alpha_2)} \longrightarrow \Sigma(\Omega\Sigma X)^{(\alpha_1)} \wedge (\Omega\Sigma Y)^{(\alpha_2)} \xrightarrow{\alpha'} \Sigma X \vee \Sigma Y,$$

where the left map is the suspension $\Sigma j^{(\alpha_1)} \wedge j^{(\alpha_2)}$.

Proof of Theorem 1.2 Suppose A and B are homotopy coassociative co-H spaces. Fix $\alpha \in \mathcal{I}$. Let N_α be the space $L_{\alpha_1+\alpha_2}$ obtained by applying Proposition 1.4 to $\Sigma(\Omega A)^{(\alpha_1)} \wedge (\Omega B)^{(\alpha_2)}$. Then N_α is homotopy coassociative, is homotopy cocommutative as well if at least one of A or B is, and $\Sigma^{\alpha_1+\alpha_2-1} N_\alpha \simeq A^{(\alpha_1)} \wedge B^{(\alpha_2)}$. As well, there is a "Whitehead product"

$$g_\alpha\colon N_\alpha \longrightarrow \Sigma(\Omega A)^{(\alpha_1)} \wedge (\Omega B)^{(\alpha_2)} \xrightarrow{\alpha'} A \vee B.$$

This is the analogue of f_α in the suspension case and Lemma 2.1 combined with Lemma 4.4 show that Ωg_α has the analogous image in homology as Ωf_α. Multiplying the Ωg_α's then implies the map

$$\prod_{\alpha \in \mathcal{I}} \Omega N_\alpha \to \Omega(A \vee B)$$

is a homology isomorphism and hence a homotopy equivalence.

The naturality in Proposition 1.4 and the naturality of the Whitehead product imply that both maps in the composite defining g_α are natural with respect to co-H maps $A \to C$ and $B \to D$ between coassociative co-H spaces. Thus g_α is natural and hence so is the product decomposition of $\Omega(A \vee B)$.

One should also note that Theorem 1.2 is nothing new if $A = \Sigma A'$ and $B = \Sigma B'$. For then the definition of g_α shows it is really f_α for A' and B', and the decomposition in Theorem 1.2 becomes the standard Hilton-Milnor decomposition of $\Omega(\Sigma A' \vee \Sigma B')$. ∎

7 Idempotents on M_k

This section constructs p-local wedge decompositions of the spaces M_k in Theorem 1.1. We show the symmetric group Σ_k on k letters acts on M_k and use this to construct idempotents which give the splittings.

Let σ be an element in Σ_k. If X is a space then we can define a map $\sigma\colon X^{(k)} \to X^{(k)}$ by permuting the factors in the smash product. An analogous self-map $M(\sigma)$ can be defined on M_k by the composite

$$M(\sigma)\colon M_k \longrightarrow \Sigma(\Omega A)^{(k)} \xrightarrow{\Sigma\sigma} \Sigma(\Omega A)^{(k)} \longrightarrow M_k.$$

We wish to show that $M(\sigma)$ is a co-H map and satisfies the composition property $M(\sigma_1 \circ \sigma_2) \simeq M(\sigma_1) \circ M(\sigma_2)$. We again use $X = \Omega A$. Let $\sigma \in \Sigma_k$. We first line up the definition of $M(\sigma)$ with a similar one for $\Sigma X^{(k)}$. Let $X(\sigma)$ be the composite

$$X(\sigma)\colon \Sigma X^{(k)} \xrightarrow{\Sigma E^{(k)}} \Sigma(\Omega\Sigma X)^{(k)} \xrightarrow{\Sigma\sigma} \Sigma(\Omega\Sigma X)^{(k)} \longrightarrow \Sigma X^{(k)}.$$

Note that the naturality of σ implies that $\sigma \circ E^{(k)} \simeq E^{(k)} \circ \sigma$, and so $X(\sigma)$ is homotopic to $\Sigma X^{(k)} \xrightarrow{\Sigma\sigma} \Sigma X^{(k)}$. Thus if $\sigma_1, \sigma_2 \in \Sigma_k$ then $X(\sigma_1) \circ X(\sigma_2) \simeq X(\sigma_1 \circ \sigma_2)$.

Let $\sigma \in \Sigma_k$. Consider the diagram

$$\begin{tikzcd}
M_k \arrow[r] \arrow[d, "s"] & \Sigma(\Omega A)^{(k)} \arrow[r, "\Sigma\sigma"] \arrow[d] & \Sigma(\Omega A)^{(k)} \arrow[r] \arrow[d] & M_k \arrow[d, "s"] \\
\Sigma X^{(k)} \arrow[r, "\Sigma E^{(k)}"] & \Sigma(\Omega\Sigma X)^{(k)} \arrow[r, "\Sigma\sigma"] & \Sigma(\Omega\Sigma X)^{(k)} \arrow[r] & \Sigma X^{(k)}.
\end{tikzcd}$$

The middle square homotopy commutes by the naturality of σ. The left and right squares homotopy commute by the diagram of telescopes in Lemma 3.2 applied, as we have seen before, to the idempotents γ_k and λ_k. Also, the map $\Sigma(\Omega A)^{(k)} \to \Sigma(\Omega\Sigma X)^{(k)}$ is a suspension and has a left homotopy inverse, so by Lemma 3.3, the induced map s of telescopes is a co-H map and has a left homotopy inverse. Finally, note that the top row of this diagram is the definition of $M(\sigma)$ while the bottom row is the definition of $X(\sigma)$.

Lemma 7.1 *Let $\sigma_1, \sigma_2 \in \Sigma_k$. Then as self-maps of M_k we have $M(\sigma_1 \circ \sigma_2) \simeq M(\sigma_1) \circ M(\sigma_2)$.*

Proof The diagram preceding the Lemma shows that for any $\sigma \in \Sigma_k$ we have $X(\sigma) \circ s \simeq s \circ M(\sigma)$. Further, if $\sigma_1, \sigma_2 \in \Sigma_k$ then preceding the three squares in the diagram for σ_1 with the three squares for σ_2 shows that $s \circ M(\sigma_1) \circ M(\sigma_2) \simeq X(\sigma_1) \circ X(\sigma_2) \circ s$. When defining $X(\sigma)$ we noted that $X(\sigma_1 \circ \sigma_2) \simeq X(\sigma_1 \circ \sigma_2)$. Thus $s \circ M(\sigma_1) \circ M(\sigma_2) \simeq X(\sigma_1 \circ \sigma_2) \simeq s \circ M(\sigma_1 \circ \sigma_2)$. Composing this homotopy with a left homotopy inverse for s then proves the lemma. ■

Remark If $A = \Sigma A'$ then by Lemma 4.6 $M_k \to \Sigma(\Omega A')^{(k)}$ is equivalent to the inclusion $\Sigma(A')^{(k)} \to \Sigma(\Omega\Sigma A')^{(k)}$. So in this case $M(\sigma)$ is homotopic to the permutation $\Sigma\sigma$ on $\Sigma(A')^{(k)}$.

Lemma 7.2 *Let $\sigma \in \Sigma_k$. Then the map $M_k \xrightarrow{M(\sigma)} M_k$ is a co-H map.*

Proof Recalling that $X(\sigma)$ is homotopic to the map $\Sigma X^{(k)} \xrightarrow{\Sigma\sigma} \Sigma X^{(k)}$, the diagram preceding Lemma 7.1 (and the comments following it) shows there is a homotopy commutative diagram

$$\begin{tikzcd}
M_k \arrow[r, "M(\sigma)"] \arrow[d, "s"] & M_k \arrow[d, "s"] \\
\Sigma X^{(k)} \arrow[r, "\Sigma\sigma"] & \Sigma X^{(k)},
\end{tikzcd}$$

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

where s is a co-H map.

Let $\Delta_M\colon M_k \to M_k \vee M_k$ and $\Delta_X\colon \Sigma X^{(k)} \to \Sigma X^{(k)} \vee \Sigma X^{(k)}$ be the respective comultiplications. Let $t\colon \Sigma X^{(k)} \to M_k$ be a left homotopy inverse of s. Recall that by definition, $M(\sigma) = t \circ \Sigma\sigma \circ s$. Now consider what happens when the square above is followed by the composite

$$\Sigma X^{(k)} \xrightarrow{\Delta_X} \Sigma X^{(k)} \vee \Sigma X^{(k)} \xrightarrow{t\vee t} M_k \vee M_k.$$

On the one hand, since both s and $\Sigma\sigma$ are co-H maps we have $(t\vee t)\circ\Delta_X\circ(\Sigma\sigma\circ s) \simeq (t\vee t)\circ(\Sigma\sigma\vee\Sigma\sigma)\circ(s\vee s)\circ\Delta_M \simeq \big(M(\sigma)\vee M(\sigma)\big)\circ\Delta_M$. On the other hand, since s is co-H we have $(t\vee t)\circ\Delta_X\circ(s\circ\Delta_M) \simeq (t\vee t)\circ(s\vee s)\circ\Delta_M\circ M(\sigma) \simeq \Delta_M\circ M(\sigma)$. As the diagram implies $\Sigma\sigma\circ s \simeq s\circ M(\sigma)$, we therefore have $\big(M(\sigma)\vee M(\sigma)\big)\circ\Delta_M \simeq \Delta_M\circ M(\sigma)$, which proves that $M(\sigma)$ is a co-H map. ∎

We next use elements in the symmetric group to construct idempotents on M_k. Let p be a prime and let $\mathbf{Z}_{(p)}$ be the p-local integers. Let $R_k = \mathbf{Z}_{(p)}[\Sigma_k]$ be the group ring of Σ_k. The k-th Dynkin-Specht-Wever element β_k in R_k is defined inductively by $\beta_2 = 1-(1,2)$ and $\beta_k = \big(1-(k,k-1,\dots,2,1)\big)(1\otimes\beta_{k-1})$. By [J] we have $\beta_k\beta_k \simeq k\beta_k$. So if $(p,k)=1$ and we let $e_k = \frac{1}{k}\beta_k$ then e_k is an idempotent.

Let X be a space. The symmetric group acts on $X^{(k)}$. Suspending so we can add, the Dynkin-Specht-Wever element β_k determines a map $\bar\beta_k\colon \Sigma X^{(k)} \to \Sigma X^{(k)}$. As well, β_k acts on $\bigotimes_{i=1}^k \tilde H_*(X;\mathbf{Z}/p\mathbf{Z})$ by $\beta_k(x_1\otimes\cdots\otimes x_k) = \Big[x_1,\big[x_2,[\cdots[x_{k-1},x_k]\cdots]\big]\Big]$. The map $\bar\beta_k$ induces the suspension of this action in homology.

Just as we defined a self-map $M(\sigma)$ of M_k from an element of Σ_k by using $\Sigma(\Omega A)^{(k)}$ as an intermediary, so can we define a map $M(\beta_k)$. Precisely, $M(\beta_k)$ is the composite

$$M_k \longrightarrow \Sigma(\Omega A)^{(k)} \xrightarrow{\bar\beta_k} \Sigma(\Omega A)^{(k)} \longrightarrow M_k.$$

Lemma 7.1 tells us this is homotopic to inductively defining $M(\beta_k)$ by $M(\beta_2) = 1-M\big((1,2)\big)$ and $M(\beta_k) = \Big(1-M\big((k,k-1,\dots,2,1)\big)\Big)\circ M\big((1)(\beta_{k-1})\big)$. Since $\Sigma(\Omega A)^{(k)}$ is coassociative but not cocommutative one cannot perform the arithmetic (distributivity in particular) to show that $\bar\beta_k\circ\bar\beta_k \simeq k\cdot\bar\beta_k$. On the other hand, if A is homotopy coassociative and cocommutative then so is M_k and $M_k \to \Sigma(\Omega A)^{(k)}$ is a co-H map. Now it is possible to perform the arithmetic in the abelian group $[M_k, M_k]$ which shows that $M(\beta_k)\circ M(\beta_k) \simeq k\cdot M(\beta_k)$.

Suppose spaces and maps have been localized at a prime p. If $(p,k)=1$ then we can define a map $M(e_k)\colon M_k \to M_k$ by $M(e_k) = \frac{1}{k}M(\beta_k)$. We have $M(e_k)\circ M(e_k) \simeq M(e_k)$ and so $M(e_k)$ is an idempotent. Let U_k and V_k be the telescopes of $M(e_k)$ and $1-M(e_k)$ respectively. Then $M_k \simeq U_k \vee V_k$.

Since M_k is a co-H space, both U_k and V_k have co-H structures determined by their retractions off M_k. However, much more is true in this case. Recall that two co-H spaces are co-H equivalent if there is a co-H map between them which is also a homotopy equivalence. We will show in Lemma 7.4 that there is a co-H equivalence $M_k \to U_k \vee V_k$, and this lets us determine that U_k and V_k are homotopy coassociative and cocommutative. First we need the following Lemma.

Lemma 7.3 *When $(p,k)=1$ the maps $M(\beta_k)$, $1-M(\beta_k)\colon M_k\to M_k$ are both co-H maps.*

Proof Since M_k is both homotopy coassociative and cocommutative, sums and differences of co-H maps with M_k as their domain are also co-H maps. Lemma 7.2 then implies that the map $M_k \xrightarrow{\bar\beta_k} M_k$ is a composition of co-H maps and so is co-H. If $(p,k)=1$, then the map $k\colon M_k\to M_k$ is a co-H map and a homotopy equivalence, so an inverse $\frac{1}{k}\colon M_k\to M_k$ is also a co-H map. Thus $M(\beta_k)=\frac{1}{k}\cdot M(\bar\beta_k)$ is a co-H map. ∎

Lemma 7.4 *When $(p,k)=1$ the telescope maps $M_k\to U_k$ and $M_k\to V_k$ are co-H maps and add to give a co-H equivalence $M_k\to U_k\vee V_k$. Further, this equivalence implies both U_k and V_k are homotopy coassociative and cocommutative.*

Proof By Lemma 7.3, the idempotent $M(\beta_k)\colon M_k\to M_k$ is a co-H map and so by Lemma 3.1 the map to the telescope $M_k\to U_k$ is a co-H map. Similarly for $1-M(\beta_k)$ and $M_k\to V_k$. Since M_k is homotopy coassociative and cocommutative the sum of two co-H maps is again a co-H map, so the homotopy equivalence $M_k\to U_k\vee V_k$ is also a co-H map. The assertions about the homotopy coassociativity and cocommutativity of U_k and V_k follow from Lemma 2.6. ∎

Summarizing the results of this section so far we have:

Proposition 7.5 *Localize at a prime p. Suppose A is a homotopy coassociative, cocommutative co-H space. By Theorem 1.1 $\Sigma\Omega A\simeq\bigvee_{k=1}^{\infty}M_k$ where each M_k is a coassociative, cocommutative co-H space. If $(p,k)=1$ then there is a co-H equivalence $M_k\simeq U_k\vee V_k$, and each of U_k and V_k are coassociative, cocommutative co-H spaces.* ∎

We next explicitly state the homological consequences of Proposition 7.5. In what follows we take homology with mod-p coefficients. If $(p,k)=1$ the Dynkin-Specht-Wever element $\frac{1}{k}\beta_k$ determines an idempotent on $\bigotimes_{i=1}^{k}\Sigma^{-1}\tilde H_*(A)$ by permutation. Let D_A be the direct limit. This gives a sequence

$$\bigotimes_{i=1}^{k}\Sigma^{-1}\tilde H_*(A)\xrightarrow{c}D_A\xrightarrow{d}\bigotimes_{i=1}^{k}\Sigma^{-1}\tilde H_*(A),$$

where $d\circ c$ equals $\frac{1}{k}\beta_k$ and $c\circ d$ equals the identity map. Note that $c(a_1\otimes\cdots\otimes a_k)=\Big[a_1,\big[a_2,[\cdots[a_{k-1},a_k]\cdots]\big]\Big]$. By Corollary 4.5, $\tilde H_*(M_k)\cong\Sigma\big(\bigotimes_{i=1}^{\infty}\tilde H_*(A)\big)$. By construction, the action of $\frac{1}{k}\beta_k$ on the space M_k induces the suspension of the action of $\frac{1}{k}\beta_k$ on $\bigotimes_{i=1}^{k}\Sigma^{-1}\tilde H_*(A)$. Thus:

Lemma 7.6 *If $(p,k)=1$ then $H_*(U_k)\cong\Sigma D_A$ and the map $M_k\to U_k$ induces Σc in homology.* ∎

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

8 p-Local Homotopy Decompositions of ΩA

Throughout this section assume that all spaces and maps have been localized at a prime p. All homology calculations will be with mod-p coefficients. Assume that A is a homotopy coassociative, cocommutative co-H space so that each M_k is as well. Finally, we will only consider those values of k such that $(p,k)=1$.

The splitting of M_k in Proposition 7.5 will be used to construct homotopy decompositions of ΩA. We begin by defining some maps. For a space Y, let $W_k\colon \Sigma(\Omega Y)^{(k)} \to Y$ be the k-fold Whitehead product of the evaluation map $\Sigma\Omega Y \xrightarrow{\mathrm{ev}} Y$ with itself. Note that if Y is a suspension, $Y=\Sigma Z$, then the composite $w_k\colon \Sigma Z^{(k)} \to \Sigma(\Omega\Sigma Z)^{(k)} \xrightarrow{W_k} \Sigma Z$ is homotopic to the k-fold Whitehead product of the identity map on ΣZ with itself.

For the coassociative co-H space A, we again let $X=\Omega A$ and use the co-H map $A \to \Sigma X$. As in Section 6 there is a "Whitehead product" $\bar{w}_k\colon M_k \to \Sigma(\Omega A)^{(k)} \xrightarrow{W_k} A$. The naturality of $\bar{w}_k$ implies there is a homotopy commutative diagram

$$\begin{tikzcd} M_k \arrow[r, "\bar{w}_k"] \arrow[d] & A \arrow[d] \\ \Sigma X^{(k)} \arrow[r, "w_k"] & \Sigma X. \end{tikzcd}$$

To build up to the proof of the decomposition in Theorem 8.4 we need to record some homological properties resulting from this diagram. This is done in Lemmas 8.1 and 8.3.

Lemma 8.1 *The image of $(\Omega\bar{w}_k)_*$ is the subalgebra of $H_*(\Omega A) \cong T\big(\Sigma^{-1}\tilde{H}_*(A)\big)$ generated by $P_kH_*(\Omega A)$, the submodule of primitives of tensor length k.*

Proof If the statement of the lemma replaced $M_k \xrightarrow{\bar{w}_k} A$ by $\Sigma X^{(k)} \xrightarrow{w_k} \Sigma X$ then the conclusion would be clear. The proof of the lemma is simply a modification to the coassociative case of the usual one for suspensions.

We use the definition of $\bar{w}_k$ as the composition $M_k \xrightarrow{m_k} \Sigma(\Omega A)^{(k)} \xrightarrow{W_k} A$. By Corollary 4.5, $(m_k)_*$ is the suspension of the tensor algebra inclusion

$$t\colon \Sigma\Big(\bigotimes_{i=1}^{k} \Sigma^{-1}\tilde{H}_*(A)\Big) \to \Sigma\Big(\bigotimes_{i=1}^{k} T(\Sigma^{-1}\tilde{H}_*(A))\Big).$$

Since m_k is a co-H map, $(\Omega m_k)_*$ restricted to the generating set $\bigotimes_{i=1}^{k}\Sigma^{-1}\tilde{H}_*(A)$ of $H_*(\Omega M_k)$ is the inclusion t.

If $a_1,\dots,a_k \in H_*(\Omega\Sigma\Omega A)$ then the effect of the map $\Omega\Sigma\big((\Omega A)^{(k)}\big) \xrightarrow{\Omega W_k} \Omega A$ in homology is $(\Omega W_k)_*(a_1\otimes\cdots\otimes a_k) = k\cdot\Big[a_1,\big[a_2,\big[\cdots[a_{k-1},a_k]\cdots\big]\big]\Big]$. (Note that k is a unit since we have assumed $(p,k)=1$.) In particular, if $a_1\otimes\cdots\otimes a_k$ includes into $H_*(\Omega\Sigma\Omega A)$ through t then the image of $(\Omega W_k)_*(a_1\otimes\cdots\otimes a_k)$ is a primitive of tensor length k in $H_*(\Omega A)$.

Putting the last two paragraphs together, $(\Omega\bar{w}_k)_* = \big(\Omega(m_k \circ W_k)\big)_*$ sends the generating set $\bigotimes_{i=1}^{k} \Sigma^{-1}\tilde{H}_*(A)$ of $\tilde{H}_*(\Omega M_k)$ isomorphically onto $P_k H_*(\Omega A)$. Since $(\Omega\bar{w}_k)_*$ is multiplicative, the Lemma follows. ∎

Since the space U_k of Proposition 7.5 is a co-H space, by Lemma 7.6 we have $H_*(\Omega U_k) \cong T(D_A)$. Lemmas 7.6 and 8.1 combined then imply:

Lemma 8.2 *The composite*

$$D_A \longrightarrow H_*(\Omega U_k) \longrightarrow H_*(\Omega M_k) \xrightarrow{(\Omega\bar{w}_k)_*} H_*(\Omega A)$$

maps D_A isomorphically onto $P_k H_*(\Omega A)$. ∎

For a space Y, let $H_k \colon \Omega\Sigma Y \to \Omega\Sigma Y^{(k)}$ be the k-th James-Hopf invariant.

Lemma 8.3 *The composite*

$$\Omega M_k \xrightarrow{\Omega\bar{w}_k} \Omega A \longrightarrow \Omega\Sigma X \xrightarrow{H_k} \Omega\Sigma X^{(k)}$$

is multiplicative in homology.

Proof Cohen and Tayor [CT, 5.1] examine the composite $\Omega\Sigma Y^{(k)} \xrightarrow{\Omega w_k} \Omega\Sigma Y \xrightarrow{H_k} \Omega\Sigma Y^{(k)}$ for any space Y. They first consider $(H_k)_*$ and determine three properties of the primitives of $H_*(\Omega\Sigma Y)$. These properties allow them to immediately conclude that if $H_*(\Omega\Sigma Y)$ were isomorphic as a Hopf algebra to $T\big(\tilde{H}_*(Y)\big)$ then $H_k \circ \Omega w_k$ is multiplicative in homology. (Here, $T\big(\tilde{H}_*(Y)\big)$ is given the standard Hopf algebra structure by requiring $\tilde{H}_*(Y)$ to be primitive and then multiplicatively extending to the whole tensor algebra.) Such a Hopf algebra isomorphism happens, for example, if Y is a suspension.

In our case, $X = \Omega A$ is not a suspension, so $H_*(\Omega\Sigma X)$ is not isomorphic to $T\big(\tilde{H}_*(X)\big)$ as a Hopf algebra, the isomorphism is only as an algebra. On the other hand, A is a coassociative, cocommutative co-H space so by Lemma 2.2 $H_*(\Omega A)$ is isomorphic as a Hopf algebra to $T\big(\Sigma^{-1}\tilde{H}_*(A)\big)$. The co-$H$ inclusion $A \xrightarrow{s} \Sigma X$ lets one restate Cohen and Taylor's three properties about the primitives of $H_*(\Omega\Sigma X)$ and $(H_k)_*$ in terms of $H_*(\Omega A)$ and $(H_k \circ \Omega s)_*$. Now we can make the same conclusion, that $H_k \circ \Omega s \circ \Omega\bar{w}_k$ is multiplicative in homology. ∎

Let ϕ_k be the composite

$$\phi_k \colon U_k \longrightarrow M_k \xrightarrow{\bar{w}_k} A.$$

We now prove the decomposition in Theorem 1.3, restated as follows.

https://doi.org/10.4153/CJM-2003-008-5 Published online by Cambridge University Press

Theorem 8.4 *Suppose A is a homotopy coassociative, cocommutative co-H space. Then there is a homotopy decomposition*

$$\Omega A \simeq \Omega U_k \times F_k,$$

where F_k is the homotopy fiber of ϕ_k. This is natural for co-H maps $A \to B$ between coassociative, cocommutative co-H spaces.

Proof Consider the homotopy commutative diagram

$$\begin{tikzcd}
\Omega U_k \arrow[r] & \Omega M_k \arrow[r, "\Omega \overline{w}_k"] \arrow[d] & \Omega A \arrow[d] & & & \\
& \Omega\Sigma X^{(k)} \arrow[r, "\Omega w_k"] & \Omega\Sigma X \arrow[r, "H_k"] & \Omega\Sigma X^{(k)} \arrow[r] & \Omega M_k \arrow[r] & \Omega U_k.
\end{tikzcd}$$

Let $q\colon \Omega U_k \to \Omega U_k$ be the map from one end of the diagram to the other. By Lemma 8.3, q is multiplicative in homology.

By Lemma 8.2, the restriction of $(\Omega\bar{w})_*$ to the generating set D_A of $H_*(\Omega U_k)$ is an isomorphism onto $P_k H_*(\Omega A)$. This is a submodule of $P_k H_*(\Omega\Sigma X)$. The James-Hopf invariant $(H_k)_*$ sends the submodule of tensor algebra elements of length k isomorphically onto the set of homology generators of $H_*(\Omega\Sigma X^{(k)})$. The map then to $H_*(\Omega M_k)$ picks off the homology generators which came from tensor algebra elements of length k in $H_*(\Omega A)$. The further map to $H_*(\Omega U_k)$ picks off the primitive tensor elements of length k. Thus q_* is an isomorphism when restricted to the generating set D_A of $H_*(\Omega U_k)$. Since q_* is multiplicative, it is therefore an isomorphism. Hence q is a a homotopy equivalence.

The naturality of the decomposition follows because all the maps in the above diagram are natural. ■

References

[B1] I. Berstein, *A note on spaces with nonassociative comultiplication*. Proc. Cambridge Philos. Soc. **60**(1964), 353–354.

[B2] ———, *On cogroups in the category of graded algebras*. Trans. Amer. Math. Soc. **115**(1965), 257–269.

[BH] I. Berstein and J. R. Harper, *Cogroups which are not suspensions*. Algebraic Topology (Arcata 1986), 63–86, Lecture Notes in Math. **1370**(1989).

[CT] F. R. Cohen and L. R. Taylor, *The homology of function spaces*. Math. Z. **198**(1988), 299–316.

[CW] F. R. Cohen and J. Wu, *A remark on the homotopy groups of* $\Sigma^n \mathbf{R}P^2$. The Čech Centennial, 65–81, Contemp. Math. **181**(1995).

[Ga] T. Ganea, *Cogroups and suspensions*. Invent. Math. **9**(1970), 185–197.

[Gr1] B. Gray, EHP *spectra and periodicity I: geometric constructions*. Trans. Amer. Math. Soc. **340**(1993), 595–616.

[Gr2] ———, *Associativity in two-cell complexes*. Geometry and Topology: Aarhus (1998), 185–196, Contemp. Math. **258**(2000).

[J] N. Jacobson, *Lie Algebras*. Dover, 1962.

[R] J. W. Rutter, *The suspension of the loops on a space with comultiplication*. Math. Ann. **209**(1974), 69–82.
[T] S. D. Theriault, *A reconstruction of Anick's fibration*. Submitted.

University of Virginia
Charlottesville, Virginia 22904
U.S.A.
e-mail: st7b@virginia.edu

Current address:
University of Aberdeen
Aberdeen AB24 3UE
United Kingdom
email: s.theriault@maths.abdn.ac.uk